Molière Angular Distribution Under The Process With Ionization

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Abstract

The diffusion equation to get Molière angular distribution with ionization is solved analytically using Kamata and Nishimura formulation of the theory. The shape of the distribution is reduced to the traditional Molière function with some smaller expansion parameter B and with some larger unit of Molière angle compared with those without ionization. The new distribution is compared with the traditional one and the scale factor ν characterizing the new theory is indicated in figures.

1 Introduction:

Charged particles traversing through materials receive multiple Coulomb scattering and change their directions of motion successively. Many theoretical works have been proposed to describe the angular distribution of the particles after traversing matters. Among them the Molière theory [1, 2] is regarded most advanced taking account single and plural scatterings in the theory.

In spite of high accuracy of the theory, almost no improvements have been achieved to the Molière theory. There exists another formulation of Molière theory proposed by Kamata and Nishimura [3, 4], equivalent in mathematics [5]. Using the method we have succeeded in getting the Molière angular distribution with ionization. The results will be helpful in reliable designings and analyses of experiments concerning charged particles, as well as accurate and effective tracing of charged particles and basic tests of Monte Carlo simulations [6, 7].

2 The Angular Distribution With Ionization Under The Relativistic Condition:

The diffusion equation for the angular distribution of charged particles with ionization can be represented [3, 4] as

$$\frac{\partial f}{\partial t} = \iint \{f(\vec{\theta} - \vec{\theta'}) - f(\vec{\theta})\}\sigma(\vec{\theta'})d\vec{\theta'} + \varepsilon\frac{\partial f}{\partial E} \tag{1}$$

under the small angle approximation [8], where we measure t in radiation lengths [9]. The last term of the right-hand side means that the charged particles with the initial energy E_0 dissipate their energy in constant rate, ε in unit radiation length [9]:

$$E = E_0 - \varepsilon t. \tag{2}$$

We assume axial symmetry of Coulomb scatterings, then Hankel transforms of Eq. (1) gives

$$\frac{\partial \tilde{f}}{\partial t} = -2\pi\tilde{f}\int_0^\infty [1 - J_0(\zeta\theta)]\sigma(\theta)\theta d\theta + \varepsilon\frac{\partial \tilde{f}}{\partial E} \tag{3}$$

According to the Kamata-Nishimura formulation of Molière theory, this equation becomes

$$\frac{\partial \tilde{f}}{\partial t} = -\frac{\zeta^2}{w^2}\tilde{f}\{1 - \frac{1}{\Omega}\ln\frac{\zeta^2}{w^2/\beta^2}\} + \varepsilon\frac{\partial \tilde{f}}{\partial E}, \tag{4}$$

where w is the scattering coefficient defined in Rossi-Greisen [9], with scattering energy E_s replaced by Kamata-Nishimura's K [3, 4]:

$$w = 2pv/K. \tag{5}$$

If we take the relation (2) into account, the last term of the right hand side of Eq. (4) vanishes. In the relativistic condition, or in case where the energy E of charged particle is enough greater than the rest energy (referred to as *mass-less approximation*), we get the following relations

$$pv = E\{1 - (mc^2/E)^2\} \simeq E \quad \text{and} \quad w \simeq 2E/K,. \tag{6}$$

so that we get the Kamata-Nishimura equation [3, 4]:

$$\frac{\partial \tilde{f}}{\partial t} = -\frac{K^2\zeta^2}{4E^2}\tilde{f}\{1 - \frac{1}{\Omega}\ln\frac{K^2\zeta^2}{4E^2}\}, \tag{7}$$

where E depends on t, or using the relation (2) we get

$$\varepsilon\frac{\partial \ln \tilde{f}}{\partial E} = \frac{K^2\zeta^2}{4E^2}\{1 - \frac{1}{\Omega}\ln\frac{K^2\zeta^2}{4E^2}\}. \tag{8}$$

The solution satisfying the initial condition of $\tilde{f} = 1/(2\pi)$ at $E = E_0$ is expressed by

$$\tilde{f} = \frac{1}{2\pi}\exp\{-\frac{K^2\zeta^2 t}{4E_0E}[1 + \frac{1}{\Omega}(2 - \frac{E_0+E}{E_0-E}\ln\frac{E_0}{E}) - \frac{1}{\Omega}\ln\frac{K^2\zeta^2}{4E_0E}]\}. \tag{9}$$

