STOCHASTIC COOLING THEORY*

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1. OPEN LOOP

Feedback from kicker back to PU via the beam is neglected. Consider momentum cooling as proposed by Thorndahl.[1] (See Fig. 1). A particle passing the PU induces a pulse which is amplified, filtered, and synchronized to arrive at the gap with the particle, correcting its energy by ΔE_c volts/turn. In the frequency domain, the single particle of charge e has a DC current ef_o and AC components $2ef_o$ at each harmonic of the particle's revolution frequency f_o (Fig. 2).

The periodic notch filter $H(\omega)$ insures that particles with too much energy are decelerated while particles with too little energy are accelerated, compressing the beam energy spread into the notches. If $G_n(E)$ is the voltage gain from preamp input to final amplifier output for harmonic n, including the filter, then

$$\Delta E_c = 2\, ef_o\, R \sum \mathrm{Re}\, G_n \qquad \text{(volts)} \tag{1}$$

where $R = \sqrt{R_{pu} R_k}$ is the mean of the PU and gap resistance, and the summation is over the harmonics in the system bandwidth.

The kicks from the other particles and the amplifier noise contribute heating terms. The noise density referred to the preamp input is shown in Fig. 3. It consists of white noise $2KT = 8.2\times10^{-21}$ watts/Hz, assuming a 3 dB noise figure, and Schottky noise of $2\, e^2 f_o^2 N R_{pu}$ watts per Schottky band, where N is the number of particles in the beam and Δf_o is the spread in revolution frequencies. A single particle is driven only by the noise at harmonics of its own revolution frequency. Summing the noise density over harmonics, we find the rms energy charge per turn

$$\Delta E_{ic}^2 = \underbrace{2KT\, f_o R_K \sum |G_n|^2}_{\text{amplifier noise}} + \underbrace{2e^2 f_o^3 R^2 \frac{N}{\Delta f_o} \sum \frac{|G_n|^2}{n}}_{\text{Schottky noise}} \quad (\text{volts})^2. \tag{2}$$

For non-square distributions, replace $N/\Delta f_o$ in (2) by dN/df_o. If the Schottky bands overlap (perfect mixing), $n\Delta f_o$ in the summation should be replaced by f_o.

*Shortened version of ISR report (1978) with same title.

The evolution of the particle distribution

$$\psi\,(E,t) = \frac{dN}{dE} \tag{3}$$

is governed by the Fokker-Planck equation

$$\frac{\partial\psi}{\partial t} = \underbrace{\frac{\partial}{\partial E}\left(\frac{\Delta E_c}{T_o}\psi\right)}_{\text{cooling}} + \underbrace{\frac{1}{2}\frac{\partial}{\partial E}\left(\frac{\Delta E_{ic}^2}{T_o}\frac{\partial\psi}{\partial E}\right)}_{\text{heating}} \tag{4}$$

where T_o is the revolution period. This equation is nonlinear because ΔE_{ic}^2 depends on ψ, so a general analytic solution is ruled out, although stationary solutions are easy to find. The equation has been integrated numerically with the measured filter characteristics to compare with the ICE experiment at CERN.

For an ideal linear filter,

$$\sum \mathrm{Re}\, G_n = n_\ell G$$
$$\sum |G_n|^2 = n_\ell G^2 \tag{5}$$
$$\sum \frac{|G_n|^2}{n} = \Lambda G^2$$

where $\Lambda = \sum \frac{1}{n} \sim 1$

and n_ℓ is the number of harmonics in the system bandwidth Δf, namely $n_\ell = \Delta f/f_o$. Let

$$\frac{\Delta E_c}{T_o} = \frac{2e\, f_o R n_\ell G}{T_o} = -\frac{E}{\tau_o} \quad \text{volts/sec} \tag{6}$$

where τ_o is the cooling time for a single particle with no noise present. Then with noise,

$$\Delta E_{ic}^2 = \frac{1}{2}\frac{N\, T_o{}^2}{n_\ell \tau_o{}^2}\left[\frac{\Lambda f_o}{n_\ell \Delta f_o} + \underbrace{\frac{2\, kT\, f_o}{2e^2 f_o^2 R_{pu} N}}_{\eta}\right] E^2 \tag{7}$$

