Introduction to Accelerator Theory

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Abstract

The theory of alternating-gradient synchrotrons is described together with the principle of AG focusing in beam transport systems and storage rings. Betatron and synchrotrn oscillations are treated in a linear approximation. The theory of non-linear oscillations is sketched in connection with the synchrotron oscillation. Historical development of accelerators is briefly described.

CONTENTS

Part I. Introduction

§1. Historical Development

History of particle accelerators started when Cockcroft and Walton succeeded in artificially disintegrating nuclui by the bombardment of a proton beam in 1932. Since then, accelerators attaining higher and higher energy and producing higher intensity have been developed. We classify the accelerator development into the following five phases and briefly sketch the basic principles.

1) direct voltage accelerator
2) resonance accelerator
3) synchronous accelerator
4) alternating-gradient focusing
5) colliding beam machine (storage ring)

1) Direct Voltage Accelerator

A first type of accelerator, which is the simplest in principle, is to apply a DC potential between the ion source and beam exit as shown in Fig.1. Let the potential difference be V, then the energy gain is eV, where e is the electronic charge. It is convenient to express energy in terms of eV (electron volt) and the following auxiliary units are used as shown in Table 1.

Table 1

1 eV (electron volt)		
10^3	KeV	(kilo-)
10^6	MeV	(mega-)
10^9	GeV, BeV	(giga-, billion-)
10^{12}	TeV	(tera-)

This type of accelerators include a Cockcroft-Walton accelerator, a Van de Graaf accelerator etc. In these accelerators, special methods are used to produce a high DC potential. The maximum attainable energy, however, is limited by the insulation breakdown to about 8 MeV.

2) Resonance Accelerator

A proposal to cope with the insulation breakdown of the DC accelerator is to apply a high frequency (radio frequency or RF) electric field in series in resonance with the particle motion. The first type of such an accelerator is a linear accelerator as shown in Fig.2. An array of coaxial cylindrical electrodes of increasing length is aligned along the axis of a long glass vacuum chamber. The electrodes are connected alternately to two bus bars extending along the length of the chamber and supplied by the radio frequency power source. The separation L between accelerating gaps is the distance traversed by the particles during the half-cycle of the applied electric field whence

$$L = \frac{1}{2}\frac{v}{f}\,, \qquad (1.1)$$

where v is the particle velocity and f is the frequency. Each time a particle of charge e crosses a gap, it sees a field which gives it an energy increment $eV \sin\phi$, where V is the maximum gap voltage and ϕ is the phase at which the particle crosses the gap. The early machines were designed to accelerate only the particles which arrived at a phase close to 90°, so the energy gain at each gap was about eV. If the length of the accelerator can be made long, the maximum attainable energy is limited only by economical considerations.

Further development of linear accelerators must have awaited the development of microwave technology during World War II from a technical point of view and the detailed application of the principle of phase stability from a theoretical point of view.

Another application of the principle of resonance acceleration is to use a magnetic field in addition to an RF electric field. The motion of a particle in a magnetic field is determined by the balance between the Lorentz force and the centrifugal force

$$evB = \frac{mv^2}{\rho}\,, \qquad (1.2)$$

where B is the magnetic field, m is the particle mass and ρ is the radius of curvature. Then, the angular frequency ω is expressed as

$$\omega = \frac{v}{\rho} = \frac{eB}{m} , \qquad (1.3)$$

and constant as long as B and m are constants. Such a machine is called a cyclotron and shown in Fig.3. If an RF field is applied in resonance with the particle motion, a single accelerating gap called a "dee" can accelerate a particle beam to a final energy.

The maximum energy of the cyclotron is limited to about 20 MeV for protons because the resonance condition ω = constant in eq.(1.3) breaks down as the particle energy becomes relativistic. The condition also breaks down because the magnetic field is made weaker at the periphery of the magnet than at the center to provide vertical focusing. This difficulty was overcome by the invention of the principle of "phase stability".

Before going to a discussion of phase stability, we remark a general theorem inherent to circular accelerators. Consider an accelerator as shown in Fig.4. We asks whether it is possible for a static electric field to accelerate a particle. In a static field,

$$\mathrm{rot}\ E = 0, \qquad (1.4)$$

from the Maxwell equation. Then, the energy gain ΔE for one revolution is

$$E = \oint E\ ds = \oint \mathrm{rot}\ EdS = 0, \qquad (1.5)$$

so that the energy increase in the accelerating gap is completely cancelled by the field outside and there is no net energy change. Thus, to increase a particle energy, a high frequency electric field based on the principle of resonant acceleration is essential.

3) Synchronous Accelerator

In a cyclotron, as the relativistic mass increases, the angular velocity decrease as indicated by eq.(1.3). In order to cope with this, the radio frequency should be decreased in phase with the particle motion. Further in this case, the accelerating phase should be on the falling side of the RF electric field to ensure "phase stability". This principle of phase stability invented by Veksler and McMillan is clearly exhibited in a paper of McMillan. "Consider, for example, a particle whose energy is such that its angular velocity is just right to match the frequency of the electric field. This will be called the equilibrium energy. Suppose further that the particle crosses the accelerating gap just as the electric field passes through zero, changing in such a sense that an earlier arrival of the particle would result in an acceleration. This orbit is obviously stationary. To show that it is stable, suppose that a displacement in phase is made such that the particle arrives at the gaps too early. It is then accelerated: the increase in energy causes a decrease in angular velocity, which makes the time of arrival tend to become later. A similar argument shows that a change of energy from the equilibrium value tends to correct itself. These displaced orbits will continue to oscillate, with both pahse and energy varying about their equilibrium values." These oscillations are called synchrotron oscillations or longitudinal oscillations.

The cyclotron based on this principle is called a synchrocyclotron (or an FM cyclotron). Another machine using this principle is a synchrotron, in which the magnetic field also changes with time. Since the volume of magnets required for the synchrotron is smaller, the synchrotron is quite suitable to produce high energy. Present linear accelerators are also based on the principle of phase stability. The equilibrium phase should be on the rising side of the electric field in this case.

4) Weak Focusing and Strong Focusing

Particles injected into an accelerator have a finite spread in position and angle, so that it is very important to confine the particles during the acceleration period. Otherwise, particle will be lost and the final intensity will be quite small. In general, particles will oscillate about some equilibrium orbit and thus stability is obtained. This oscillation is called a "transverse oscillation" or a "betatron oscillation" because this oscillation mode was first analyzed with a machine called a betatron.

A betatron does not use an RF electric field, but an induced electric field due to changing magnetic flux. From the Maxwells equation

$$\mathrm{rot}\ E = -\frac{\partial B}{\partial t}\ , \qquad (1.6)$$

the energy gain per revolution ΔE is given by

$$\Delta E = \oint E ds = -\frac{d\Phi}{dt} , \qquad (1.7)$$

where Φ is the magnetic flux enclosed by the particle trajectory. An important feature of the betatron is to shape the magnetic field so that

$$B(r) \propto r^{-n} \qquad (1.8)$$

where $\quad 0 < n < 1. \qquad (1.9)$

The condition (1.9) ensures stability both in horizontal and vertical planes.

This principle of transverse focusing was adopted in synchrotrons and is now called the principle of "weak focusing". It was later discovered that a stronger focusing could be achieved if sections of large positive n-values and large negative n-values (focusing and defocusing sections) were provided alternatingly. This scheme is called "alternating gradient or AG focusing" or "strong focusing". This possibility will be studied in detail in this lecture.

5) Colliding Beam Machine (Storage Ring)

In a stationary target experiment, the available center of mass energy W is expressed as

$$W^2 = -(p + q)^2 \qquad (1.10)$$

$$= 2E_L M + 2M^2 , \qquad (1.11)$$

where p,q are four-momenta of incident and target particles, E_L is the energy of the incident particle and the two particles are assumed to have the same mass M. If the two beams are made to collide head-on with energy E, the available center-of-mass energy is

$$W = 2E . \qquad (1.12)$$

So, the colliding beam machine of energy E corresponds to a conventional machine of a corresponding energy E_L given by

$$E_L \approx 2 \cdot \frac{E}{M} \cdot E. \qquad (1.13)$$

The biggest conventional accelerator is the FNAL 500 GeV proton synchrotron, whereas the biggest colliding beam machine is the CERN 30 GeV × 30 GeV ISR. The latter corresponds, in energy, to a conventional accelerator of about 1.8 TeV.

§2. Scope of This Lecture

This lecture is concentrated on the principle of alternating gradient focusing in synchrotrons, beam transport systems and storage rings and its consequences on synchrotron oscillations. The effects of various perturbations such as magnetic field imperfections, gas scattering etc. are outside of the scope of this lecture. Further, the particles are treated independent; the effect of mutual interactions between particles, related to space charge effects and instabilities, are not dealt with. For further reference, a list of reference books on accelerator theory is given selectively at the end of this note.

Part II. Betatron Oscillation

§1. Equation of Motion

The basic equations which determine the motion of charged particles in an accelerator are the (relativistic) Newton's equation of motion

$$\frac{dp}{dt} = F, \quad (2.1)$$

and the Lorentz force equation

$$F = e[v \times B] + eE. \quad (2.2)$$

Since the magnetic force (the first term of eq.(2.2)) is directed perpendicular to velocity, it does not change the particle energy. The electric field (the second term) is applied in a direction of particle motion, and it increases the particle energy. In all existing synchrotrons, the energy gain per revolution is much smaller than the particle energy so that, in treating the dynamics of particle motion in a magnetic field, the particle energy can be considered to be constant and the magnetic field can also be considered to be constant. The small change of a particle energy is taken into account as a perturbation (which causes an adiabatic damping of oscillations). Further, the transverse oscillations (betatron oscillations) governed by the magnetic force term and the longitudinal oscillations (synchrotron oscillations) effected by the electric field term can be treated independently because their oscillation frequencies differ considerably from each other. Thus, the effect of the magnetic term and particularly the stability of particle motion in a static magnetic field will be treated in this part.

We first consider a particle moving in a uniform magnetic field. The Lorentz force is constant and the motion is a circular one with radius ρ. The motion is determined by the balance between the magnetic Lorentz force and the centrifugal force

$$\frac{mv^2}{\rho} = evB. \quad (2.3)$$

Thus

$$p = mv = eB\rho, \quad (2.4)$$

where p denotes the momentum of a particle. A useful numerical relation worth memorizing is

$$p(\mathrm{GeV}/c) \simeq 0.3\ B(\mathrm{T})\rho(\mathrm{m}), \tag{2.5}$$

where p is measured in GeV/c, B in T (1 Tesla = 10^4 Gauss) and ρ in m.

In actual accelerators, and magnetic field is not always uniform, but all the accelerators are designed so that the trajectory is circular for a given momentum (design momentum) as determined by eq.(2.4). In addition to circular portions, all the modern accelerators comprise straight sections. Thus, the trajectory of a particle having the design momentum is circular in some parts and straight in the other parts. The trajectory of particles which have a momentum other than the design value is complicated and should be determined by solving the equation of motion as will be shown later.

To go further, it is convenient to introduce a cylindrical coordinate system (r, θ, z) for a curved section and a cartesian coordinate system (x, z, s) for a straight section as shown in Fig.5. (Note that the sign of θ is opposite to the usual definition. This definition is used because the magnetic field in the z-direction and the positive charge moving in the θ direction appropriately define the radius vector r.) The equations of motion (2.1) and (2.2) are written in the cylindrical coordinate system in the form

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 = -er\dot{\theta}B_z + e\dot{z}B_\theta + eE_r \tag{2.6}$$

$$\frac{d}{dt}(m\dot{z}) = -e\dot{r}B_\theta + er\dot{\theta}B_r + eE_z \tag{2.7}$$

$$\frac{1}{r}\frac{d}{dt}(mr^2\dot{\theta}) = -e\dot{z}B_r + e\dot{r}B_z + eE_\theta\ . \tag{2.8}$$

In the cartesian coordinate system

$$\frac{d}{dt}(m\dot{x}) = e\dot{z}B_s - e\dot{s}B_z + eE_x \tag{2.9}$$

$$\frac{d}{dt}(m\dot{z}) = e\dot{s}B_x - e\dot{x}B_s + eE_z \tag{2.10}$$

$$\frac{d}{dt}(m\dot{s}) = e\dot{x}B_z - e\dot{z}B_x + eE_s\ . \tag{2.11}$$

The third equations (2, 8) and (2, 11) are related to acceleration and will be discussed in Part III. The electric field is applied in the direction of motion so that

$$E_r = E_x = E_z = 0 .$$

Further, it is convenient to use θ and s as independent variables instead of t. Then

$$\begin{aligned} \frac{d}{d\theta}\left(m\,\dot{\theta}\frac{dr}{d\theta}\right) - mr\dot{\theta} &= -erB_z + e\frac{dz}{d\theta}B_\theta \\ \frac{d}{d\theta}\left(m\,\dot{\theta}\frac{dz}{d\theta}\right) &= -e\frac{dr}{d\theta}B_\theta + erB_r , \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} \frac{d}{ds}\left(m\dot{s}\,\frac{dx}{ds}\right) &= e\frac{dz}{ds}B_s - eB_z \\ \frac{d}{ds}\left(m\dot{s}\,\frac{dz}{ds}\right) &= eB_x - e\frac{dx}{ds}B_s . \end{aligned} \tag{2.13}$$

