Mechanism of Depth-Variation of Molière Angular Distribution

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Abstract

Mechanism of depth-variation of Molière angular distribution is investigated under the mass-less approximation using Kamata-Nishimura formulation of the Molière theory. Investigations are achieved in both the frequency space and the angular space. Depth-variation of the shape comes from the difference of increases of width between the central gaussian distribution and the asymptotic single scattering one. We have found change of the threshold angle discriminating moderate-angle scattering and large-angle one among the single scattering quantitatively explains the monotonous increase of B with increase of traversed thickness in case of fixed energy process and the monotonous decrease of the scale factor ν in case with ionization loss.

1   Introduction:

Under the multiple scattering process with ionization [1], it shows a distinctly different feature compared with that under the fixed energy process. The shape of angular distribution becomes young again, or more precisely the expansion parameter B becomes small again, before particles dissipate their whole energies. We examined the mechanism of depth-variation of Molière angular distribution under the mass-less approximation [1] with and without ionization using Kamata-Nishimura formulation of Molière theory [2, 3]. The theory separates the single scattering into moderate scattering and low-frequent large angle scattering, by the former we get the angular distribution of Williams type [4] and by the latter we get the effects of single and plural scatterings. The adjustment of discriminating angle between the two realizes the invariance of asymptotic single-scattering term and leads to the monotonous increase of B under the fixed energy process and becoming small again of B under the ionization process.

2   Depth-Variation of Molière Angular Distribution:

The Molière angular distribution is determined by the two parameters, the expansion parameter B and the unit of Molière angle θ_{M}, among them the shape of the distribution is determined by the former. According to our investigation [1], those in case with ionization are determined by

$$B - \ln B = \Omega - \ln\{\Omega/(\nu t)\} \quad \text{and} \quad \theta_{\mathrm{M}} = \theta_{\mathrm{G}}\sqrt{B/\Omega} \tag{1}$$

under the mass-less approximation, where ν and θ_{G} are expressed as

$$\nu = e^2 (E/E_0)^{(E_0+E)/(E_0-E)} \quad \text{and} \quad \theta_{\mathrm{G}} = K\sqrt{t}/\sqrt{E_0 E}. \tag{2}$$

In case without ionization we have $\nu = 1$ and $\theta_{\mathrm{G}} = K\sqrt{t}/E$.

We show depth-variation of B in Fig. 1 for various incident energies. In fixed-energy process, B increases monotonously with t. It means the weight of single and plural scatterings decreases gradually against multiple scattering. In case with ionization, B shows smaller value, so that the shape of angular distribution shows younger one than that in fixed-energy process compared at the same t. Before charged particles reach to their ranges, B becomes small again so that the shape of angular distribution becomes young again. This fact does not mean central structure of angular distribution becomes narrow again. The width of central gaussian distribution, which is well

represented by θ_M, increases monotonously with t and shows larger value than that in fixed-energy process as indicated in Fig. 2. The relative increase of θ_M to θ_G becomes small again at the final stage of their passages, which makes the shape young again.

3   Interpretation of Depth-Variation in The Frequency Space:

The diffusion equation for Molière angular distribution is described [2, 3] as

$$\frac{\partial f}{\partial t} = \iint \{f(\vec{\theta} - \vec{\theta'}) - f(\vec{\theta})\}\sigma(\vec{\theta'})d\vec{\theta'}. \tag{3}$$

$\sigma(\vec{\theta})$ denotes the single scattering formula. Under the azimuthally symmetrical condition, it becomes

$$\sigma(\theta)2\pi\theta d\theta dt = (\pi\Omega)^{-1}(K^2/E^2)\theta^{-4}2\pi\theta d\theta dt \quad \text{with} \quad \theta > \sqrt{e}\chi_a. \tag{4}$$

Applying Hankel transforms, Kamata-Nishimura rewrote the equation [2, 3] as

$$\frac{\partial \tilde{f}}{\partial t} = -\frac{K^2\zeta^2}{4E^2}\tilde{f}\{1 - \frac{1}{\Omega}\ln\frac{K^2\zeta^2}{4E^2}\}. \tag{5}$$

This equation is integrated as

$$\tilde{f} = \frac{1}{2\pi}\exp\{-\langle\frac{K^2\zeta^2}{4E^2}\rangle_{\text{av}}t + \frac{1}{\Omega}\langle\frac{K^2\zeta^2}{4E^2}\ln\frac{K^2\zeta^2}{4E^2}\rangle_{\text{av}}t\}, \quad \text{where} \quad \langle x\rangle_{\text{av}} \equiv t^{-1}\int_0^t x dt. \tag{6}$$