where η is the average noise-to-signal ratio in each revolution frequency band. The second-moment of the Fokker-Planck equation becomes

$$\frac{1}{\sigma}\frac{d\sigma}{dt} = -\frac{1}{\tau_o} + \frac{3}{4}\frac{NT_o}{\eta_\ell \tau_o^2}\underbrace{\left[\frac{\Lambda f_o}{\eta_\ell \Delta f_o} + \eta\right]}_{\substack{= 1 \text{ for perfect mixing} \\ > 1 \text{ for bad mixing}}} \tag{8}$$

where σ is the rms energy spread, and the decrease of Δf_o with time has been neglected. If the PU resistance is large enough, the amplifier heating term η can be neglected, and (8) becomes

$$\frac{1}{\sigma}\frac{d\sigma}{dt} = -\frac{1}{\tau_o} + \frac{3}{4}\frac{N\Lambda}{\eta_\ell^2 \tau_o^2 \Delta f_o}. \tag{9}$$

The Schottky heating is minimized by increasing the number of harmonics η_ℓ and the revolution frequency spread Δf_o via the machine dispersion.

The same analysis for betatron cooling yields

$$\frac{1}{\sigma}\frac{d\sigma}{dt} = -\frac{1}{\tau_o} + \frac{1}{2}\frac{NT_o}{\eta_\ell \tau_o^2}\underbrace{\left[\frac{\Lambda f_o}{\eta_\ell \Delta f_o} + \eta\right]}_{= 1 \text{ for perfect mixing}} \tag{10}$$

which is the usual result for the perfect mixing limit.[2] Here $\sigma^2 = \overline{x^2}$ and τ_o is the simple-particle damping time for no noise. The average noise to-signal ratio,[3] $\eta = r^2/\sigma^2$, increases as σ is cooled because the amplifier noise is not filtered in this case, while the spread Δf_o in revolution frequencies remains constant.

2. CLOSED LOOP

So far, feedback from the kicker back to the PU via the beam has been neglected. When the Schotthy bands overlap (perfect mixing), any coherent modulation produced by the kicker on the beam smears out before it arrives at the PU, so the open-loop damping rates apply. For bad mixing, the coherent modulation does not decay, but remains approximately constant around the machine circumference. In this case, the damping rates and system stability are modified.

Consider the betatron damping system shown in Fig. 4. The amplifier noise x_n and Schottky noise x_s are assumed to be injected into the loop as shown, while x_B is the coherent signal on the beam due to the force F, $\omega_\beta = Q\,\omega_o$ is the betatron frequency, and Im $\Delta\omega$ is the damping rate for the coherent modes of beam oscillation. It is related to the single-particle damping rate by

$$\mathrm{Im}\,\Delta\omega = \frac{N}{2n_\ell}\frac{1}{\tau_o}. \tag{11}$$

A simple particle with revolution frequency Ω_i responds to the force F as

$$\ddot{x}_i + Q^2\Omega_i^2 x_i = F \tag{12}$$

where the dot signifies the co-moving derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \Omega_i\frac{\partial}{\partial\phi}. \tag{13}$$

The single-particle response is therefore

$$\tilde{x}_i = \frac{\tilde{F}}{2\omega_\beta(\omega_i - \omega)} \tag{14}$$

where the tilda indicates the Fourier transform and $\omega_i = (n \pm Q)\Omega_i$. The coherent signal on the beam is then

$$\tilde{x}_B = \frac{1}{N}\sum\tilde{x}_i = \frac{\tilde{F}}{2\omega_\beta}\underbrace{\frac{1}{N}\sum\frac{1}{\omega_i - \omega}}_{G(\omega)}, \tag{15}$$

or

$$\tilde{x}_B = \frac{-\Delta\omega\,G(\omega)}{1 + \Delta\omega\,G(\omega)}(\tilde{x}_n + \tilde{x}_s) \tag{16}$$

while the force acting on the beam is

$$\tilde{F} = \frac{-2\,\omega_\beta\,\Delta\omega}{1 + \Delta\omega\,G(\omega)}(\tilde{x}_n + \tilde{x}_s). \tag{17}$$

The denominators in (16) and (17) are the usual transverse coasting-beam dispersion relations. A typical plot of the inverse of the beam transfer function for real ω is shown in Fig. 5. It deviates from the real axis when ω is within the band of incoherent frequencies ω_i, that is, within a Schottky band. The system is unstable if $\Delta\omega$ lies to the right of the hatched line.