We now consider the properties of the magnetic field. We put $r = \rho + x$, where ρ is the design value, and consider x and z to be small. Then,

$$\begin{aligned} B_z(r,\theta,z) &= B_z(\rho) + \frac{\partial B_z}{\partial r}x + \frac{\partial B_z}{\partial z}z + \cdots \\ B_r(r,\theta,z) &= B_r(\rho) + \frac{\partial B_r}{\partial r}x + \frac{\partial B_r}{\partial z}z + \cdots \\ B_\theta(r,\theta,z) &= B_\theta(\rho) + \frac{\partial B_\theta}{\partial r}x + \frac{\partial B_\theta}{\partial z}z + \cdots . \end{aligned} \tag{2.14}$$

These fields must satisfy the Maxwell equations

$$\begin{aligned} \mathrm{div}\, B &= 0 \\ \mathrm{rot}\, B &= 0 . \end{aligned} \tag{2.15}$$

The last equation results because there is no current in the vacuum chamber where the beam travels and the induction current can be neglected since the change of the magnetic field is very slow. Then,

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z} = 0$$

$$\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{1}{r}\frac{\partial B_r}{\partial \theta} = 0 \ .$$

We assume that the magnetic field is independent of θ. We further assume that

$$B_\theta(r, \theta, z) = 0$$

$$B_r(\rho) = 0$$

$$\frac{\partial B_z}{\partial z} = \frac{\partial B_r}{\partial r} = 0.$$

All these assumptions are consistent with eq.(2.16), and a relation

$$\frac{\partial B_r}{\partial z} = \frac{\partial B_z}{\partial r} \tag{2.17}$$

results. Similar considerations can be made in the cartesian coordinate system to give

$$\frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} \ . \tag{2.18}$$

This field gradient is utilized to focus the particle beam. Further, in all accelerators, the magnets are designed so that only this gradient appears and the expansion of eq.(2.14) is strictly valid except for unavoidable errors.

The field is independent of θ or s, and the problem is a two-dimensional one. Further, the curvature of the orbit can be neglected in the present problem and we can use a cartesian coordinate system. The desired field is given by

$$B_x = az$$
$$B_z = ax \tag{2.19}$$
$$a = \frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x}.$$

Since rot B = 0, B is expressed by a scalar potential

$$B = -\mathrm{grad}\ \Psi \tag{2.20}$$

and

$$\Psi = -axz. \tag{2.21}$$

We produce this potential by a suitable boundary shape of iron yokes. The boundary condition at the iron yoke is from the equation rot H = 0 and divB = 0 (see Fig.6),

$$\begin{aligned} H_{1t} &= H_{2t} \\ B_{1n} &= B_{2n} \, , \end{aligned} \tag{2.22}$$

where t designates a transverse component and n designates a normal component. Since the permeability of iron is very large H_{2t} in iron can be taken to be zero. Thus, the boundary condition is that the transverse field is zero at the boundary. The problem is identical to the problem of an electrostatic field bounded by a conductor, and the desired field is obtained by the iron yokes whose shape is the equipotential surface given by eq.(2.21). Such a magnet is called a quadrupole magnet and is shown in Fig.7a.

To bend a particle as well as to focus it, it is possible to pass a beam off-axis as shown in Fig.7(b). The bending field is

$$B_z = \pm\frac{\partial B_z}{\partial x}\ x_0$$

and the effective gradient (often called a profile parameter n/ρ) is

$$\frac{n}{\rho} = -\frac{1}{B_z}\frac{\partial B_z}{\partial x} = \mp\frac{1}{x_0} \quad . \tag{2.23}$$

In this case, the left half is unnecessary and is replaced by a neutral pole as shown in Fig.7(c). Further, since the neutral pole is inconvenient for injection, extraction, replacement for vacuum chambers etc., the neutral pole is replaced by "shims" as shown in Fig.7(d). Computer programs to calculate proper pole face shapes for producing a desired field have been developed in the laboratories around the world. These include SIBYL, LINDA, RINDA, TRIM, MAGNET, etc. and are available at KEK.

We now go back to the equation of motion. Firstly, the velocity v of a particle is expressed as

$$v = \sqrt{(r\dot{\theta})^2 + \dot{r}^2 + \dot{z}^2} \ ,$$

or

$$v = \sqrt{\dot{s}^2 + \dot{x}^2 + \dot{z}^2} \ .$$

Since we consider the first order theory, $\dot{r}$, $\dot{x}$, and $\dot{z}$ are neglected compared to $r\dot{\theta}$ or $\dot{s}$. Then

$$v \simeq r\dot{\theta}, \qquad \text{or} \qquad v \simeq \dot{s} \ .$$

We further assume that the mass is constant (the effect of change in mass will be discussed in §6). Then in a cylindrical coordinate system,

$$m\dot{\theta}\,\frac{d^2x}{d\theta^2} - mv = -e\rho B_z(\rho) - exB_z(\rho) - e\rho\,\frac{\partial B_z}{\partial x}\,x \qquad (2.24)$$

$$m\dot{\theta}\,\frac{d^2z}{d\partial^2} = e\rho\,\frac{\partial B_r}{\partial z}\,z. \qquad (2.25)$$

We put

$$p = mv = p_0 + \Delta p,$$

where p_0 is the design momentum given by $p_0 = eB_z(\rho)\rho$. Then eq.(2.24) is

$$m\dot{\theta}\,\frac{d^2x}{d\theta^2} - \Delta p = -\,x\,\frac{p_0}{\rho} - e\rho\,\frac{\partial B_z}{\partial x}\,x.$$

Now, $m\dot{\theta} = \dfrac{p_0}{\rho}$ and the transformation gives

$$\frac{d^2x}{d\theta^2} + (1 - n)x = \rho\,\frac{\Delta p}{p_0} \ , \qquad (2.26)$$

where

$$n = -\,\frac{\rho}{B_z}\,\frac{\partial B_z}{\partial x} \ .$$

The suffix 0 in p_0 will be omitted in the followings for brevity. The quantity n is called a field index. Similarly, eq.(2.25) is transformed as

$$\frac{d^2z}{d\theta^2} + nz = 0 \qquad (2.27)$$

The equations of motion in a cartesian coordinate system are derived in a similar way. For a quadrupole magnet

$$\frac{d^2x}{ds^2} + \frac{e}{p}\frac{\partial B_z}{\partial x} x = 0$$

(2.28)

$$\frac{d^2z}{ds^2} - \frac{e}{p}\frac{\partial B_x}{\partial z} z = 0 .$$

For a drift space

$$\frac{d^2x}{ds^2} = 0$$

(2.29)

$$\frac{d^2z}{ds^2} = 0 .$$

We introduce a variable $s = \rho\theta$ for a curved section, where s consistently denotes a path length of the central design orbit in a machine. Then, eqs.(2.26) and (2.27) are rewritten as

$$\frac{d^2x}{ds^2} + \frac{1-n}{\rho^2} x = \frac{1}{\rho}\frac{\Delta p}{p}$$

(2.30)

$$\frac{d^2z}{ds^2} + \frac{n}{\rho^2} z = 0 .$$

Eqs. (2.28) - (2.30) are the basic equations of motion in a linear theory.

§2. Motion in an Azimuthally Constant Field

We now solve the equations of motion (2.28) - (2.30) under the assumption that ρ, $\frac{dB_z}{\partial x}$ etc. are constant azimuthally, i.e. dependent of s. This assumption is correct for all practical accelerators and transport systems in that all elements, though they may differ from element to element, have specified values of ρ, $\frac{\partial B_z}{\partial x}$ etc. The most difficult equation is the first of eq.(2.30), i.e.

$$\frac{d^2x}{ds^2} + Kx = \frac{1}{\rho}\frac{\Delta p}{p} ,$$

where

$$K = \frac{1-n}{\rho^2}$$

We assume K>0 and first solve the homogeneous equation

$$\frac{d^2x}{ds^2} + Kx = 0 \ .$$

The solution is

$$x = A \cos\sqrt{K}\, s + B \sin\sqrt{K}\, s, \tag{2.32}$$

where A and B are arbitrary constants. Since eq.(2.31) is a second order differential equation, its solution is completely specified by giving the initial values $x(s_0)$ and $x'(s_0)$ at $s = s_0$. Then

$$A = x(s_0),$$

$$B = \frac{1}{\sqrt{K}}\, x'(s_0) \ . \tag{2.33}$$

We then solve the inhomogeneous equation by the method of variation of constants. Put

$$x' = -A\sqrt{K} \sin\sqrt{K}\, s + B\sqrt{K} \cos\sqrt{K}\, s, \tag{2.34}$$

with a subsidiary condition

$$A' \cos\sqrt{K}\, s + B' \sin\sqrt{K}\, s = 0 \ . \tag{2.35}$$

Then, the equation of motion is

$$-A'\sqrt{K} \sin\sqrt{K}\, s + B'\sqrt{K} \cos\sqrt{K}\, s = \frac{1}{\rho} \frac{p}{p} \ . \tag{2.36}$$

Combining eqs.(2.35) and (2.36), we obtain the equations for A and B

$$A' = -\frac{1}{\rho} \frac{\Delta p}{p} \frac{1}{\sqrt{K}} \sin\sqrt{K}\, s$$

$$B' = \frac{1}{\rho} \frac{\Delta p}{p} \frac{1}{\sqrt{K}} \cos\sqrt{K}\, s.$$

The solution is

$$x = \left(+\frac{1}{\rho K} \frac{\Delta p}{p} \cos\sqrt{K}\, s + a\right) \cos\sqrt{K} s + \left(\frac{1}{\rho K} \frac{\Delta p}{p} \sin\sqrt{K} s + b\right) \sin\sqrt{K} s,$$

where a and b are arbitrary constants. The initial conditions $x(s_0) = x'(s_0) = 0$ determine a and b, and the solution is

$$x = \frac{1}{\rho K} \frac{\Delta p}{p} \{1 - \cos\sqrt{K} s\} \ . \tag{2.37}$$

The complete solution is thus

$$x = x_0 \cos\sqrt{K}s + \frac{x_0{}'}{\sqrt{K}} \sin\sqrt{K}s + \frac{1}{\rho K}\frac{\Delta p}{p}\{1 - \cos\sqrt{K}s\} . \tag{2.38}$$

As is easily seen, the general form of the solution for y (y is a generic symbol of x or z) is written in the form

$$\begin{aligned} y &= a(s)y_0 + b(s)y_0{}' + e(s)\frac{\Delta p}{p} \\ y' &= a'(s)y_0 + b'(s)y_0{}' + e'(s)\frac{\Delta p}{p} \\ &= c(s)y_0 + d(s)y_0{}' + f(s)\frac{\Delta p}{p} . \end{aligned} \tag{2.39}$$

It is convenient to express a solution in a matrix form

$$\begin{pmatrix} y(s) \\ y'(s) \\ \frac{\Delta p}{p} \end{pmatrix} = \begin{pmatrix} a(s) & b(s) & e(s) \\ c(s) & d(s) & f(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_0{}' \\ \frac{\Delta p}{p} \end{pmatrix} . \tag{2.40}$$

This matrix is called a transfer matrix and denoted by $M(s|s_0)$. It is easily verified that

$$M(s|s_0) = M(s|s_1)\, M(s_1|s_0), \tag{2.41}$$

so that the tracing of a trajectory can be done easily by matrix multiplication. The transfer matrices for various elements are summarized in Appendix. It is to be noted that

$$\det M = 1. \tag{2.42}$$

This is a general law derived from the fact that the equation of motion has no damping term (a term proportional to the first derivative y'), and expresses the fact that the phase space area in (y,y') is conserved by the transformation. (This point will be discussed later.)

§3. Stability of Motion

We first neglect a momentum spread of the beam and assume that all the particles have a design momentum. The effect of momentum spread will be discussed in §7. For a particle of design momentum, there is a closed curve called an "equilibrium orbit". For a particle to lie on the equilibrium orbit, however, the particles should be injected strictly into the equilibrium orbit. The inejcted particles have a finite spread of position and angle, and those particles should oscillate around the equilibrium orbit in order for the motion to be "stable". If the particles depart form the equilibrium orbit as they travel in the machine, the motion is called "unstable". The oscillation mentioned above is called the "transverse oscillation" or the "betatron oscillation". We consider the stability of such an oscillation in this section.

First, take a case of an azimuthally symmetric machine (weak-focusing machine). The equations of motion are, from eq.(2.30),

$$\frac{d^2x}{ds^2} + \frac{1-n}{\rho^2}x = 0$$

$$\frac{d^2z}{ds^2} + \frac{n}{\rho^2} z = 0,$$

where n and ρ are constants. It will be easily seen that the motions are sinusoidal and stable in both planes if

$$0 < n < 1 . \qquad (2.43)$$

It is a great invention of Christofilos and Courant, Livingston and Snyder that the stability of motion is obtained if sections of large positive n and large negative n are alternately provided in a periodical manner. This principle is called the "alternating-gradient" or "AG" principle. In discussing the stability of motion based on the AG principle, it is quite profitable to use the matrix formalism described in §2.

