LONG RUN STABILITY AND THE NOISE

S. Heifets
University of Houston
and
Texas Accelerator Center

The purpose of this paper is two-fold. The first part contains discussion on the effects which limit the lifetime of a particle. The long-run tracking is necessary to observe these effects. The second part deals with the noise implemented in a tracking program that is considered as the means to make the tracking more efficient and in particular, independent of the initial conditions.

Let us assume that the magnet field multipoles b_m are small enough, so that the typical dimensionless parameter describing the effect of random multipoles

$$\lambda_m = (2\pi/\sqrt{N})\beta_\perp \sigma^{m-1} b_m$$

of the order 0.1 for an amplitude $\sigma \sim 0.5$cm (N is the number of independent random multipoles, $\beta_\perp$ is average beta function).

As a result, all nonlinear effects within the ''linear'' apperture are small, yielding the smear less than or about 10%, the tune spread of order of 10^{-3} etc., and a perturbation theory (canonical transformation, Lie algebra) is applicable. For small λ_m there is a hierarchy of the nonlinear resonances: the width of the resonances of the K-th order (driven by the terms of order of b_m^K) decreased as λ_m^K. Because the number of these resonances increases only as k^2, there is not overlapping of the higher order resonances, though the net of the resonances is dense, provided the main resonances do not overlap. So, according to Chirikov's criterion, there is no chaotic motion in the linear apperture, except within the thin stochastic layers in the vicinity of the separatrixes.

On the other hand, for $\lambda_m > 1$ there is no hierarchy of resonances and the overlapping results in chaotic motion and fast losses of the particles. If multipoles b_m are important up to m=6, the differences in the size of the linear apperture $\sigma_L = 5$mm, where $\lambda_6 \sim \sigma_L^5 \sim 0.1$, and the apperture $\sigma = 8$mm, where $\lambda_6 \sim 1.0$ is rather small. This means, that given b_m it makes no sense to make the physical apperture much larger than the linear apperture, (enlarged for an off momentum particle, including the closed orbit distortion, injection errors, etc.), and that a particle's life-time is given mostly by the life time of the particle inside the linear apperture, because outside of it the particles get lost very fast. The main effects which determine a particle's life-time in the linear apperture are well known: Arnold's diffusion, synchro-betatron resonances, weak modulation diffusion and the noise in the system.

The rate of Arnold's diffusion

$$D_A \sim \exp(-4\pi/\Delta\nu)$$

is expotentially small because the resonance width $\Delta\nu$ has to be much less than the tune spread 10^{-3} for the linear apperture [1], and the effect is negligible. The synchro-betatron resonances in the worst case of their overlapping could cause chaotic motion in a band with a width equal to the amplitude of the tune modulation. For the SSC this hardly takes place. The side-band resonances, caused by the periodic momentum variation with synchrotron frequency $Q_s = 2\times10^{-3}$, stay apart in tune on $\Delta\nu = Q_s$, what corresponds to the change of emittance

$$\Delta\varepsilon/\varepsilon \sim (Q_s/\varepsilon)(\partial\nu/\partial\varepsilon)^{-1} \sim 4\times10^{+3}$$

larger, than the whole emittance variation within the linear apperture $(5\text{mm}/0.1\text{mm})^2 \sim 2500$. In addition to that, the width of the side-band resonances decreases rapidly with the number of the resonance, provided that the linear chromaticity is canceled out, so that the tune modulation $\delta\nu \sim (\Delta p/p)^2 \lesssim Q_s$. The modulation of the Twiss parameters with $(\Delta p/p)$ also gives side-band resonances with the width proportional to $(\Delta p/p)^{K/2}$ for K-th resonance. The ripples of the power supply give the tune modulation of the first order $\delta \sim (\Delta B/B)$, but the most dangerous low-frequency components can be absent or suppressed. Therefore, there are only few side-band resonances and they do not overlap. This conclusion agrees with the preliminary results of the tracking with synchrotron modulations [2]. The weak modulation diffusion along the lines of the non-linear resonances, driven by the chaotic motion in the exponentially thin stochastic layers seem to be negligible, unless noise or damping deliver new particles into stochastic layers, though more careful consideration is needed of the structure of the resonances net inside the linear apperture.

The observation of the above mentioned exponentially small effects in tracking can demand a lot of CPU time. The situation is even worse than that, because the diffusion can be expected only for the particles close to the thin stochastic layers, so the results of tracking might critically depend on the choice of the initial conditions. To be representative, the long-run tracking has to be done with a large number of particles, which is almost impossible to do in practice. Below we intend to show that noise, implemented in tracking, can be a mean to overcome this difficulty.