Applying our translation formula [5] we get the probability density in Molière series:

$$f(\vartheta) = f^{(0)}(\vartheta) + B^{-1}f^{(1)}(\vartheta) + B^{-2}f^{(2)}(\vartheta) + \ldots. \tag{10}$$

The expansion parameter B is determined by

$$B - \ln B = \Omega - \ln\{\Omega/(\nu t)\} \quad \text{where} \quad \nu = e^2(E/E_0)^{(E_0+E)/(E_0-E)} \tag{11}$$

and the Molière angle ϑ is measured in the unit $\theta_{\rm M}$:

$$\vartheta = \theta/\theta_{\rm M} \quad \text{and} \quad \theta_{\rm M} = \theta_{\rm G}\sqrt{B/\Omega}. \tag{12}$$

$\theta_{\rm G}$ shows the well-known root mean square angle derived from the Fermi-Yang theory [9, 10, 11, 12] with E_s replaced by K:

$$\theta_{\rm G} = K\sqrt{t}/\sqrt{E_0E}. \tag{13}$$

The angular distributions with ionization are compared with those without ionization in Fig. 1.

3 Molière Angular Distribution With Ionization:

We should remind equation (7) of Kamata-Nishimura formulation is written under the relativistic condition or the mass-less approximation. At finite energies, we have to start with Eq. (4). We assume the solution of Eq. (4) to be

$$\tilde{f} = \frac{1}{2\pi}\exp\{-\frac{\theta_{\rm G}^2\zeta^2}{4}(1 - \frac{1}{\Omega}\ln\frac{\theta_{\rm G}^2\zeta^2}{4\nu t/\beta^2})\} \quad \text{with} \quad \theta_{\rm G}^2 = \int_0^t \frac{4}{w^2}dt \tag{14}$$

and introduce a new unknown function ν. Under the ionization process of a constant rate, gaussian mean square angle becomes

$$\theta_{\rm G}^2=\frac{K^2}{2\varepsilon}\{\frac{1}{pv}-\frac{1}{p_0v_0}+\frac{1}{2mc^2}\ln\frac{(E_0-mc^2)/(E-mc^2)}{(E_0+mc^2)/(E+mc^2)}\}. \tag{15}$$

If we could determine ν, the Molière angular distribution would be reduced to the traditional Molière expansion by using the translation formula indicated in [5], where B is determined by

$$B-\ln B=\Omega-\ln\{\Omega/(\nu t/\beta^2)\} \tag{16}$$

and the Molière angle (12) by (15).

4 Derivation of the New Scale Factor ν:

If we regard the dependence of E on t by the relation (2), then the last term of Eq. (4) vanishes. Substituting Eq. (14) into Eq. (4), we get the ordinary differential equation for ν:

$$\frac{\partial}{\partial t}\ln(\nu t/\beta^2)+\frac{4}{w^2\theta_{\rm G}^2}\ln(\nu t/\beta^2)=\frac{4}{w^2\theta_{\rm G}^2}(1-\ln\frac{4\beta^2}{w^2\theta_{\rm G}^2}). \tag{17}$$

The solution of this equation with initial value 1 of ν at $t=0$ becomes

$$\begin{aligned}\ln(\nu t/\beta^2) &= \exp\{-\int\frac{4dt}{w^2\theta_{\rm G}^2}\}\int_0^t[\frac{4}{w^2\theta_{\rm G}^2}(1-\ln\frac{4\beta^2}{w^2\theta_{\rm G}^2})\exp\{\int\frac{4dt}{w^2\theta_{\rm G}^2}\}]dt\\ &= 1-\frac{1}{w^2}\int_0^t\frac{4}{\theta_{\rm G}^2}\ln\frac{4\beta^2}{w^2\theta_{\rm G}^2}dt.\end{aligned} \tag{18}$$

Thus we get

$$\ln(\nu/\beta^2)=\ln\frac{\theta_{\rm G}^2}{4t}+\frac{4}{\theta_{\rm G}^2}\int_0^t\frac{\ln(w^2/\beta^2)}{w^2}dt, \tag{19}$$

hence

$$\nu/\beta^2=\frac{\theta_{\rm G}^2}{4t}\exp\{\frac{4}{\theta_{\rm G}^2}\int_0^t\frac{\ln(w^2/\beta^2)}{w^2}dt\}. \tag{20}$$

The scale factor ν obtained this time is a function of $E_0/(mc^2)$ and $E/(mc^2)$. Variation of ν with t is indicated in Fig. 2.