The open-loop signals are reduced by the factor

$$T_n(\omega) = \frac{1}{1 + \Delta\omega\,G(\omega)} = \frac{G^{-1}(\omega)}{\Delta\omega + G^{-1}(\omega)} = \frac{\mathrm{Num}}{\mathrm{Den}} \tag{18}$$

which is the ratio of the vectors shown in Fig. 5. The numerator is typically ~1/4 S where S is the total frequency spread for the Schotthy band in question, so significant signal reductions require coherent damping rates in excess of $\Delta\omega \sim 1/4$ S. In fact, large reductions in the Schotthy signals are commonly observed when operating the betatron cooling systems at the ISR[3] or the ICE experiment, particularly for the lower frequency bands where the frequency spread is small. Even the relatively small feedback via parasitic coupling impedances produces a noticeable effect in the ISR, in this case reducing the area of the stable bands compared with the unstable bands. The noise heating terms in (4) should thus be multiplied by $|T_n|^2$ while the cooling term is multiplied by ReT_n. Detailed calculations have not been performed yet.

Similar signal reduction occurs for momentum cooling (Fig. 6). In addition, system stability is more critical because of the periodic notch filter. The poles and zeroes and response for a first-order filter are shown in Fig. 7. The high gain and changing phase near the poles may cause instability unless the total loop gain KHG including the beam is less than unity everywhere.

For bad mixing, the density and energy modulation on the beam due to the gap voltage u is

$$\tilde{I}_n(\omega) = e N \omega_o \int \tilde{\psi}_n \, dE \tag{19}$$

$$\tilde{\psi}_n(E,\omega) = - j \frac{e}{T_o} \tilde{u} \frac{\frac{d\psi_o}{dE}}{\omega - n\omega_o - nkE}$$

where the stationary beam distribution $\psi_o(E)$ is normalized to unity. Thus the beam transfer function is

$$G(\omega) = - 2\pi j \frac{Ne^2 R}{T_o^2} \int \frac{\frac{d\psi_o}{dE}\, dE}{\omega - n\omega_o - nkE} \tag{20}$$

which is approximately (exact for a Lorentz distribution)

$$G(\omega) \approx j\, nk \frac{N e^2 R}{(\omega - n\omega_o - j\sigma)^2 T_o^2} \tag{21}$$

with Landau damping included. Here σ is the rms frequency spread for the nth Schottky band. The beam thus has second-order poles near each harmonic of the revolution frequency, and the beam response falls off as the square of the frequency outside the Schotthy bands.

The pole-zero diagram for the closed-loop transfer function is shown in Fig. 8. As the gain is increased, the poles move on the paths indicated. Eventually the system pole crosses into the right-hand-plane and instability results. As the pole approaches the axis, the noise power near the resonance increases, possibly saturating the amplifier. The simple pole in Fig. 7 is replaced by a pole-zero cluster, which is responsible for the signal reduction within the Schottky bands. This reduction is probably beneficial since it is largest for the low frequency Schottky bands which contribute most of the heating. This may explain why the momentum cooling observed in the ICE experiment is faster than expected from the open-loop transfer function.