The noise might be one of the mechanisms of particle losses. The sources of the noise mentioned in [1] (intrabeam scattering, synchrotron radiation, residual gas, rf noise) give the emittance growth rate of order of

$$d\varepsilon/dt \sim (10^{-15} \quad 10^{-17})\ \text{m/sec}$$

For $(\partial\nu/\partial\varepsilon) \sim 10^4\ \text{m}^{-1}$ it gives the tune shift

$$\delta\nu \sim (\partial\nu/\partial\varepsilon)\cdot(d\varepsilon/dt)\cdot t \sim 10^{-6}$$

for the luminosity lifetime $t \sim 1$ day.

The comparable noise is caused by the fluctuations of the displacement (δr) of one colliding bunch relative to another one. The fluctuation (δr) can be of order of 10% of rms beam size σ. This gives the emittance growth rate

$$d\varepsilon/dt \sim (4\pi\xi)^2\ (\delta r^2/2\sigma^2)^2\ \varepsilon\ N_{IR}\ f_o$$

where $\xi = N_B r_p/4\pi\varepsilon_n$ is the linear tune shift due to the beam interaction, N_{IR} is the number of interactions per ring, f_o is the revolution frequency. For the SSC the rate is $d\varepsilon/dt \sim 2\cdot10^{-15}$ m/sec.

Other sources of noise (Schottky noise, electron trapping) give less. So, the noise in the system is very small and for this reason it has not been included in the tracking simulations of the beam stability.

Nevertheless, it may be beneficial to include the controllable white noise in tracking. The main advantage is that noise eliminates the dependence of the trajectory behavior on the initial conditions, solving the dilemma: how many particles have to be tracked versus number of revolutions, if their product is given by the available amount of CPU time. Practically this can be done very easily by adding some random kicks in the tracking programs with an rms value that corresponds to the diffusion across the linear apperture during the tracking.

For stable dynamic system the diffusion rate is the diffusion rate due to the noise only. For stochastic systems where trajectories diverge exponentially, the effective diffusion rate is close to the intrinsic diffusion rate of the system. The noise only displays the average stochastic property of the system, which can be missed in tracking with a given trajectory.

To demonstrate this approach, the tracking of a single FODO cell has been done with an octupole kick and periodic tune modulation

$$\nu(s) = \nu_o + \delta\nu \cdot \mathrm{Cos}(Q_s\, s/R)$$

The model is not the simplest possible one, but it reproduces the tracking situation. The tune ν_o was chosen close to 1/4 resonance and the amplitude of the tune modulation was chosen to assure the crossing of the resonance, but small enough not to interfere with other nonlinear resonances. In the dimensionless variables the linear transformation for one revolution is just a rotation in the plane (ξ, η) on the angle $(2\pi\nu)$ and the octupole kick gives

$$\bar{\xi} = \xi\,;\ \bar{\eta} = \eta + K\bar{\xi}^3$$

The side band resonances are located at the resonance amplitudes $\varepsilon_r(k)$,

$$(1)\qquad \varepsilon_r(k) = (\nu_o - 1/4 - k\cdot Q_s/4)\left[2|V(k)| - \partial\nu/\partial\varepsilon\right]^{-1}$$

where $V(k) = VJ_k(4\delta\nu/Q_s)$, $V = k/16\pi$,

$$\partial\nu/\partial\varepsilon = 3k/8\pi$$

The amplitude width of the k-th resonance is

$$(2)\qquad |\varepsilon - \varepsilon_r(k)| = \varepsilon_r(k)\ \sqrt{4|V(k)|/(\partial\nu/\partial\varepsilon)}$$

The model on certain conditions can describe very complicated motion. In fig.1 the maximum value of $\varepsilon_{eff} = (\xi^2 + \eta^2)$ during 20 synchrotron cycles is shown versus the inverse synchrotron tune $N_s = Q_s^{-1}$ in the range $660 < N_s < 700$. The initial conditions were $\xi(o) = 1.0$, $\eta(o) = -1.0$. Other parameters were $K = 0.4353 \times 10^{-2}$, $\nu_o = 0.252$, $\delta\nu = 0.007$. Without synchrotron motion ($\delta = 0$) the 1/4 resonance in this case is located at $\nu = 0.24965$ and has the width $\Delta\nu = 1.0\times10^{-3}$. For small N_s (large Q_s) the motion is stable, and ε_{eff} remains close to the initial conditions. For small Q_s the motion becomes unstable, the amplitude blows up to the value, that corresponds to the tune shift of order of the tune modulation $\delta\nu$, as expected. But the dependence on Q_s is not smooth, there are sharp oscillations, which are very sensitive to the change of the initial conditions. It can be seen from the fig. 2, where maximum ε_{eff} during a synchrotron cycle is given versus the cycle number for total 75 cycles. The synchrotron tune $Q_s^{-1} = 665$, the total number of revolutions is 49875, $\nu_o = 1/4$, $\delta\nu = 0.007$, the kick is set in such a way, that initial conditions $\xi(o) = \eta(o) = 1.0$ correspond to the fixed point of the k=1 synchro-betatron resonance.