In case with ionization, the terms in Eq. (6) are determined by

$$\langle\frac{K^2\zeta^2}{4E^2}\rangle_{\text{av}} = \frac{K^2\zeta^2}{4E_0E} \quad \text{and} \quad \langle\frac{K^2\zeta^2}{4E^2}\ln\frac{K^2\zeta^2}{4E^2}\rangle_{\text{av}} = \frac{K^2\zeta^2}{4E_0E}\ln\frac{K^2\zeta^2}{4E_0E\nu}, \tag{7}$$

where ν is defined in Eq. (2). Thus we get

$$\tilde{f} = \frac{1}{2\pi}\exp\{-\frac{K^2\zeta^2t}{4E_0E}(1 - \frac{1}{\Omega}\ln\frac{K^2\zeta^2}{4E_0E\nu})\} = \frac{1}{2\pi}\exp\{-\frac{K^2\zeta^2t}{4E_0E}(1 + \frac{1}{\Omega}\ln\nu t - \frac{1}{\Omega}\ln\frac{K^2\zeta^2t}{4E_0E})\}. \tag{8}$$

Expanding the right-hand side up to the first degree of Ω^{-1} and using θ_G of (2), we have

$$2\pi\tilde{f} \simeq \exp\{-(1 + \frac{1}{\Omega}\ln\nu t)\frac{\theta_G^2\zeta^2}{4}\} - \frac{1}{\Omega}\exp\{-\frac{\theta_G^2\zeta^2}{4}\}\frac{\theta_G^2\zeta^2}{4}\ln\frac{\theta_G^2\zeta^2}{4} + \cdots, \tag{9}$$

so that we get the angular distribution

$$f2\pi\theta d\theta \simeq f^{(0)}\left(\frac{\theta/\theta_G}{\sqrt{1 + \Omega^{-1}\ln\nu t}}\right)\frac{\theta d\theta/\theta_G^2}{(1 + \Omega^{-1}\ln\nu t)} + \frac{1}{\Omega}f^{(1)}\left(\frac{\theta}{\theta_G}\right)\frac{\theta d\theta}{\theta_G^2} + \cdots, \tag{10}$$

where $f^{(0)}$ and $f^{(1)}$ denote the Molière functions [5, 6].

In fixed-energy process, we should substitute $\nu = 1$ in Eq. (10). The first term shows the central gaussian distribution and the second the asymptotic single-scattering distribution. There exists a little difference between units of the two distributions. The unit of the former varies a slight more rapid than the latter due to the factor of $(1 + \Omega^{-1}\ln t)^{1/2}$, which causes the quantitative correction to the central structure of angular distribution by the single scattering, mentioned in the section VII.B of Scott [7] and the section 4 of Kamata-Nishimura [2].

In case with ionization, ν decreases monotonously, so that the ratio of the unit of the first term to the latter decreases at the last stage of their passages, which causes the shape of the angular distribution young again. It should be noted that asymptotic feature of the second term also shows accumulation of single scattering distribution at large enough angle. In fact, we have

$$2\pi\theta d\theta\int_0^t \sigma(\theta)dt' = \frac{2\theta d\theta}{\Omega}\theta^{-4}\int_0^t \frac{K^2}{E'^2}dt' = \frac{1}{\Omega}\left(\frac{\theta}{\theta_G}\right)^{-4}d\left(\frac{\theta}{\theta_G}\right)^2 = \frac{1}{B}\left(\frac{\theta}{\theta_M}\right)^{-4}d\left(\frac{\theta}{\theta_M}\right)^2, \tag{11}$$

at the observation level of t.

4 Interpretation of Depth-Variation in The Angular Space:

We introduce a discrimination angle θ_s, below which the cross section means the moderate angle scattering σ_M and above which it means the low-frequency large angle scattering σ_L:

$$\theta_s = e^{\Omega/2}\sqrt{e}\chi_a = (K/E)e^{1-C} \quad \text{and} \quad \sigma(\theta) = \sigma_M(\theta) + \sigma_L(\theta). \tag{12}$$