Now, the machine is divided into N identical sectors called a "period" or a "cell". Denoting the total circumference of the machine by C and putting L = C/N, the periodicity condition is expressed as

$$M(s + L|s) = M(s + 2L|s + L)$$

and

$$M(s + kL|s) = [M(s + L|s)]^k. \qquad (2.44)$$

We define

$$M(s) = M(s + L|s). \tag{2.45}$$

Referring to eq.(2.40), we separate the matrix M as follows.

$$M = \begin{pmatrix} \bar{M} & D \\ 0 & 1 \end{pmatrix}, \tag{2.46}$$

where

$$\bar{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$D = \begin{pmatrix} e \\ f \end{pmatrix},$$

$$0 = (0, 0) .$$

The matrix multiplication is done as follows.

$$\begin{aligned} M_1 M_2 &= \begin{pmatrix} \bar{M}_1 & D_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{M}_2 & D_2 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \bar{M}_1 \bar{M}_2 , & \bar{M}_1 D_2 + D_1 \\ 0 , & 1 \end{pmatrix} . \end{aligned} \tag{2.47}$$

Since the 2-by-2 matrix multiplication is influenced only by the matrix $\bar{M}$ and not by D and since we neglect the momentum dispersion, we can discuss the betatron oscillation in terms of only the 2-by-2 matrix $\bar{M}$. We omit the bar from $\bar{M}$ in the followings for brevity.

To study the stability of motion, it is convenient to introduce an eignvalue of M(s). We put

$$MY = \lambda Y. \tag{2.48}$$

The eigenvalues are the solutions of the determinantal equation

$$|M - \lambda I| = o, \tag{2.49}$$

or

$$\lambda^2 - (a + d) + 1 = 0, \tag{2.50}$$

where we have made use of the fact that detM = ad - bc = 1.

If we write

$$\cos\mu = \frac{1}{2}\,\mathrm{TrM} = \frac{1}{2}(a + d), \qquad (2.51)$$

the two solutions of (2.50) are

$$\lambda = \cos\mu \pm i\,\sin\mu = e^{\pm i\mu}. \qquad (2.52)$$

The quantity μ will be real if $|a + d| \le 2$ and imaginary or complex if $|a + d| > 2$.

Let us now assume $|a + b| \neq 2$. We define the variables α, β and γ by

$$\begin{aligned} a - d &= 2\alpha\,\sin\mu \\ b &= \beta\sin\mu \\ c &= -\gamma\sin\mu\ . \end{aligned} \qquad (2.53)$$

The condition detM = 1 becomes

$$\beta\gamma - \alpha^2 = 1. \qquad (2.54)$$

For $|a + d| = 2$, $\sin\mu = 0$, and β and γ are indefinite, but this case does not occur in actual accelerators (except a long straight section for matching) and we do not consider this case. We resolve the ambiguity of the sign of $\sin\mu$ by requiring β to be positive if $|\cos\mu| < 1$ and by requiring $\sin\mu$ to be positive imaginary if $|\cos\mu| > 1$. The definition of μ is still ambiguous to the extent that any multiple of 2π may be added to μ without changing the matrix. This ambiguity will be resolved later.

The matrix M may now be written as

$$M = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix} = I\cos\mu + J\sin\mu, \qquad (2.55)$$

where I is the unit matrix and

$$J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \qquad (2.56)$$

is a matrix with zero trace and unit determinant, satisfying

$$J^2 = -I\ . \qquad (2.57)$$

It should be noted that the trace of M , and therefore μ, is independent of the reference point s. For, by virtue of eq.(2.41), we have for any s_1 and s_2

$$M(s_2 + L|s_1) = M(s_2)M(s_2|s_1) = M(s_2|s_1)M(s_1)$$

so that

$$M(s_2) = M(s_2|s_1)M(s_1)[M(s_2|s_1)]^{-1} . \qquad (2.58)$$

Thus $M(s_1)$ and $M(s_2)$ are related by a similarity transformation, and therefore have the same trace and the same eigenvalues. On the other hand, the matrix M(s) as a whole depend on the reference point s. Thus the elements α, β, γ of the matrix J are functions of s, periodic with period L.

Because of eq.(2.57), the combination $I\cos\mu + J\sin\mu$ has properties similar to those of the complex exponential $e^{i\mu} = \cos\mu + i\sin\mu$; in particular, it is easily seen that, for any μ_1 and μ_2

$$(I\cos\mu_1 + J\sin\mu_1)(I\cos\mu_2 + J\sin\mu_2)$$

$$= I\cos(\mu_1 + \mu_2) + J\sin(\mu_1 + \mu_2) . \qquad (2.59)$$

The k-th power of the matrix M is thus

$$M^k = (I\cos\mu + J\sin\mu)^k = I\cos k\mu + J\sin k\mu , \qquad (2.60)$$

and the inverse is

$$M^{-1} = I\cos\mu - J\sin\mu. \qquad (2.61)$$

It follows from eq.(2.60) that if μ is real, the matrix elements of M^k do not increase indefinitely with increasing k, but rather oscillate; on the other hand, if μ is not real, $\cos k\mu$ and $\sin k\mu$ increase exponentially, and therefore the matrix elements do the same. Therefore, the motion is stable if μ is real, i.e. if $|a + d| < 2$ and unstable if $|a + d| > 2$.

In alternating gradient synchrotrons, the simplest magnet arrangement is an FD structure, where F denotes a radially focusing sector and D denotes a radially defocusing sector:

$$\rho = \text{const} = R$$

$$n = n_1 \quad , \; 0 < s < \frac{\pi R}{N} \tag{2.62}$$

$$n = -n_2 \quad , \; \frac{\pi R}{N} < s < \frac{2\pi R}{N} \; .$$

Computing the trace of this structure,

$$\cos\mu_z = \cos\phi_z \cosh\Psi_z - \frac{n_1 - n_2}{2(n_1 n_2)^{1/2}} \sin\phi_z \sinh\Psi_z \; , \tag{2.63}$$

where $\quad \phi_z = \pi n_1^{1/2}/N$ and $\Psi_z = \pi n_2^{1/2}/N$,

and

$$\cos\mu_x = \cos\phi_x \cosh\Psi_x - \frac{2 - n_1 + n_2}{2[(n_2 + 1)(n_1 - 1)]^{1/2}} \sin\phi_x \sinh\Psi_x \; ,$$

where $\phi_x = \pi(n_2 + 1)^{1/2}/N$ and $\Psi_x = \pi(n_1 - 1)^{1/2}/N$. (2.64)

If $n_1 >> 1$ and $n_2 >> 1$, the stability criteria depend only on n_1/N^2 and n_2/N^2. Both modes are stable provided n_1/N^2 and n_2/N^2 lie within the shaded region of Fig.8. This diagram is called a "necktie diagram" because of its shape.

§4. Amplitude of Betatron Oscillation

The general form of the equation of motion is a Hill's equation

$$y'' + K(s)y = 0,$$

with $\quad K(s + L) = K(s),$ (2.65)

which is a linear second-order differential equation with a periodic coefficient. We attempt to find solutions of (2.65) which have the form

$$y_1(s) = w(s)e^{i\Psi(s)} \; . \tag{2.66}$$

Another linearly independent solution is naturally

$$y_2(s) = w(s)e^{-i\Psi(s)} \; . \tag{2.67}$$

Since the Wronskian Δ defined as

$$\Delta = y_1 y_2{}' - y_1{}' y_2 \qquad (2.68)$$

is constant in s, we normalize y_1 and y_2 in a way

$$\Delta = 2i \ . \qquad (2.69)$$

Then

$$\Psi' = \frac{1}{w^2} \qquad (2.70)$$

Inserting eqs.(2.66) and (2.67) into eq.(2.65), we obtain

$$w'' + Kw - \frac{1}{w^3} = 0. \qquad (2.71)$$

Eqs.(2.70) and (2.71) are the equations determining w and Ψ.

Any solution of eq.(2.65) is a linear combination of y_1 and y_2 so that

$$y(s) = ay_1(s) + by_2(s),$$

and

$$y'(s) = ay_1{}'(s) + by_2{}'(s) \ . \qquad (2.72)$$

Then

$$y(s_2) = aw_2 e^{i\Psi_2} + bw_2 e^{-i\Psi_2} \ ,$$

$$y'(s_2) = a(w_2{}' + i\Psi_2{}'w_2)e^{i\Psi_2} + b(w_2{}' - i\Psi_2{}'w_2)e^{-i\Psi_2}, \qquad (2.73)$$

and

$$y(s_1) = aw_1 e^{i\Psi_1} + bw_1 e^{-i\Psi_1}$$

$$y'(s_1) = a(w_1{}' + i\Psi_1{}'w_1)e^{i\Psi_1} + b(w_1{}' - i\Psi_1{}'w_1)e^{-i\Psi_1} \ . \qquad (2.74)$$

Solving a and b in eq.(2.74) and inserting into eq.(2.73), we express $y(s_2)$ and $y'(s_2)$ in terms of $y(s_1)$ and $y'(s_1)$. Then, the transfer matrix is expressed as

$$M(s_2|s_1) = \begin{pmatrix} \frac{w_2}{w_1}\cos\Psi - w_2 w_1{}'\sin\Psi & w_1 w_2 \sin\Psi \\ -\frac{1 + w_1 w_1{}' w_2 w_2{}'}{w_1 w_2}\sin\Psi - \left(\frac{w_1{}'}{w_2} - \frac{w_2{}'}{w_1}\right)\cos\Psi & \frac{w_1}{w_2}\cos\Psi + w_1 w_2{}'\sin\Psi \end{pmatrix}$$

(2.75)

where Ψ stands for $\Psi(s_2) - \Psi(s_1)$, w_1 for $w(s_1)$ etc.

We now consider the case where $s_2 - s_1$ is just one period of K(s), i.e. $s_2 - s_1 = L$. The matrix M is then identical with the matrix (2.55). If we now require that w(s) be a periodic function of w (This is a consequence of the Floquet's theorem), then $w_1 = w_2$ and $w_1' = w_2'$, and the forms (2.75) and (2.55) are identical provided we make the identifications

$$\Psi(s_2) - \Psi(s_1) = \mu, \tag{2.76}$$

$$w^2 = \beta, \tag{2.77}$$

$$ww' = -\alpha, \tag{2.78}$$

from which follows automatically

$$\frac{1 + (w'w)^2}{w^2} = \frac{1 + \alpha^2}{\beta} = \gamma.$$

This identification is legitimate if we can show that $\beta^{1/2}$ - which is, of course, periodic - satisfies the differential equation (2.71) and that

$$\beta' = -2\alpha \tag{2.79}$$

To prove this, consider the matrix for the transformation from s + ds to s + L + ds. This matrix is, by eq.(2.58)

$$M(s + ds) = M(s + ds|s)\ M(s)\ [M(s + ds|s)]^{-1}. \tag{2.80}$$

For infinitesimal ds

$$M(s + ds|s) = \begin{pmatrix} 1 & ds \\ -K(s)ds & 1 \end{pmatrix}. \tag{2.81}$$

Substituting (2.81) and (2.55) in (2.80) we find

$$M(s + ds) = M(s) + \begin{pmatrix} (K\beta-\gamma)\sin\mu & -2\alpha\sin\mu \\ -2K\alpha\ \sin\mu & -(K\beta-\gamma)\sin\mu \end{pmatrix} ds, \tag{2.82}$$

so that (2.79) is indeed valid, and furthermore

$$\alpha' = -\tfrac{1}{2}\beta'' = K\beta-\gamma = K\beta - \frac{1 + \alpha^2}{\beta} \tag{2.83}$$

and

$$\gamma' = 2K\alpha. \tag{2.84}$$

With the aid of (2.79) and (2.83) it is easily verified that $\beta^{1/2}$ does indeed satisfy (2.71), and is therefore a periodic solution of that equation. Now (2.77) and (2.78) are justified, while (2.76) becomes the very important relation

$$\mu = \int_0^L \frac{ds}{\beta} . \tag{2.85}$$

Eq. (2.85) may be regarded as the definition of μ. It is consistent with the previous definition (2.51), but has the advantage of being unambiguous, while (2.51) only defines μ modules 2π.

If we consider an accelerator of circumference $C = NL$ with N identical unit cells (periods), the phase change per revolution is, of course, $N\mu$. A useful number is

$$\nu = \frac{N\mu}{2\pi} = \frac{1}{2\pi} \int_s^{s+C} \frac{ds}{\beta} . \tag{2.86}$$

This is the number of betatron oscillation wavelengths in one revolution. (In the European literature on accelerators this number is often denoted by Q.) A useful intepretation of ν is as the frequency of betatron oscillations measured in units of the frequency of revolution; we shall refer to ν simply as the frequency of betatron oscillations.