Line number 4 gives ε_{eff} for the particle located initially at the k=1 fixed point, line number 5 gives ε_{eff} for the particle with initial conditions $\xi(o) = \eta(o) = 1.3$, all other parameters are the same. In the first case the amplitude is almost not changed in the second case ε_{eff} blows up 40 times. Fig. 3 shows, how the amplitude is built up to turn by turn. There were totally 33200 revolutions, all parameters are the same, as for the line number 5 on fig.2 . The first 25000 turns the emittance is almost constant and increases very slowly up to 3 times at the end of this period, and after that blows up. Later the emittance fluctuates on the level that corresponds to the random motion within the band $|\nu(\varepsilon) - \nu_o| < \delta\nu$.

This example shows that any predictions of the particle's stability based on the short run tracking might be unreliable. Fig. 4 gives the phase plot for the same case. The structure of the phase space and the amplitude blow-up follows from the f. (1) and (2). The side band resonances overlap, if

$$\varepsilon_r(k)\left[1 + \sqrt{4|V(k)|\ \mathrm{x}\ (\partial\nu/\partial\varepsilon)^{-1}}\right] > \varepsilon_r(k+1)$$

or

with $\varepsilon_r(k=1) = 1$ and $\nu_o = 0.25$, for

$$k^2 > (3/2)\sqrt{2\pi N_s\ \delta\nu}$$

For $N_s = 665$ and $\delta v = 7\mathrm{x}10^{-3}$ it gives k>3, which corresponds to the emittance three times larger than the initial one. There are $k_{max} = 4\delta\nu/Q_s \sim 18$ side-band resonances with about the same width. The resonances with k>3 overlap, giving a stochastic layer. The central part of the band contains several isolated synchro-betatron resonances, so the motion with small initial amplitudes is stable in short-run sense. The slow creep of the amplitude up to $\varepsilon \sim 3\varepsilon_o$ can be explained by existence of thin stochastic layers in the central part of the band. We used this model to demonstrate how artificially introduced noise can make tracking of such systems independent of the initial conditions.

The results of tracking with random kicks are shown on fig. 2. The random numbers r_j uniformly distributed on the interval (-0.5,+0.5) were used to generate independent kicks on each revolution:

$$\bar{\eta} = \eta + F\cdot r_j$$

Factor F was equal to 1.9×10^{-2}. The initial seed was chosen after 15 runs with different seed to give average emittance growth. Line number 3 shows the amplitude grows due to the noise and the octupole kick, but without the synchrotron modulations. There is no blow up of the amplitude in the stable system noise and does not change anything drastically. If there are nonlinear kicks and the tune modulation then noise makes the instability of the system visible (line number 1), though the initial conditions were chosen the same, as for line 4, which does not indicate any stochastic motion.

In principle, the enhancement of the observed in simulation diffusion rate in comparison with the rate due to the noise only can give useful information on the average density of the nonlinear resonance and their width.

The enhancement of the diffusion is given by the effective Hamiltonian

$$H_{ef} = (n\nu_o - k)\varepsilon + (\partial\nu/\partial\varepsilon)(\varepsilon^2/2) + V_n \mathrm{Cos}\phi + (\beta d/n)\phi$$

where d gives rms of the random force.

$$\langle f(s)\rangle = 0; \quad \langle f(s)\, f(s')\rangle = d\ \delta(s-s')$$

This has the same form, as the Hamiltonian describing the crossing of the isolated resonance [3].
If the rate of crossing

$$v = (\beta d/\ ^{2})(\partial\nu/\partial\varepsilon)$$

is large $v \gg 1$, then the additional diffusion rate

$$\langle(\Delta\varepsilon)^2\rangle = \pi N\langle|v|^2\rangle/\beta\nu' d$$

depends on the average parameters of the resonance and their density. The enhancement of the diffusion rate was discussed also before [4]. In practice, however, it could be difficult to extract the structure of the phase space in this way because statistical fluctuations of the results of tracking can obscure the enhancement factor.