5 Limiting Cases:

At the limit of $\varepsilon\to 0$ we should get the result without ionization loss. In fact Eq. (20) reaches to

$$\nu\to 1 \qquad \text{with} \qquad \theta_{\rm G}\to K\sqrt{t}/(pv), \tag{21}$$

and the results are reduced to the original Molière-Bethe distribution [1, 2].

Under the relativistic condition or the mass-less approximation, the results of section 3 and 4 should reach to those of section 2. In fact, at the limit of $mc^2\to 0$ Eq. (20) reaches to

$$\nu\to e^2(E/E_0)^{(E_0+E)/(E_0-E)} \qquad \text{with} \qquad \theta_{\rm G}\to K\sqrt{t}/\sqrt{E_0E}. \tag{22}$$

If we imagine artificial material where Kamata-Nishimura's constant K is finite and Ω diverges, then the diffusion equation (4) becomes identical with that from gaussian approximation. Then we get

$$\tilde{f}=\frac{1}{2\pi}\exp\{-\frac{\theta_{\rm G}^2\zeta^2}{4}\} \quad \text{and} \quad fd\vec{\theta}=\frac{1}{\pi\theta_{\rm G}^2}\exp\{-\frac{\theta^2}{\theta_{\rm G}^2}\}d\vec{\theta}, \tag{23}$$

where $\theta_{\rm G}^2$ is the mean square angle derived from the gaussian approximation indicated in Eq. (15).

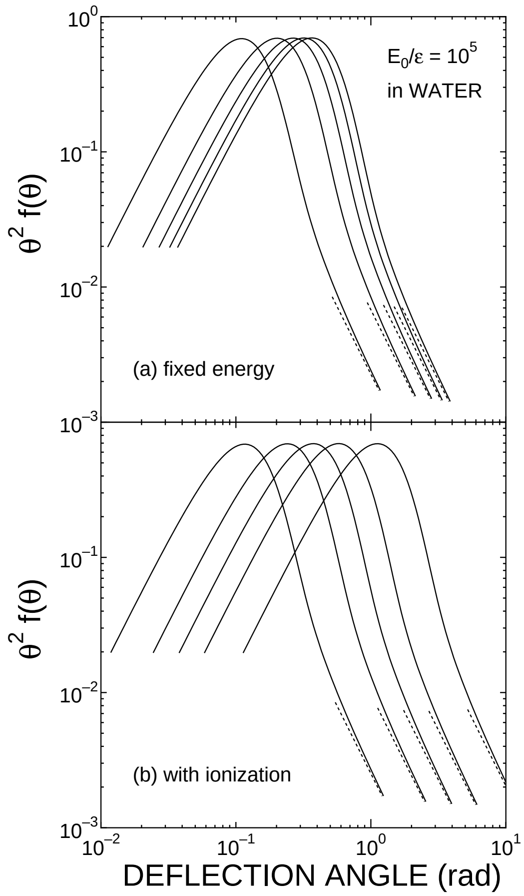

Figure 1: Angular distributions multiplied by θ^2. Solid lines correspond to those at $t/(E_0/\varepsilon)$ of 0.1, 0.3, 0.5, 0.7, and 0.9 from left to right. Dot lines indicate accumulations of single scatterings.

6 Conclusions:

Multiple scattering theory to describe Molière angular distribution is improved to take into account ionization, using Kamata-Nishimura formulation of the theory. Traditional angular distributions derived by Molière-Bethe, Kamata-Nishimura and Fermi are indicated as limiting cases of our result and are unified to our present result. The new distribution will improve the accuracy and the reliability in tracing charged particles at experimental analyses and Monte Carlo simulations.

Acknoledgments

The author is indebted to Prof. Jun Nishimura for valuable discussions through this investigation.

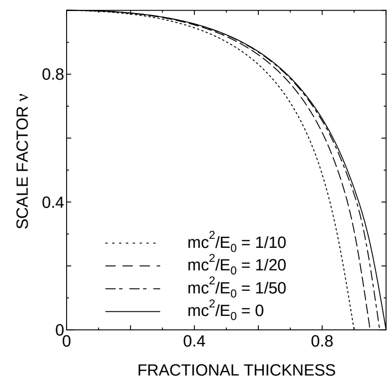

Figure 2: Variation of the scale factor ν against t. Abscissa means $t/(E_0/\varepsilon)$. The curves correspond to incident energies $E_0/(mc^2)$ of 10, 20, 50, and ∞.

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