For stack cooling, third-order or higher order filters are required to shield the accumulated beam from its own Schottky noise, yet provide enough gain on the injection orbit to compress the newly injected pulse within a few seconds. Since the beam response decreases as the square of the frequency, while the filter response increases as the cube, the overall gain increases with frequency, and eventually unity gain is likely to be exceeded. Quantitatively, the unity gain restriction

$$1 \geqslant |K H G| \tag{22}$$

can be written as

$$1 \geqslant 2\pi \frac{N e^2 R}{T_o^2} KH(\omega) \int \frac{\frac{d\psi_o}{dE}\, dE}{\omega - n\omega_o - nkE} \tag{23}$$

or

$$1 \geqslant \frac{Ne}{2n_\ell T_o} \frac{V(E)}{nkE^2} \tag{24}$$

where V(E) is the required single-particle energy change per turn,

$$V(E) = 2\, n_\ell \frac{e}{T_o} RKH(\omega) \tag{25}$$

and the energy deviation E rather than frequency $\omega = n\omega_o + nkE$ is used. Requirement (24) cannot be satisfied for practical $\bar{p}$ collection schemes, thus ruling out the filter method for stack cooling.

This problem is less serious with the Palmer method of momentum cooling. The horizontal PU can be shaped as shown in Fig. 9 with a sensitivity F(E) that decreases approximately exponentially in the stack region, with

$$V(E) = 2\, n_\ell \frac{e}{T_o} F(E). \tag{26}$$

The unity gain requirement is now

$$1 \geqslant 2\pi \frac{N e^2}{T_o^2} \int \frac{F(E) \frac{d\psi_o}{dE}}{\omega - n\omega_o - nkE} dE$$

or (27)

$$1 \geqslant \pi \frac{Ne}{n_\ell T_o} \int \frac{V(E) \frac{d\psi_o}{dE}}{\omega - n\omega_o - nkE} dE,$$

with the filtering inside the integral. The overall response decreases linearly with frequency. Van der Meer has pointed out that (27) is always satisfied to within a factor of order unity as long as the beam is cooled, that is, provided the cooling term in (4) is larger than the Schottky heating term.[5] Linear filters are also required in this case to reduce the amplifier noise in the stack region.

In summary, it seems that a complete theory of stochastic cooling that includes the effects of bad mixing is now available. Detailed calculations of cooling rates and system stability remain to be done.

ACKNOWLEDGMENT

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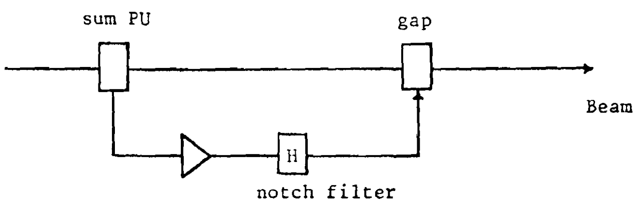