CONCLUSION

The preliminary estimation shows, that the observation of the very small effects, responsible for the finite life time of particles at the SSC, can demand a very elaborate long run tracking with necessarily large number of particles. Introduction of a noise in tracking can make such simulations less time consuming and more reliable.

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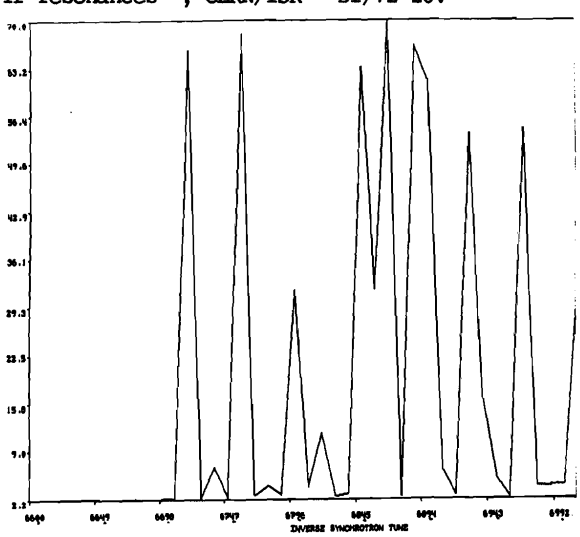

Figure 1.

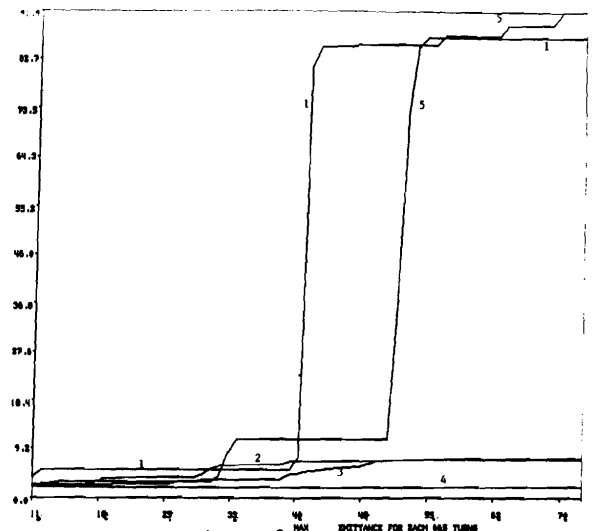

Figure 2

(1) noise+ synchrotron oscillations + octunole kick
(2) noise + octunole kick $\xi_x = \xi_z = 1.0$
(3) noise only
(4) no noise, synchrotron oscillations + octunole kick $\xi_x = \xi_z = 1.0$
(5) no noise, synchrotron oscillations + octupole kick, $\xi_x = \xi_z = 1.3$

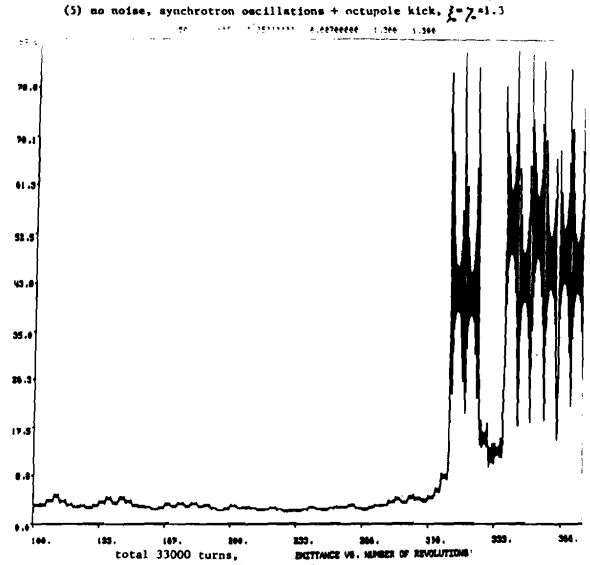

Figure 3.

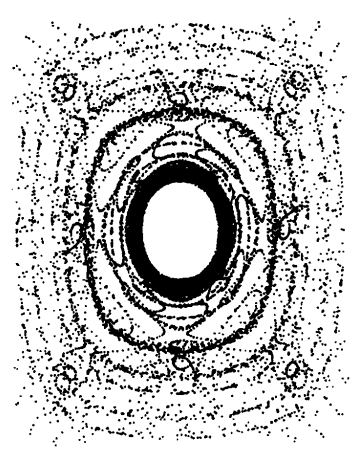

Figure 4.