Using the formula (14) of Bethe [3]

$$I_1(x) \equiv 4\int_x^\infty t^{-3}[1 - J_0(t)]dt \simeq 1 + \ln 2 - C - \ln x + O(x^2), \tag{13}$$

and neglecting higher order than $O(x^2)$, we find the diffusion factor corresponding to the Rutherford scattering of finite range, from $\theta_{\min}$ to $\theta_{\max}$, depends only on the ratio $\theta_{\max}/\theta_{\min}$ as

$$\int_{\theta_{\min}}^{\theta_{\max}} [1 - J_0(\theta\zeta)]\sigma(\theta)2\pi\theta d\theta \simeq \frac{1}{\Omega}\frac{K^2\zeta^2}{4E^2}\ln(\theta_{\max}/\theta_{\min})^2, \tag{14}$$

so that we get the following diffusion factors corresponding to the respective scatterings:

$$\int_0^\infty [1 - J_0(\theta\zeta)]\sigma_M(\theta)2\pi\theta d\theta \simeq \frac{K^2\zeta^2}{4E^2} \quad \text{and} \quad \int_0^\infty [1 - J_0(\theta\zeta)]\sigma_L(\theta)2\pi\theta d\theta \simeq -\frac{1}{\Omega}\frac{K^2\zeta^2}{4E^2}\ln\frac{K^2\zeta^2}{4E^2}. \tag{15}$$

We get the solution of Eq. (5) in power series of Ω^{-k}, according to Kamata-Nishimura [2, 3]

$$\tilde{f} = \tilde{f}_0 + \Omega^{-1}\tilde{f}_1 + \Omega^{-2}\tilde{f}_2 + \cdots, \tag{16}$$

where we distinguished f_k from $f^{(k)}$ of Molière-Bethe [5, 6]. f_k can be got successively from

$$\frac{\partial \tilde{f}_k}{\partial t} + \frac{K^2\zeta^2}{4E^2}\tilde{f}_k = \frac{K^2\zeta^2}{4E^2}\tilde{f}_{k-1}\ln\frac{K^2\zeta^2}{4E^2}. \tag{17}$$

Thus we have

$$\tilde{f}_0 = \frac{1}{2\pi}\exp\{-\int_0^t \frac{K^2\zeta^2}{4E^2}dt\} = \frac{1}{2\pi}\exp\{-\langle\frac{K^2\zeta^2}{4E^2}\rangle_{av}t\}, \tag{18}$$

$$\tilde{f}_1 = \frac{1}{2\pi}\exp\{-\int_0^t \frac{K^2\zeta^2}{4E^2}dt\}\int_0^t \frac{K^2\zeta^2}{4E^2}\ln\frac{K^2\zeta^2}{4E^2}dt = \frac{1}{2\pi}\exp\{-\langle\frac{K^2\zeta^2}{4E^2}\rangle_{av}t\}\langle\frac{K^2\zeta^2}{4E^2}\ln\frac{K^2\zeta^2}{4E^2}\rangle_{av}t. \tag{19}$$

In fixed-energy process, we get $\tilde{f}_1$ by using the property of σ_L:

$$\tilde{f}_1 = -\frac{\Omega t}{2\pi}\exp\{-\frac{\theta_G^2\zeta^2}{4}\}\int_{\theta_s}^\infty [1 - J_0(\zeta\theta)]\sigma(\theta)2\pi\theta d\theta = -\frac{1}{\pi}\exp\{-\frac{\theta_G^2\zeta^2}{4}\}\int_{e^{1-C}/\sqrt{t}}^\infty [1 - J_0(\theta_G\zeta x)]\frac{dx}{x^3}. \tag{20}$$

The last expression shows that $\tilde{f}_1$ with the lower boundary of integration multiplied by $\sqrt{t}$ would be the function of $\theta_G\zeta$, so that f_1 be of θ/θ_G, not depending on t explicitly. Thus, separating the integral to two parts, we get

$$\tilde{f}_1 = -\frac{1}{2\pi}\exp\{-\frac{\theta_G^2\zeta^2}{4}\}\Omega t\{\int_{\theta_s}^{\sqrt{t}\theta_s} + \int_{\sqrt{t}\theta_s}^\infty\} = \frac{1}{2\pi}\exp\{-\frac{\theta_G^2\zeta^2}{4}\}\{-\frac{\theta_G^2\zeta^2}{4}\ln t + \frac{\theta_G^2\zeta^2}{4}\ln\frac{\theta_G^2\zeta^2}{4}\}. \tag{21}$$