Fig. 1

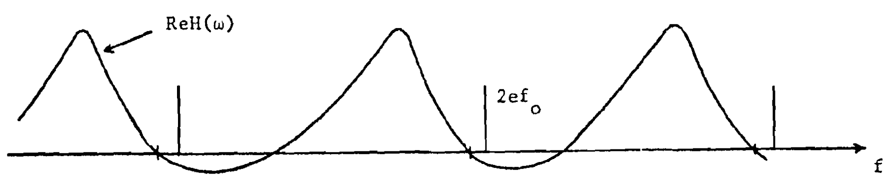

Fig. 2

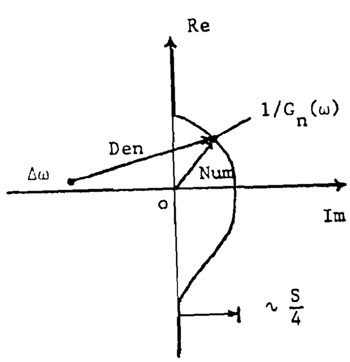

Fig. 3

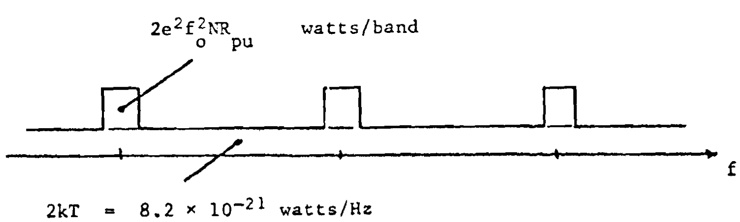

Fig. 4

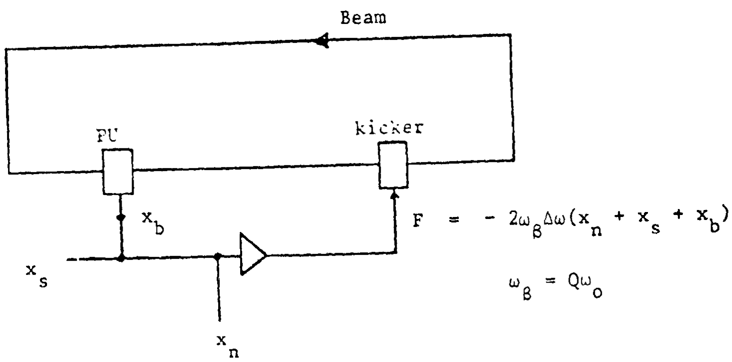

Fig. 5

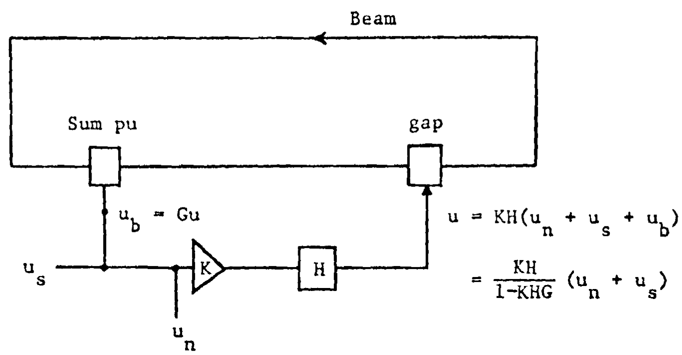

Fig. 6

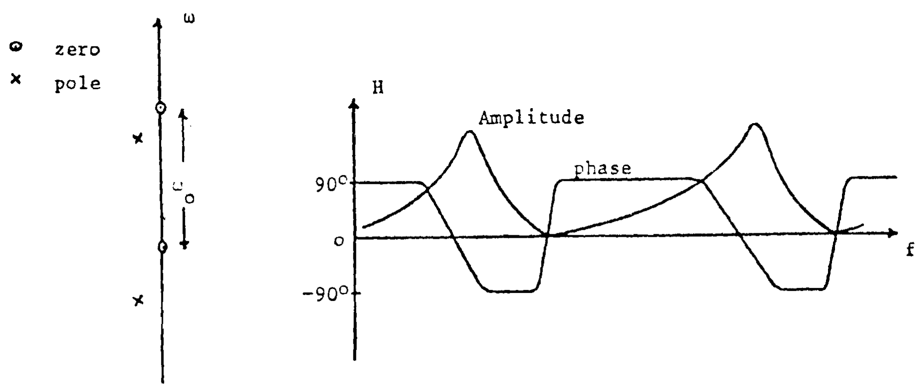

Fig. 7

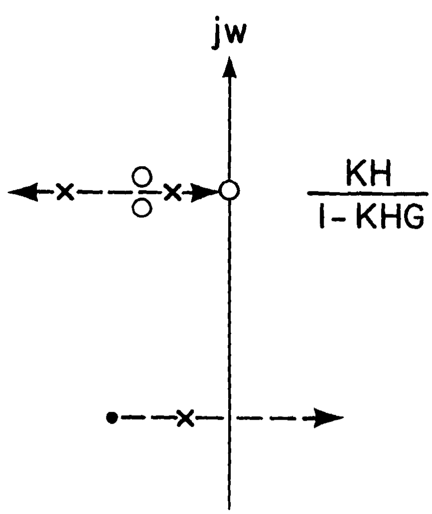

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Fig. 8

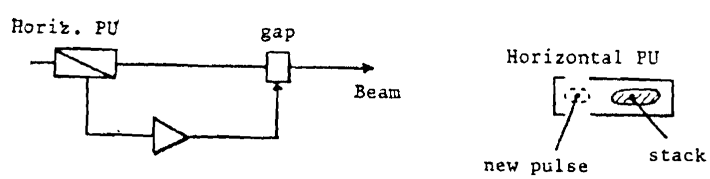

Fig. 9