The two particular solutions y_1 and y_2 may now be written a

$$y_{\substack{1\\2}} = \beta^{1/2}(s)\ e^{\pm i\nu\phi(s)} , \tag{2.87}$$

where

$$\phi(s) = \int \frac{ds}{\nu\beta} \tag{2.88}$$

is a function which increases by 2π every revolution, and whose derivative is periodic. The general solution is

$$y(s) = a\ \beta(s)^{1/2} \cos\left[\nu\phi(s) + \delta\right] , \tag{2.89}$$

where a and δ are arbitrary constants. This is a pseudo-harmonic oscillation with varying amplitude $\beta^{1/2}(s)$ and varying instantaneous wavelength

$$\lambda = 2\pi\beta(s). \tag{2.90}$$

In the treatment given here it has been tacitly assumed that $\beta(s)$ never vanishes, so that there are no singularities or ambiguities in the integral $\int ds/\beta$. This is the case when the motion is stable, i.e. when $|\cos\mu| < 1$. For then α, β and γ are real and finite: it then follows from (2.52) that β (and γ) cannot vanish.

From the form (2.89) of the solution of the equation of motion it follows that the quantity

$$W = \frac{1}{\beta}\,[y^2 + (\alpha y + \beta y')^2] = \gamma y^2 + 2\alpha yy' + \beta y'^2 = a^2 \tag{2.91}$$

is constant, independent of s. (This is a consequence of a more general theorem called the Liouville's theorem.) In the space (y, y') (the phase space), the particle lying on the ellipse (2.91) is transformed onto another ellipse (2.91) with different α, β and γ, but the same W. The ellipse expressed by eq.(2.91) is shown in Fig.9. Now, πW is the area of the ellipse so that the area of the ellipse is conserved during the particle motion. The maximum beam size is given by $\sqrt{\beta W}$ and the maximum beam divergence is given by $\sqrt{\gamma W}$. The particles in the beam occupy the points in the ellipse of the phase space. In proton accelerators, the area πW is determined by the properties of the injector and the operating energy while β, α and γ are determined by the lattice structure. The quantity πW is thus very important in discussing the properties of the beam as a whole and is called an "emittance" of the beam.

In a given accelerator, the motion is restricted by the walls of the vacuum chamber or other obstructions to a certain region around the equilibrium orbit, let us say to $|y| < a$. Then all particles whose initial conditions are such that

$$W < W_0 = \frac{a^2}{\beta_{max}}$$

will perform oscillations that remain within the vacuum chamber. We define the "admittance" or "acceptance" of the system as the area of that region of (y, y') phase space for which any particles injected with initial values within the region will remain within the vacuum chamber. In accelerator design, the acceptance of the machine should equal to or larger than the emittance of the beam.

In order to decrease the aperture of magnets for a given acceptance, it is required to make β_{max} small. A useful approximate relation, valid for many accelerators, is

$$\beta_{av} \approx C/2\pi\nu = R/\nu, \tag{2.92}$$

where $R = C/2\pi$ is the average radius of the accelerator. In the general case, the maximum value of β will exceed β_{av} by some factor, which we call the "form factor"

$$F = \beta_{max}/\beta_{av} \approx \nu\beta_{max}/R \ . \tag{2.93}$$

The form factor F can generally be kept fairly small (say about 1.5 ~ 2), and therefore the acceptance of an alternating gradient machine is mainly governed by the oscillation frequency ν.

In conventional accelerators $\nu = n^{1/2}$ for vertical and $(1 - n)^{1/2}$ for horizontal oscillations; both these frequencies are less than 1. In alternating gradient accelerators, we can make ν large, thus achieving a larger acceptance for a given aperture or alternatively a small aperture for given acceptance.

§5. Twiss Parameters

Notion of the emittance and the functions β, α and γ are quite useful in accelerators, beam transport systems and storage rings because we can treat the properties of the beam such as a beam size as a whole and not by tracing the individual particles. To apply this concept to beam transport systems and storage rings, the functions β, α and γ which are defined in the periodic system should be generalized to nonperiodic systems. In this case, these functions are called "Twiss parameters".

We assume that β_1, α_1 and γ_1 at point s_1 are defined and consider the values β_2, α_2 and γ_2 at point s_2, where the transfer matrix from s_1 to s_2 is

$$M(s_2|s_1) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.94}$$

The values β_2, α_2 and γ_2 are defined by eq.(2.75) with the replacement $w^2 = \beta$, $ww' = -\alpha$ and $1 + \alpha^2 = \beta\gamma$.

Determining equations are

$$\frac{w_2}{w_1} \cos\Psi - w_2 w_1' \sin\Psi = a$$

$$\frac{w_1}{w_2} \cos\Psi + w_1 w_2' \sin\Psi = d \qquad (2.95)$$

$$w_1 w_2 \sin\Psi = b.$$

Solving (2.95) for w_2, w_2' and Ψ, we obtain

$$\beta_2 = a^2\beta_1 - 2ab\alpha_1 + b^2\gamma_1, \qquad (2.96)$$

$$\alpha_2 = -ac\beta_1 + (2bc + 1)\alpha_1 - bd\gamma_1 \ . \qquad (2.97)$$

$$\tan\Psi = \frac{b}{a\beta_1 - b\alpha_1} \ . \qquad (2.98)$$

That Ψ given by (2.98) is consistent with the previous definition

$$\Psi = \int^s \frac{ds}{\beta} \ , \qquad (2.99)$$

is shown by taking the derivative of (2.98)

$$\frac{d^2\Psi}{ds} = \frac{1}{\beta_2} \ . \qquad (2.100)$$

This is consistent with eq.(2.99).

We now inquire whether the expression (2.91) is still valid. Introducing the matrices

$$Y = \begin{pmatrix} y \\ y' \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix}$$

eq.(2.91) is expressed as

$$Y_1^T \Sigma_1 Y_1 = W \ . \qquad (2.101)$$

Now $Y_2 = M(s_2|s_1)Y_1$, and inserting into (2.101)

$$Y_2^T \, (M^{-1})^T \, \Sigma_1 M^{-1} Y_2 = W. \tag{2.102}$$

Putting

$$\Sigma_2 = (M^{-1})^T \, \Sigma_1 \, M^{-1} \, , \tag{2.103}$$

eq.(2.102) is an equation of an ellipse. From eq.(2.103), it follows that

$$\beta_2 = a^2\beta_1 - 2ab\alpha_1 + b^2\gamma_1 ,$$

$$\alpha_2 = -ca\beta_1 + (1 + 2bc)\alpha_1 - bd\gamma_1 ,$$

$$\gamma_2 = c^2\beta_1 - 2dc\alpha_1 + d^2\gamma_1 \; .$$

These are consistent with eqs.(2.96) and (2.97) and it is shown that the Twiss parameters introduced in this section have all the properties of the functions β, α and γ described in §3 and 4.

§6. Adiabatic Damping

We now consider the effect of slow variation of energy or mass of a particle. Going back to eqs.(2.6) through (2.11), the change of energy is taken into account by the replacement

$$\frac{d}{dt}(m\dot{y}) = v \frac{d}{ds} \left(mv \frac{dy}{ds}\right)$$

$$= vp \frac{d^2y}{ds^2} + v \frac{dp}{ds}\frac{dy}{ds} \quad .$$

Then, the general equation of mition becomes

$$\frac{d^2y}{ds^3} + \frac{\frac{dp}{ds}}{p}\frac{dy}{ds} + K(s)y = 0 \quad . \tag{2.104}$$

We solve this equation by the method of variation of constants and averaging. Put the solution in the form

$$y = A\sqrt{\beta}\,\sin(\Psi + \delta)$$

$$\frac{dy}{ds} = \frac{A}{\sqrt{\beta}}\left\{\frac{\beta'}{2}\sin(\Psi+\delta) + \cos(\psi + \delta)\right\} \qquad (2.105)$$

with a subsidiary condition

$$A'\sqrt{\beta}\,\sin(\Psi + \delta) + A\sqrt{\beta}\,\delta'\cos(\Psi + \delta) = 0\ . \qquad (2.106)$$

The prime denotes differentiation with respect to s. Inserting (2.105) into (2.104), we get

$$A + \frac{\frac{dp}{ds}}{p} A\left\{\frac{\beta'}{2}\sin(\Psi + \delta)\cos(\Psi + \delta) + \cos^2(\Psi + \delta)\right\} = 0 \qquad (2.107)$$

We consider the effect averaged over s. Then, using

$$\frac{1}{L}\int_0^L \beta'\sin(\Psi+\delta)\cos(\Psi + \delta)ds = \frac{1}{L}\int_0^L \frac{\beta'}{2}\sin 2(\Psi + \delta)ds$$

$$= \frac{1}{L}\left\{\left[\frac{\beta}{2}\sin 2(\Psi + \delta)\right]_0^L - \int_0^L \cos 2(\Psi + \delta)ds\right\}$$

$$= 0\ ,$$

and

$$\frac{1}{L}\int_0^L \cos^2(\Psi + \delta)ds = \frac{1}{2}\ ,$$

we get

$$\frac{dA}{ds} + \frac{1}{2}\frac{1}{p}\frac{dp}{ds}A = 0\ , \qquad (2.108)$$

The solution of (2.108) is

$$A = \frac{A_0}{\sqrt{p}}\ . \qquad (2.109)$$

Eq.(2.109) shows that the amplitude of betatron oscillation decreases with energy as $p^{-\frac{1}{2}}$ and that the emittance of the beam decreases as p^{-1}. This is a consequence of a more general theorem called the Liouville's theorem, which states that the phase space area expressed in terms of canonically conjugute variables is conserved in a Hamiltonian system. The quantity $\pi\beta\gamma W$, where β and γ are relativistic Lorentz factors, is conserved during the acceleration period and is called a "normalized emittance" of the beam.

§7. Dispersion Function and Momentum Compaction Factor

We now consider the effect of momentum spread. The general equation of motion is (we assume only horizontal bending)

$$x'' + Kx = \frac{1}{\rho}\frac{\Delta p}{p} \quad . \tag{2.110}$$

The general solution of eq.(2.110) is a sum of a particular solution which expresses a displaced equilibrium orbit and a general solution of the homogeneous equation which expresses betatron oscillations about the displaced equilibrium orbit. The displaced equilibrium orbit is periodic and can be obtained by solving the matrix equation

$$\begin{pmatrix} x_{eq} \\ x'_{eq} \\ \frac{\Delta p}{p} \end{pmatrix} = \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{eq} \\ x'_{eq} \\ \frac{\Delta p}{p} \end{pmatrix} \tag{2.111}$$

where the transfer matrix refers to one period. The solution is

$$\begin{pmatrix} x_{eq} \\ x'_{eq} \end{pmatrix} = \frac{\Delta p/p}{2(1 - \cos\mu)} \begin{pmatrix} e + bf - de \\ f + ce - af \end{pmatrix} . \tag{2.112}$$

The quanitty $x_p = x_{eq}/\Delta p/p$ is called a dispersion function (or momentum compaction function).

Another quantity of interest is the path length difference ΔC. Referring to Fig.10, the path length is expressed as

$$dC = \sqrt{ds + x'^2_{eq}}\ ds \approx ds \tag{2.113}$$

in a straight section, and

$$dC = \sqrt{(1 + \frac{x_{eq}}{\rho})^2 ds^2 + x'^2_{eq} ds^2} \simeq (1 + \frac{x_{eq}}{\rho}) ds \qquad (2.114)$$

in a curved section. The total path length difference is

$$\Delta C = \frac{1}{\rho} \int_{\text{curved section}} x_{eq} ds \quad . \qquad (2.115)$$

A more relevant quantity is the momentum compaction factor defined as (some authors, e.g. Livingood, defines the inverse)

$$\alpha = \frac{\Delta C/C}{\Delta p/p} \quad . \qquad (2.116)$$

From eq.(2.115)

$$\alpha = \frac{1}{2\pi R} \frac{1}{\rho} \int_{\text{curved section}} x_p ds \quad . \qquad (2.117)$$

A useful approximate relation

$$\alpha \approx \frac{1}{\nu_x^2} \qquad (2.118)$$

holds for most accelerators. The momentun compaction factor is a measure of the spread of the beam due to momentum error. In a weak-focusing accelerator

$$\alpha = \frac{1}{1 - n} \quad , \qquad (2.119)$$

so that the beam size due to momentum spread as well as that due to betatron oscillations are much smaller in AG synchrotrons. This is the origin of the name "strong focusing".