In case with ionization, $\tilde{f}_1$ becomes

$$\tilde{f}_1 = -\frac{1}{2\pi}\exp\{-\frac{\theta_G^2\zeta^2}{4}\}\Omega\int_0^t dt'\int_{\theta_s}^\infty [1 - J_0(\zeta\theta)]\sigma(\theta)2\pi\theta d\theta. \tag{22}$$

This time the discriminating angle θ_s increases with t' and the coefficient in the single scattering formula varies with t'. We separate angular integration into two parts by the fixed discriminating angle $\overline{\theta_s}$ corresponding to the geometrical mean energy $\sqrt{E_0 E}$. Then we get

$$
\begin{aligned}
\tilde{f}_1 &= -\frac{1}{2\pi}\exp\{-\frac{\theta_G^2 \zeta^2}{4}\}\Omega \int_0^t dt' \{\int_{\theta_s}^{\overline{\theta_s}} + \int_{\overline{\theta_s}}^{\infty}\} \\
&= -\frac{1}{2\pi}\exp\{-\frac{\theta_G^2 \zeta^2}{4}\}\Omega \int_0^t dt' \{\frac{1}{\Omega}\frac{K^2\zeta^2}{4E'^2}\ln\frac{E'^2}{E_0 E} - \frac{1}{\Omega}\frac{K^2\zeta^2}{4E'^2}\ln\frac{K^2\zeta^2}{4E_0 E}\} \\
&= \frac{1}{2\pi}\exp\{-\frac{\theta_G^2 \zeta^2}{4}\}\{-\frac{\theta_G^2 \zeta^2}{4}\ln\nu t + \frac{\theta_G^2 \zeta^2}{4}\ln\frac{\theta_G^2 \zeta^2}{4}\}.
\end{aligned}
\tag{23}
$$

So that $\tilde{f}_0 + \Omega^{-1}\tilde{f}_1$ gives Eq. (9) and $f_0 + \Omega^{-1}f_1$ gives Eq. (10), respectively. We can understand the probability density corresponding to the first term of Eq. (23) modifies the width of central gaussian distribution as the property of Kamata-Nishimura series function of $f_1^{(1)}$ indicated in [8].
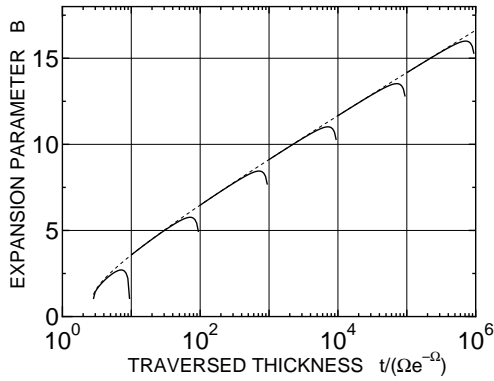

Figure 1: Depth-variation of expansion parameter B for various incident energies, E_0/ε of 10, 10^2, 10^3, 10^4, 10^5, and 10^6 in unit of $\Omega e^{-\Omega}$. Dot line indicates traditional B without ionization.
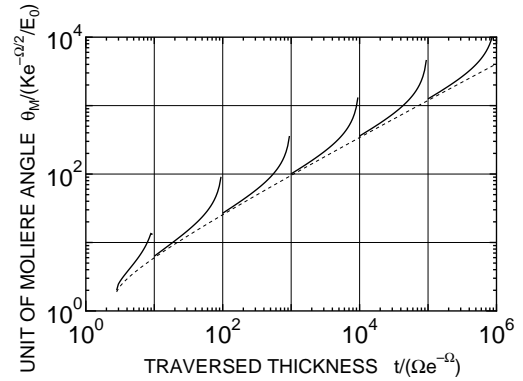
Figure 2: Depth-variation of the unit of Molière angle θ_M for various incident energies, E_0/ε of 10, 10^2, 10^3, 10^4, 10^5, and 10^6 in unit of $\Omega e^{-\Omega}$. Dot line indicates traditional θ_M without ionization.

5   Conclusions:

The shape of angular distribution becomes young again at the final stage of Molière process with ionization. The aging of the shape is oriented to relative increase of the unit of central gaussian distribution to that of asymptotic single-scattering one, both increasing with t. The mechanism can be explained by the move of Fourier component to and from the single scattering term in frequency space. In the angular space, it corresponds to the existence of discrimination angle separating the single scattering into the moderate and the large-angle scatterings. Increase of discrimination angle with t causes the monotonous aging in fixed-energy process and energy dependence of the discrimination angle in stead of a constant causes becoming young again of the shape in ionization process.

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