We further consider the case where the magnetic field has an error ΔB. The basic equation is then

$$x'' + Kx = -\frac{1}{\rho}\frac{\Delta B}{B} \quad . \qquad (2.120)$$

The particular solution is naturally

$$x = -x_p \frac{\Delta B}{B},$$

so that when there are errors in momentum and magnetic field, the displaced equilibrium orbit is

$$x_{eq} = x_p \left(\frac{\Delta p}{p} - \frac{\Delta B}{B}\right), \tag{2.121}$$

and

$$\frac{\Delta C}{C} = \left(\frac{\Delta p}{p} - \frac{\Delta B}{B}\right). \tag{2.122}$$

§8. Resonances

In an actual magnet system, the fields will differ somewhat from the ideal design. Therefore a particle which originally starts out on the ideal equilibrium orbit will, in general, not stay on that orbit, but will deviate from it. In this respect, there should exist a closed orbit, the "displaced equilibrium orbit", which the particle can follow , and which is located well within the aperture of the machine. Further, oscillations about this displaced equlibrium orbit should be stable.

Since the particle revolve around the machine many times, the repeated action in phase called a "resonance" should be avoided. The condition of resonance is

$$p\nu_x + q\nu_z = \text{integer}, \tag{2.123}$$

where $k = |p| + |q|$ is the order of resonance, and ν_x and ν_z designate frequencies of betatron oscillation in the horizontal and vertical planes. Resonances with k = 1 and 2 are "linear" resonances and treated in detail by Courant and Snyder. Higher k values designate "non-linear" resonances and these are treated by Schoch. Linear resonances are dangerous and shoule ba avoided in any accelerator. Nonlinear resonances with order k = 3 and 4 are also dangerous, but higher order resonances are not so important in accelerators. In storage rings, still higher order resonances should be taken into consideration. The method to avoid resonances is to employ frequencies of betatron oscillations ν_x and ν_y which do not satisfy the condition (2.123).

§9. Computer Programs

In designing an accelerator, the ν-value should first be determined. Higher ν-value is preferable for focusing, but the tolerances on magnets and alignments are severer. Further, higher ν-values require higher field gradients of quadrupole magnets or high profile parameters (n/ρ) of gradient magnets which are technically limited. Thus, a compromised value must be chosen. The phase advance per cell (or period) μ is usually chosen between $\frac{\pi}{4}$ and $\frac{\pi}{2}$. Then, from eq. (2.86) the number of identical cells is determined. For a given value of μ, the parameters of the magnet such as the profile parameter should be determined by use of the equations such as (2.63) and (2.64). The orbit parameters such as β, α, γ, x_p etc. should then be calculated.

In a beam transport system or an insertion of a storage ring, the initial values of β, α, γ, x_p etc. are usually given and we intend to "match" these functions to desired final values. These are done by using bending (dipole) magnets and quadrupole magnets and we should determine the parameters of these magnets. Further, the beam sizes etc. should be calculated along the beam line.

All these calculations are based on a matrix calculation as described above, and the procedure is quite cumbersome. Thus, computer programs are inevitable in these calculations. Computer programs such as SYNCH, TRANSPORT, MAGIC, etc. have been developed in the laboratories around the world. These programs are available at KEK.

§1. Standing Wave and Travelling Wave Pictures

The electric field appearing in an accelerating cavity is a standing wave, but a travelling wave picture, in which particles ride on the travelling wave, is more relevant for mathematical description. This is because a discrete energy gain is replaced by a continuous one and thus a difference equation is replaced by a differential equation.

We consider a single accelerating gap as shown in Fig.4. Generalization to many gaps is straightforward. Neglecting the length of the gap, the electric field E is expressed as

$$E = V\,\delta_p(s)\,\sin(\omega_{rf}t + \phi_s), \tag{3.1}$$

where ω_{rf} is the angular frequency of the RF and δ_p is a delta-function periodic with period C (circumference of the machine). The accelerating gap is placed at s=0 and particles pass at phase ϕ_s at t=0. The voltage V of the RF system is defined by the energy gain ΔE per revolution in the following way.

$$\Delta E = e\oint E ds = eV\sin\phi_s\ . \tag{3.2}$$

Denoting the angular frequency of the particle by ω, and the particle velocity by v,

$$C = \frac{2\pi v}{\omega}$$

$$= \frac{2\pi v}{\omega_{rf}}\cdot h\ , \tag{3.3}$$

where the resonance condition

$$\omega_{rf} = h\omega \tag{3.4}$$

is assumed. The integer h is called a "harmonic number".

Since the function $\delta_p(s)$ is periodic with period C, it can be expanded into a Fourier series,

$$\delta_p(s) = \sum_n \frac{2}{C}\cos\frac{2\pi n}{C}s. \tag{3.5}$$

Inserting this into eq.(3.1),

$$E = \sum_n \frac{2V}{C} \sin(\omega_{rf}t + \phi_s) \cos \frac{n\omega_{rf}}{vh} s$$

$$= \sum_n \frac{V}{C} [\sin(\omega_{rf}t + \phi_s + \frac{n\omega_{rf}}{hv} s) + \sin(\omega_{rf}t + \phi_s - \frac{n\omega_{rf}}{hv} s)] \ . \tag{3.6}$$

This is a representation in terms of travelling waves.

Now, the particles move with $s \approx vt$, so that only the term containing $\sin(\omega_{rf}t + \phi_s - \frac{h\omega_{rf}}{hv} s)$ will contribute to acceleration when averaged over many revolutions. Other terms will accelerate and decelerate particles alternately and the net effect will be neglected. Thus, only a term with n=h is retained and

$$E = \frac{V}{C} \sin(\omega_{rf}t + \phi_s - \frac{\omega_{rf}}{v_s} s) \ , \tag{3.7}$$

where v_s is the velocity of a "synchronous particle" which satisfies the resonance condition (3.4). In general, eq.(3.7) is expressed as

$$E = E(s) \sin(\int_0^t \omega_{rf} dt - \int_0^{s(t)} \frac{\omega_{rf}}{v_s} ds + \phi_s) \ . \tag{3.8}$$

§2. Equation of Synchrotron Oscillation

1) Phase equation

With $v_s = R\omega_s$, $\omega_{rf} = h\omega_s$ and $\theta = \frac{s}{R}$, eq. (3.7) is expressed as

$$E = \frac{V}{C} \sin \phi \ ,$$

$$\phi = \omega_{rf}t - h\theta + \phi_s \ . \tag{3.9}$$

Now,

$$\omega = \dot{\theta} = \frac{v}{R} \ , \tag{3.10}$$

and we expand ω around the synchronous energy E_s in terms of $\Delta E = E - E_s$ in a way

$$\omega = \omega_s + \frac{d\omega}{dE}\Big|_s \Delta E + \cdot \cdot \cdot \cdot \tag{3.11}$$

From (3.10)

$$\frac{\Delta\omega}{\omega} = \frac{\Delta v}{v} - \frac{\Delta R}{R} . \tag{3.12}$$

Using

$$\frac{\Delta v}{v} = \frac{1}{\beta^2\gamma^2}\frac{\Delta E}{E} ,$$

and

$$\frac{\Delta R}{R} = \alpha \frac{\Delta p}{p} = \frac{\alpha}{\beta^2}\frac{\Delta E}{E} ,$$

eq.(3.12) is

$$\frac{\Delta\omega}{\omega} = \frac{1}{\beta^2}\left(\frac{1}{\gamma^2} - \alpha\right)\frac{\Delta E}{E} . \tag{3.13}$$

Taking the derivative of (3.9), using the expansion (3.11) and using the relation (3.13)

$$\frac{d\phi}{dt} = \omega_{rf} - h\omega_s - \frac{h\omega_s}{\beta_s^2}\left(\frac{1}{\gamma_s^2} - \alpha\right)\frac{\Delta E}{E}$$

$$= \frac{h\omega_s}{\beta_s^2}\left(\alpha - \frac{1}{\gamma_s^2}\right)\frac{\Delta E}{E} , \tag{3.14}$$

where the relation $\omega_{rf} = h\omega_s$ is used.

2) Energy Gain of Synchronous Particles

Particles gain energy from an RF electric field or from an induced electric field due to changing magnetic flux (betatron acceleration). They lose energy, for example, by synchrotron radiation in electron machines. In this lecture, we consider only RF acceleration and betatron acceleration.

Betatron acceleration is due to the induced electric field given by the Maxwell's equation

$$\text{rot } E = -\frac{\partial B}{\partial t} . \tag{3.15}$$

now

$$2\pi R E_s = \int E ds = \int \text{rot} E dS = + \frac{\partial \Phi}{\partial t} ,$$

where Φ is the magnetic flux enclosed by the particle trajectory and the sign is chosen so that the direction of flux coincides with that of B. The energy gain per revolution is $e\, E_s \cdot 2\pi R = e \frac{\partial \Phi}{\partial t}$. The energy gain due to RF field is $eV\sin\phi_s$.

Now, from the equation $p = eB\rho$

$$\frac{dp}{dt} = \frac{1}{v}\frac{dE}{dt} = e\rho\frac{dB}{dt} + eB\frac{d\rho}{dt} . \tag{3.16}$$

Since $\frac{2\pi}{\omega}\frac{dE}{dt}$ is the energy gain per revolution,

$$eV\sin\phi_s + e\frac{\partial\Phi}{\partial t} = 2\pi eR\rho\frac{dB}{dt} + 2\pi eRB\frac{d\rho}{dt} . \tag{3.17}$$

Usually, $\frac{d\rho}{dt} = 0$ and the betatron acceleration is much smaller than RF acceleration so that

$$eV\sin\phi_s \approx 2\pi eR\rho\frac{dB}{dt} . \tag{3.18}$$

3) Energy equation

From the energy balance, the following equations result

$$\begin{aligned} \frac{2\pi}{\omega}\frac{dE}{dt} &= eV\sin\phi + e\frac{\partial\Phi}{\partial t} , \\ \frac{2\pi}{\omega_s}\frac{dE_s}{dt} &= eV\sin\phi_s + e\frac{\partial\Phi_s}{\partial t} , \end{aligned} \tag{3.19}$$

where s refers to synchronous particles. These equations are derived naturally from the equations of motion (2.8) and (2.11). Taking a difference of (3.19), we obtain

$$\frac{1}{\omega}\frac{dE}{dt} - \frac{1}{\omega_s}\frac{dE_s}{dt} = \frac{eV}{2\pi}\{\sin\phi - \sin\phi_s\} + \frac{e}{2\pi}\{\frac{\partial\Phi}{\partial t} - \frac{\partial\Phi_s}{\partial t}\} . \tag{3.20}$$

We now let $E = E_s + \Delta E$, assume that E is small, and expand (3.20) in terms of ΔE. Firstly,

$$\begin{aligned} \frac{1}{\omega}\frac{dE}{dt} - \frac{1}{\omega_s}\frac{dE_s}{dt} &= \frac{1}{\omega_s}\frac{dE_s}{dt} - \frac{1}{\omega_s^2}(\frac{\partial\omega}{\partial E})_s\Delta E\frac{dE_s}{dt} + \frac{1}{\omega_s}\frac{d\Delta E}{dt} - \frac{1}{\omega_s}\frac{dE_s}{dt} \\ &= \frac{1}{\omega_s}\frac{d\Delta E}{dt} - \frac{1}{\omega_s^2}(\frac{\partial\omega}{\partial E})_s\Delta E\frac{dE_s}{dt} . \end{aligned} \tag{3.21}$$

Further,

$$\frac{e}{2\pi}\left\{\frac{\partial\Phi}{\partial t} - \frac{\partial\Phi_s}{\partial t}\right\}$$

$$= \frac{e}{2\pi}\left\{\frac{\partial}{\partial t}\int_{E_s+\Delta E} BdS - \frac{\partial}{\partial t}\int_{E_s} BdS\right\}$$

$$= \frac{e}{2\pi}\int_{\Delta E}\frac{\partial B}{\partial t}\, dS$$

$$= \frac{e}{2\pi}\oint\frac{\partial B_s}{\partial t}\, x_p\,\frac{\Delta p}{p}\, ds$$

$$= \frac{e}{2\pi}\,\frac{\partial B_s}{\partial t}\,\frac{\Delta p}{p}\oint_{magnet} x_p ds$$

$$= \frac{e\rho R\alpha}{\beta_s^2}\,\frac{\Delta E}{E}\,\frac{\partial B_s}{\partial t}\,, \tag{3.22}$$

where the relation $\oint_{mag} x_p ds = 2\pi R\ \rho\alpha$ given by eq.(2.117) is used. Inserting (3.21) and (3.22) into (3.20)

$$\frac{1}{\omega_s}\frac{d\Delta E}{dt} - \left\{\frac{1}{\omega_s^2}\left(\frac{\partial\omega}{\partial E}\right)_s\frac{dE_s}{dt} + \frac{e\rho R\alpha}{\beta_s^2 E_s}\frac{\partial B_s}{\partial t}\right\}\Delta E = \frac{eV}{2\pi}\{\sin\phi - \sin\phi_s\}\,. \tag{3.23}$$

Now, $\omega = \frac{v}{R}$ and R is a function of E and B, so that

$$\frac{d\omega}{dt} = \frac{\partial\omega}{\partial E}\Big)_{B_s}\frac{dE}{dt} + \frac{\partial\omega}{\partial B}\Big)_{E_s}\frac{dB}{dt}\ . \tag{3.24}$$

From (3.12) and (2.122)

$$\frac{\Delta\omega}{\omega} = \frac{\Delta v}{v} - \frac{\Delta R}{R}$$

$$= \frac{\Delta v}{v} - \alpha\left(\frac{\Delta p}{p} - \frac{\Delta B}{B}\right)$$

$$= \frac{1}{\beta^2}\left(\frac{1}{\gamma^2} - \alpha\right)\frac{\Delta E}{E} + \alpha\frac{\Delta B}{B}\,,$$

so that

$$\frac{\partial\omega}{\partial E}\Big)_{B_s} = \frac{\omega}{\beta^2 E}\left(\frac{1}{\gamma^2} - \alpha\right) \text{ and } \frac{\partial\omega}{\partial B}\Big)_{E_s} = \frac{\omega}{B}\alpha\ . \tag{3.25}$$

Thus,

$$\frac{e\rho R\alpha}{\beta_s^2 E_s}\frac{\partial B_s}{\partial t} = \frac{B}{\omega}\frac{\partial\omega}{\partial B})_{E_s}\frac{e\rho R}{\beta_s^2 E_s}\frac{\partial B_s}{\partial t}$$

$$= \frac{pR}{\omega\beta_s^2 E_s}\frac{\partial\omega}{\partial B})_{E_s}\frac{\partial B_s}{\partial t}$$

$$= \frac{1}{\omega^2}\frac{\partial\omega}{\partial B})_{E_s}\frac{\partial B_s}{\partial t} \quad .$$

Inserting this into (3.23),

$$\frac{1}{\omega_s}\frac{d\Delta E}{dt} - \frac{1}{\omega_s^2}\{(\frac{\partial\omega}{\partial E})_s\frac{dE_s}{dt} + \frac{\partial\omega}{\partial B})_{E_s}\frac{\partial B_s}{\partial t}\}\Delta E = \frac{eV}{2\pi}\{\sin\phi - \sin\phi_s\} \tag{3.26}$$

Now

$$\frac{dB_s}{dt} = \frac{\partial B_s}{\partial t} + \frac{\partial B}{\partial x}\frac{dx}{dt})_s + \frac{\partial B}{\partial z}\frac{dz}{\partial t})_s \quad .$$

Since the equilibrium orbit does not change,

$$\frac{dx}{dt})_s = \frac{dz}{dt})_s = 0,$$

so that

$$\frac{dB_s}{dt} = \frac{\partial B_s}{\partial t} \quad .$$

Then, by use of (3.24), eq.(3.26) is transformed as

$$\frac{1}{\omega_s}\frac{d\Delta E}{dt} - \frac{1}{\omega_s^2}\frac{d\omega}{dt}\Delta E = \frac{eV}{2\pi}\{\sin\phi - \sin\phi_s\} \ ,$$

and finally we obtain the energy equation

$$\frac{d}{dt}(\frac{\Delta E}{\omega_s}) = \frac{eV}{2\pi}\{\sin\phi - \sin\phi_s\} \ . \tag{3.27}$$

The phase equation (3.14) and the energy equation (3.27) are the basic equations of synchrotron motion.

§3. Transition Energy

We consider a revolution time $T = \frac{C}{v}$. Now,

$$\frac{\Delta T}{T} = \frac{\Delta C}{C} - \frac{\Delta v}{v}$$

$$= (\alpha - \frac{1}{\gamma^2}) \frac{\Delta p}{p} . \quad (3.28)$$

Since α is usually less than unity in AG accelerators, there is an energy where $\Delta T = 0$, i.e. a revolution time does not differ for different momenta. This energy is given by

$$\gamma_t = \frac{1}{\sqrt{\alpha}} , \quad (3.29)$$

and is called a "transition energy" (divided by particle rest energy).

Below transition energy, a particle having a larger energy revolves faster, while above transition energy, that particle revolves slower. As explained in Part I, this influences phase stability. Below transition energy, the stable phase is on the rising side, while above transition the stable phase is on the falling side of the RF field. At transition, phase stability breaks down and the RF phase should be switched quickly. We will not discuss the problem of transition crossing in this lecture, but merely state that transition energies have been successfully crossed in all existing synchrotrons.

§4. Small Amplitude Oscillation

The equations of synchrotron oscillation are nonlinear, but if we put

$$\phi = \phi_s + \Delta\phi , \quad (3.30)$$

and assume that $\Delta\phi$ is small, the equations are linearized. From eq.(3.14) and (3.27)

$$\frac{d\Delta\phi}{dt} = \frac{h\omega_s}{\beta_s^2} (\alpha - \frac{1}{\gamma_s^2}) (\frac{\Delta E}{E}) ,$$

and

$$\frac{d}{dt} (\frac{\Delta E}{\omega_s}) = \frac{eV}{2\pi} \cos\phi_s \Delta\phi .$$

Solving for ΔE in the first equation and inserting into the latter, we get

$$\frac{d}{dt}\left(\frac{\beta_s^2 E_s}{\eta_s \omega_s^2}\frac{d\Delta\phi}{dt}\right) = \frac{heV}{2\pi}\cos\phi_s \Delta\phi \tag{3.31}$$

where

$$\eta_s = \alpha - \frac{1}{\gamma_s^2} \ . \tag{3.32}$$

If we assume that the kinematic factors such as β_s, E_s, etc. are constant, the solution of (3.31) is expressed as

$$\Delta\phi = A \sin(\nu_s t + \delta),$$

when $\eta_s \cos\phi_s < 0$, and the motion is stable. If $\eta_s \cos\phi_s > 0$, the solution is of an exponential form and the motion is unstable. This is consistent with the discussion of §3. Here, A and δ are arbitrary constants, and

$$\nu_s = \sqrt{-\frac{heV\ \omega_s^2 \eta_s}{2\pi\beta_s^2\ E_s}\cos\phi_s} \tag{3.33}$$

is the "frequency" of (small amplitude) synchrotron oscillation. The corresponding energy spread is

$$\frac{\Delta E}{E} = \frac{\beta_s^2 \nu_s}{h\eta_s\omega_s} A\cos(\nu_s t + \delta)$$

$$= \sqrt{-\frac{eV\beta_s^2\cos\phi_s}{2\pi h\eta_s E_s}}\ A\cos(\nu_s t + \delta). \tag{3.34}$$

It is to be noted that the trajectory in a phase space $(\Delta\phi, \frac{\Delta E}{E})$ is an ellipse.

§5. Adiabatic Damping

We now consider a slow (adiabatic) change of the kinematical factors and the RF voltage appearing in eq.(3.31). We put

$$c = \frac{\beta_s^2 E_s}{\eta_s \omega_s^2} \tag{3.35}$$

and the equation becomes

$$\frac{d^2\Delta\phi}{dt^2} + \frac{c'}{c}\frac{d\Delta\phi}{dt} + \nu_s^2\Delta\phi = 0, \qquad (3.36)$$

where the prime denotes differentiation with respect to t. We solve eq. (3.36) by the method of variation of constants and averaging. We put

$$\Delta\phi = A \sin(\nu_s t + \delta), \qquad (3.37)$$

amd

$$\frac{d\Delta\phi}{dt} = A\nu_s \cos(\nu_s + \delta), \qquad (3.38)$$

with a subsidiary condition

$$A'\sin(\nu_s t + \delta) + A(\nu_s' t + \delta')\cos(\nu_s t + \delta) = 0. \qquad (3.39)$$

Inserting (3.37) and (3.38) into (3.36), we obtain

$$\nu_s' A \cos(\nu_s t + \delta) + \nu_s A' \cos(\nu_s t + \delta) - \nu_s \nu_s' A t \sin(\nu_s t + \delta)$$

$$- \nu_s A\delta' \sin(\nu_s t + \delta) + \frac{c'}{c}\nu_s A \cos(\nu_s t + \delta) = 0. \qquad (3.40)$$

Eliminating δ' from (3.39) and (3.40), we get

$$\nu_s' A\cos^2(\nu_s + \delta) + \nu_s A' + \frac{c'}{c}\nu_s A \cos^2(\nu_s t + \delta) = 0. \qquad (3.41)$$

Now the change of A, ν_s and c is much slower than the period of synchrotron oscillation, so that we average (3.41) over synchrotron oscillations. With

$$\langle\cos^2(\nu_s t + \delta)\rangle_{av} = \frac{1}{2}, \qquad (3.42)$$

we obtain

$$\frac{1}{2}\frac{\nu_s'}{\nu_s} + \frac{A'}{A} + \frac{1}{2}\frac{c'}{c} = 0. \qquad (3.43)$$

The solution of eq. (3.42) is

$$A = \frac{\text{const}}{\sqrt{\nu_s c}} \propto \left[\frac{\eta_s}{VE_s\cos\phi_s}\right]^{\frac{1}{4}}, \qquad (3.44)$$

where (3.33) and (3.35) are used. The effect on energy spread is, from eq.(3.34)

$$\Delta E \propto [\frac{E_s V\cos\phi_s}{\eta_s}]^{\frac{1}{4}} \beta_s \quad . \tag{3.45}$$

Eqs.(3.44) and (3.45) show the adiabatic damping of the amplitude of synchrotron oscillation. It is to be noted

$$\frac{(\Delta\phi)_{max}(\Delta E)_{max}}{h\omega_s} = [\frac{\eta_s}{VE_s\cos\phi_s}]^{\frac{1}{4}} [\frac{E_s V\cos\phi_s}{\eta_s}]^{\frac{1}{4}} \frac{\beta_s}{h\omega_s}$$

$$= \frac{R\beta_s}{hc\beta_s}$$

$$= \text{const.} \tag{3.46}$$

This is a relation derived from the Liouville's theorem as decribed later.

§6. Hamiltonian Formalism

The equations of synchrotron oscillation is nonlinear. To treat nonlinear problems, it is quite useful to apply the theorems of classical dynamics. To this end, the canonical or Hamiltonian formalism is essential.

We must choose appropriate variables to express the equations of motion in a canonical form. To this end, we recall that the phase space desity expressed in terms of canonical variables is conserved in a Hamiltonian system and that energy E and time t are canonically conjugate. Then, $\Delta E \Delta t$ is in variant. Since $\phi = \omega_{rf}t$,

$$\Delta t = \frac{\Delta\phi}{\omega_{rf}} = \frac{\Delta\phi}{h\omega_s} \quad ,$$

so that

$$\Delta E \Delta t = \frac{\Delta\phi\Delta E}{h\omega_s} \tag{3.47}$$

is invariant. This ia a general representation of the result (3.46).

Introducing a variable W defined by

$$W(E) = \int \frac{E\,dE}{h\omega_s} \quad , \tag{3.48}$$

it will be guessed that ϕ and W are canonically conjugate. Actually, equations of motion (3.14) and (3.27) are expressed as

$$\frac{d\Delta W}{dt} = \frac{eV}{2\pi h} (\sin\phi - \sin\phi_s), \tag{3.49}$$

and

$$\frac{d\phi}{dt} = \frac{\eta_s h^2 \omega_s^2}{\beta_s^2 E_s} \Delta W . \tag{3.50}$$

These equations are canonical, i.e.

$$\frac{d\phi}{dt} = \frac{\partial H}{\partial(\Delta W)} , \qquad \frac{d\Delta W}{dt} = - \frac{\partial H}{\partial \phi} , \tag{3.51}$$

if we choose the Hamiltonian $H(\Delta W, \phi)$ to be

$$H(\Delta W,\phi) = \frac{1}{2} \frac{\eta_s h^2 \omega_s^2}{\beta_s^2 E_s} (\Delta W)^2 + \frac{e V}{2\pi h} (\cos\phi + \phi\sin\phi_s). \tag{3.52}$$

If the variation of parameters such η_s, V etc. with time is neglected, the Hamiltonian does not contain time explicitely and is the constant of motion. The trajectory in the phase space is expressed as

$$H(\Delta W, \phi) = \text{const.} \tag{3.53}$$

From the form of (3.52), the motion is considered to be one in a potential $U(\phi)$ given by

$$U(\phi) = (\cos\phi + \phi\sin\phi_s) . \tag{3.54}$$

§7. Stability of Motion

We assume that the parameters such as η, V etc. are constant in time. Then,the Hamiltonian (3.52) is constant and

$$\frac{\pi h^3 \eta_s \omega_s^2}{eV\beta_s^2 E_s} (\Delta W)^2 + U(\phi) = C = \text{const}, \tag{3.55}$$

or by (3.50)

$$\frac{\pi E_s \beta_s^2}{heV\omega_s^2 \eta_s} \left(\frac{d\phi}{dt} \right)^2 + U(\phi) = C, \tag{3.56}$$

where $U(\phi)$ is given by (3.54).

The potential $U(\phi)$ is shown in Fig.11(a). We consider a case where $\eta_s>0$, i.e. above transition. For case a, the motion is bounded by ϕ_{min} and ϕ_{max} and is stable. For case c, the motion is bounded only one side and is unstable. The case b is a boundary case for stable and unstable motions. The values of C are given for cases a and b in the following way.

$$C = U(\phi_{min}) = U(\phi_{max})$$

for case a,

$$C = U(\phi_1) = U(\phi_2)$$

for case b and $\frac{d\phi}{dt} = 0$ at these points.

At ϕ_1, the potential is maximum and ϕ_1 is obtained by solving the equation

$$\frac{\partial U(\phi_1)}{\partial \phi} = \sin\phi_s - \sin\phi = 0. \tag{3.57}$$

The solutions are

$$\phi = \phi_s \text{ or } \pi - \phi_s$$

and

$$\phi_1 = \pi - \phi_s . \tag{3.58}$$

At ϕ_s or $\pi - \phi_s$,

$$\frac{d\Delta W}{dt} = 0,$$

and (3.59)

$$\frac{d\phi}{dt} = 0 ,$$

so that the motion does not occur at these points and these points are "fixed points".

The motion about $\phi = \phi_s$ is sinusoidal and stable as described in §4, so that this point is called a "stable" fixed point. We put $\phi = \phi_1 + \Delta\phi$ and consider a motion about $\phi = \phi_1$. From eqs.(3.41) and (3.50),

$$\frac{d\Delta W}{dt} = \frac{eV}{2\pi h}\{\sin(\pi - \phi_s + \Delta\phi) - \sin\phi_s\}$$

$$= -\frac{eV}{2\pi h}\cos\phi_s \Delta\phi$$

and

$$\frac{d\Delta\phi}{dt} = \frac{\eta_s h^2 \phi_s^2 \Delta W}{\beta_s^2 E_s} .$$

Thus, eliminating ΔW,

$$\frac{d^2\Delta\phi}{dt} + \frac{heV\eta_s\omega_s^2}{2\pi\beta_s^2 E_s}\cos\phi_s \Delta\phi = 0 \quad . \qquad (3.60)$$

Since $\eta_s \cos\phi_s < 0$ for stable phase angle, the motion is exponential and unstable. The point $\phi = \phi_1$ is, therefore, called an "unstable" fixed point.

The trajectory in phase space is shown in Fig.11(b). The curve b which passes through the unstable fixed point ϕ_1 is a boundary curve between stable and unstable motions and is called a "separatrix".

The separatrix is given by the equation

$$C = U(\phi_1),$$

or

$$\frac{\pi E_s \beta_s^2}{heV\omega_s^2\eta_s}\left(\frac{d\phi}{dt}\right)^2 + \cos\phi + \cos\phi_s + (\phi - \pi + \phi_s)\sin\phi_s = 0. \quad (3.61)$$

It is, in general, difficult or impossible to obtain analytic solutions of the nonlinear problems and the phase-space considerations given above are quite useful.

Acknowledgement

In some part of this lecture, the author used good descriptions of the book by Livingston and Blewett and the paper by Courant and Snyder.

Reference Books on Accelerator Theory

A) Those written in Japanese

1) 熊谷寛夫, 西川哲治, 小林喜幸 : 「加速器」 核物理学講座 6 (共立出版, 1960)

2) 熊谷寛夫責任編集 : 「加速器」 実験物理学講座 28 (共立出版, 1975)

B) General

3) M.S. Livingston and J.P. Blewett: "Particle Accelerators" (McGraw-Hill, 1962)

Historical developments are fully described. Technical problems as well as underlying physics are presented.

4) S. Flüge, ed. "Nuclear Instrumentation I" in "Handbuch der Physik", volume XLIV, (Springer Verlag, 1959)

C) Cyclic Accelerators (particularly AG machines)

5) J.J. Livingood: "Principles of Cyclic Particle Accelerators" (D. van Nostrand, 1961)

Introductory description of the theory of cyclic accelerators. Suitable for a beginner. The equations of synchrotron oscillation are wrong, however, and should be corrected as done in this lecture note.

6) H. Bruck: "Accélérateurs Circulaires de Particules", (Press Universitaires de France, 1966)

Though written in French, this is a standard text book on the theory of AG synchrotrons.

7) A.A. Kolomensky and A.N. Lebedev: "Theory of Cyclic Accelerators" (North-Holland, 1966)

Highly mathematical. Suitable for the study of non-linear oscillations. Suitable for advanced readers.

8) E.D. Courant and H.S. Snyder: "Theory of the Alternating-Gradient Synchrotron", Ann. Phys. 3, 1 (1958)

A classical article describing the theory of AG synchrotrons. Should be read once by an accelerator physicist.

D) Beam Transport System

9) K.G. Steffen: "High Energy Beam Optics" (Interscience, 1965)

10) A.P. Banford: "The Transport of Charged Particle Beams" (E & F.N. Spon Ltd., 1966)

E) Collection of Papers

11) M.S. Livingston: "The Development of High-Energy Accelerators" (Dover, 1966)

Collects papers of historical importance in accelerator development.

F) Instability

Beam instability, the phenomena which appear at high intensity, was not treated in this lecture. The last part of ref.6) is devoted to this subject.

12) E.D. Courant: "Accelerators for High Intensities and High Energies" in Ann. Rev. Nucl. Scie. 18, 435 (1968)

G) Electron Storage Rings

13) M. Sands: "The Physics of Electron Storage Rings. An Introduction" in "Physics with Intersecting Storage Rings, Proc. of the Intern. School of Physics, Enrico Fermi, Course XLVI" ed. by D. Touscheck (Academic Press, 1971)

H) Non-linear Oscillation Theory

Mathematical problems of nonlinear oscillation theory are given, for example, in

14) N.N. Bogoliubov and Y.A. Mitropolsky: "Asymptotic Methods in the Theory of Non-linear Oscillations" (Hindustan Publishing Corpn., India, 1961). Japanese translation
益子正教訳「非線型振動論 ―漸近的方法」
(共立出版, 1961)

Nonlinear oscillations in AG synchrotrons are treated in

15) A. Schoch: "Theory of linear and non-linear perturbations of betatron oscillations in alternating gradient synchrotrons", CERN 57-21 (1958)

16) P.A. Sturrock: "Nonlinear Effects in Alternating-Gradient Synchrotrons", Ann. Phys. 3, 113 (1958)

Appendix Transfer Matrix

1) definition

$$\begin{pmatrix} y \\ y' \\ \frac{\Delta p}{p} \end{pmatrix} = M \begin{pmatrix} y \\ y' \\ \frac{\Delta p}{p} \end{pmatrix}_A \qquad (A.1)$$

2) Drift Length ℓ

$$M_\ell = \begin{pmatrix} 1 & \ell & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (A.2)$$

3) Pure Sector Magnet

$$M_H^s = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1 - \cos\theta) \\ -\frac{\sin\theta}{\rho} & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \qquad (A.3)$$

$$\theta = \frac{\ell}{\rho}$$

$$M_V^s = \begin{pmatrix} 1 & \ell & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (A.4)$$

4) Gradient Sector Magnet

a) Focusing Plane

$$M_F = \begin{pmatrix} \cos\sqrt{K}\ell & \frac{1}{\sqrt{K}}\sin\sqrt{K}\ell & \frac{1}{\rho K}(1 - \cos\sqrt{K}\ell) \\ -\sqrt{K}\sin\sqrt{K}\ell & \cos\sqrt{K}\ell & \frac{1}{\rho\sqrt{K}}\sin\sqrt{K}\ell \\ 0 & 0 & 1 \end{pmatrix} \qquad (A.5)$$

$$K = \begin{cases} \dfrac{|n| + 1}{\rho^2} & \text{horizontal plane} \\[2ex] \dfrac{|n|}{\rho^2} & \text{vertical plane} \end{cases}$$

b) Defocusing Plane

$$M_D = \begin{pmatrix} \cosh\sqrt{K}\ell & \frac{1}{\sqrt{K}} \sinh\sqrt{K}\ell & \frac{1}{\rho K}(\cosh\sqrt{K}\ell - 1) \\ \sqrt{K}\sinh\sqrt{K}\ell & \cosh\sqrt{K}\ell & \frac{1}{\rho\sqrt{K}} \sinh\sqrt{K}\ell \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{(A.6)}$$

$$K = \begin{cases} \dfrac{(|n| - 1)}{\rho^2} & \text{horizontal plane} \\[2ex] \dfrac{|n|}{\rho^2} & \text{vertical plane} \end{cases}$$

5) Quadrupole Magnet

a) Focusing Plane

$$M_F = \begin{pmatrix} \cos\sqrt{K}\ell & \frac{1}{\sqrt{K}} \sin\sqrt{K} & 0 \\ -\sqrt{K}\sin\sqrt{K}\ell & \cos\sqrt{K}\ell & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

b) Defocusing Plane

$$M_D = \begin{pmatrix} \cosh\sqrt{K}\ell & \frac{1}{\sqrt{K}} \sinh\sqrt{K}\ell & 0 \\ \sqrt{K} \sinh\sqrt{K}\ell & \cosh\sqrt{K}\ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{(A.8)}$$

$$K = \left| \frac{e}{P_0} \frac{\partial B_z}{\partial x} \right|$$

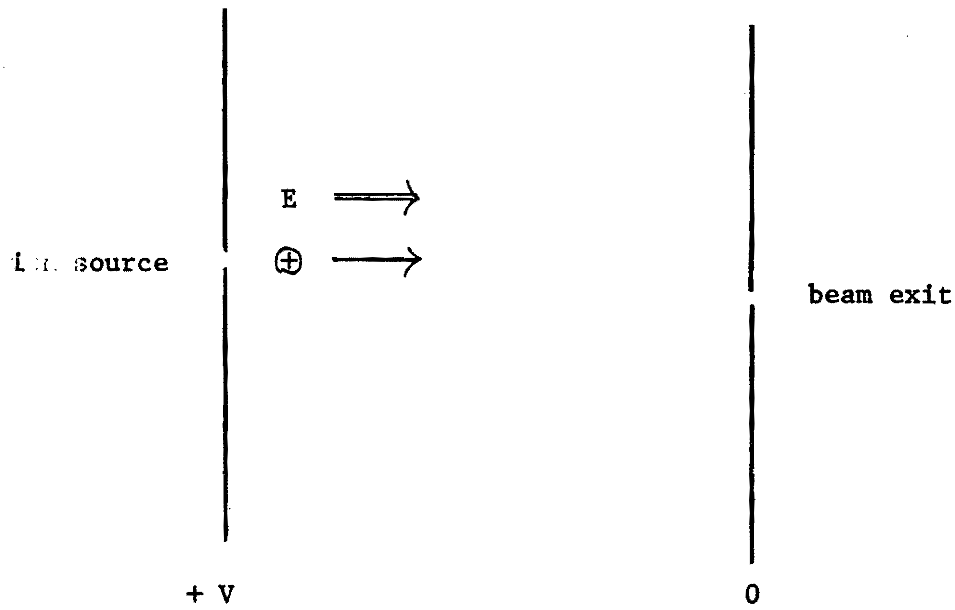


Fig. 1. Principle of direct voltage accelerator

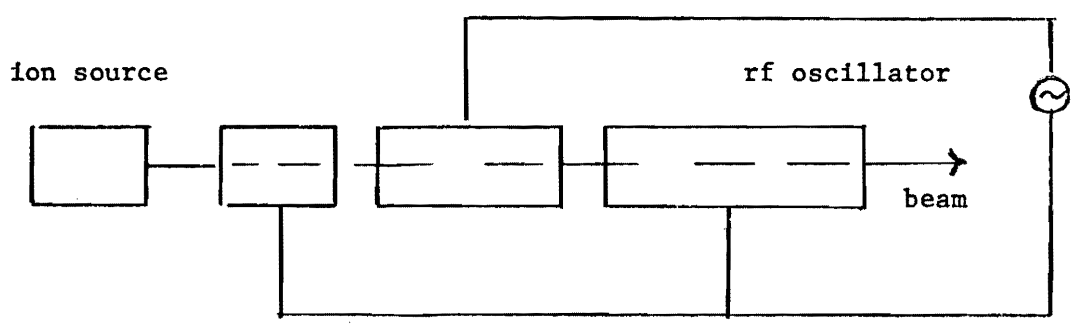


Fig. 2. Schematic diagram of early linear accelerator

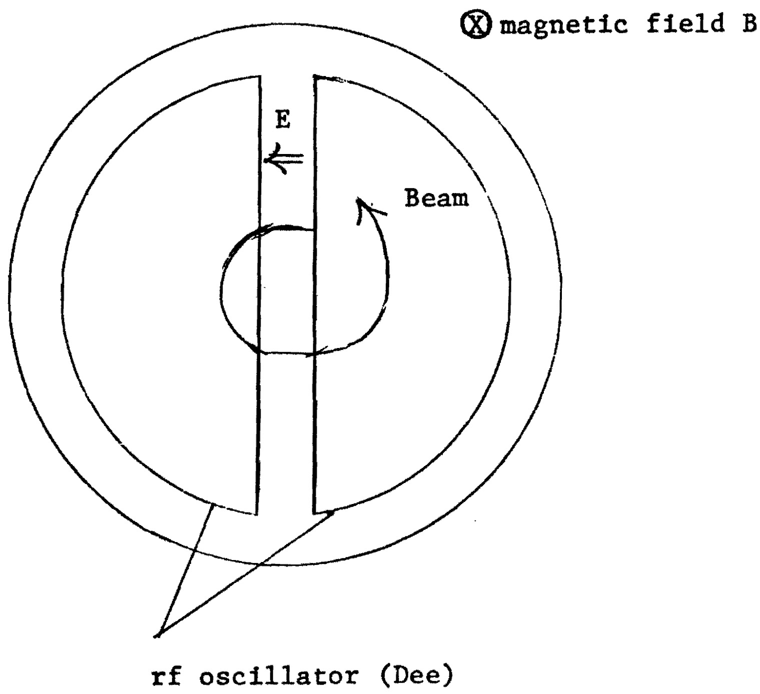


Fig. 3. Cyclotron

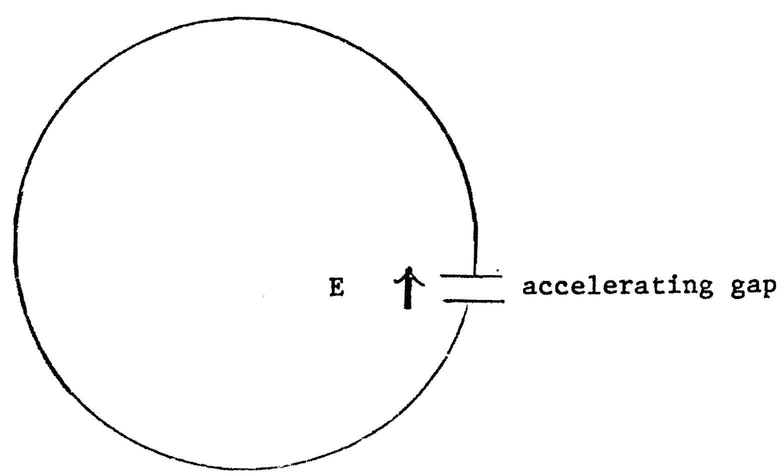


Fig. 4. Schematic diagram of accelerating system

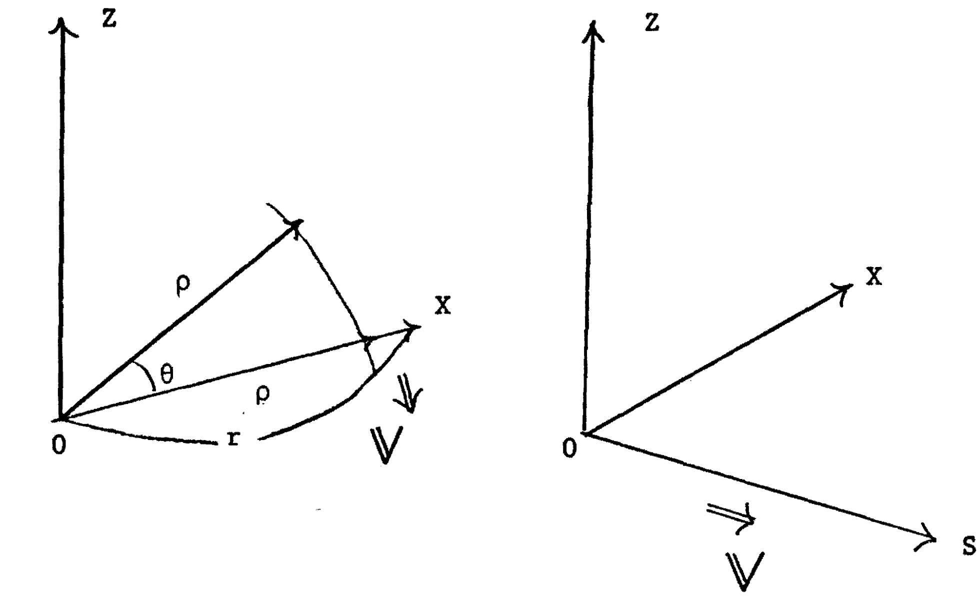


Fig. 5. Coordinate system.

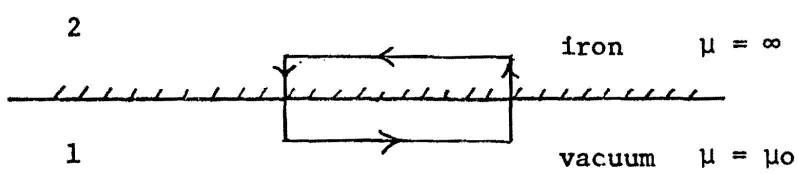


Fig. 6. Boundary of iron yokes

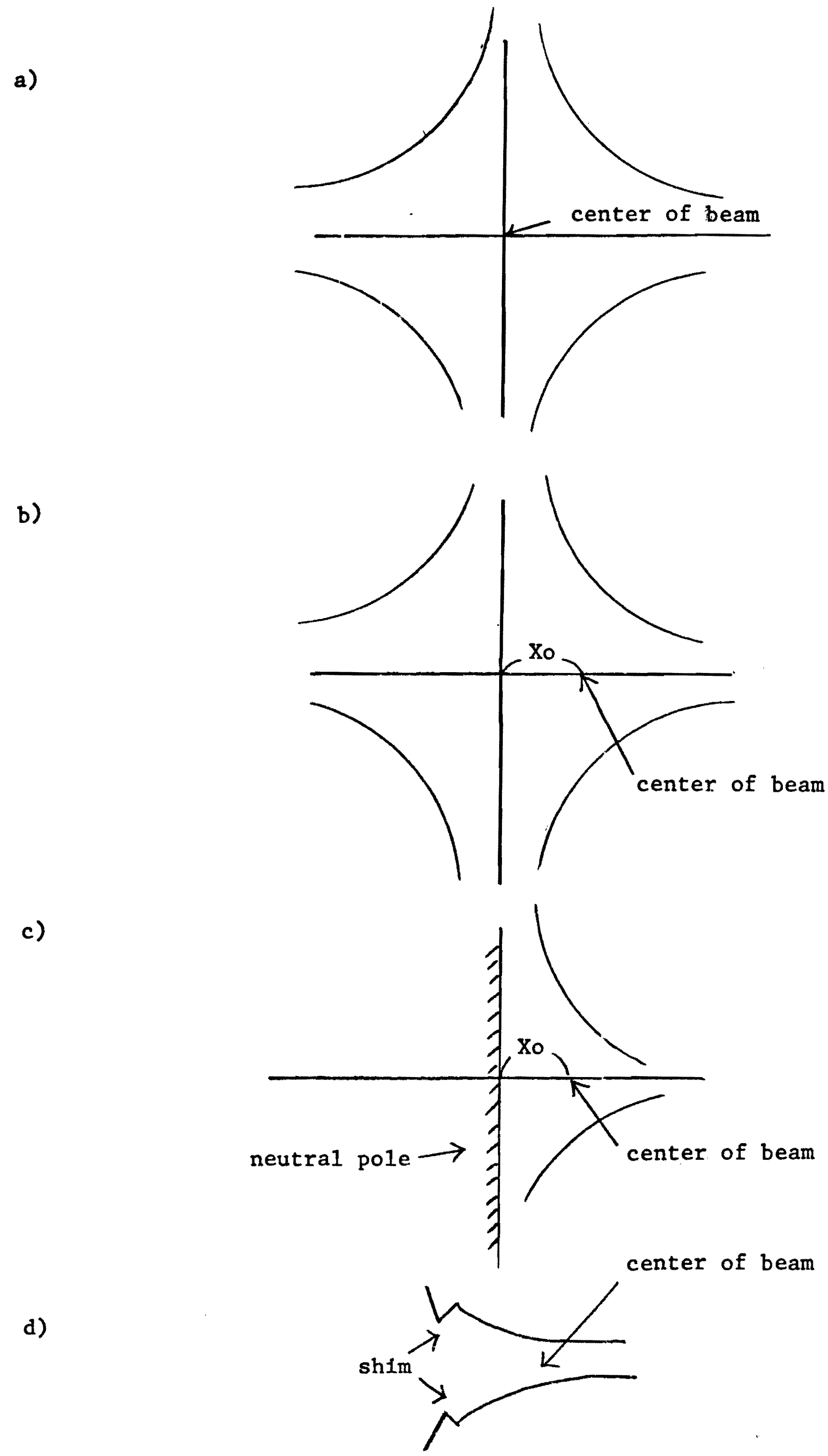


Fig. 7. Quadrupole and gradient magnets.

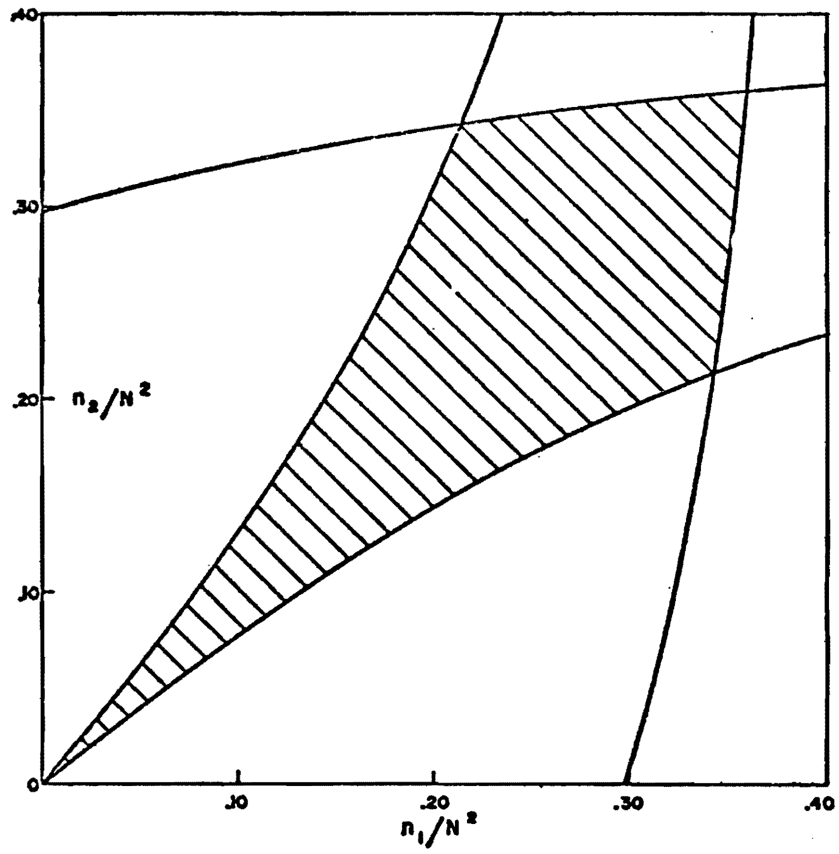


Fig. 8. Necktie diagram (after Courant and Snyder).

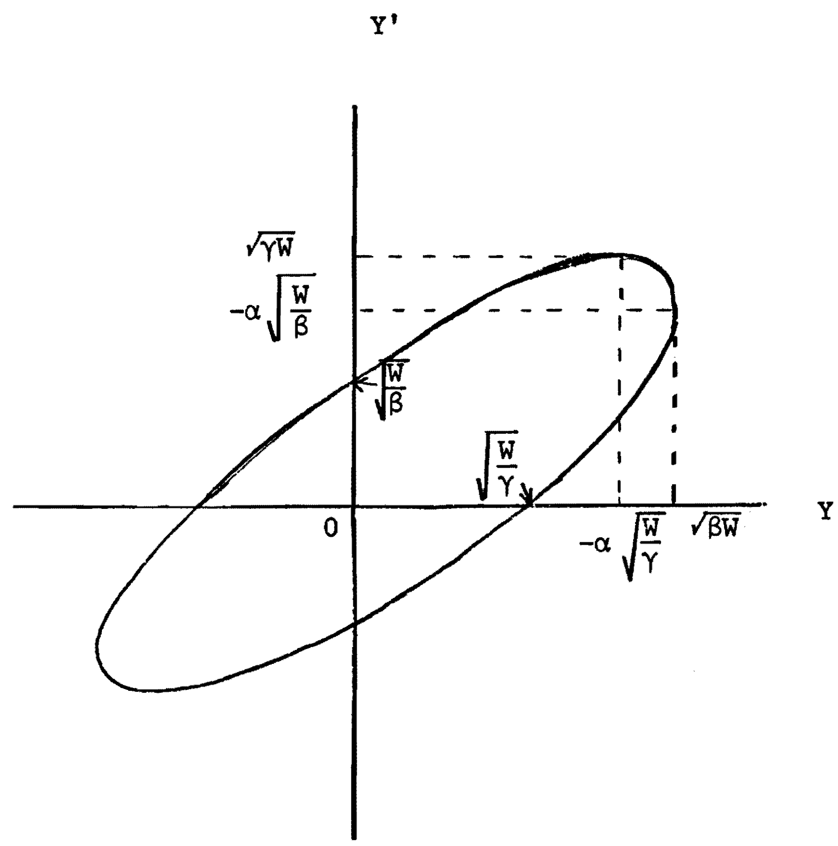


Fig. 9. Phase space ellipse

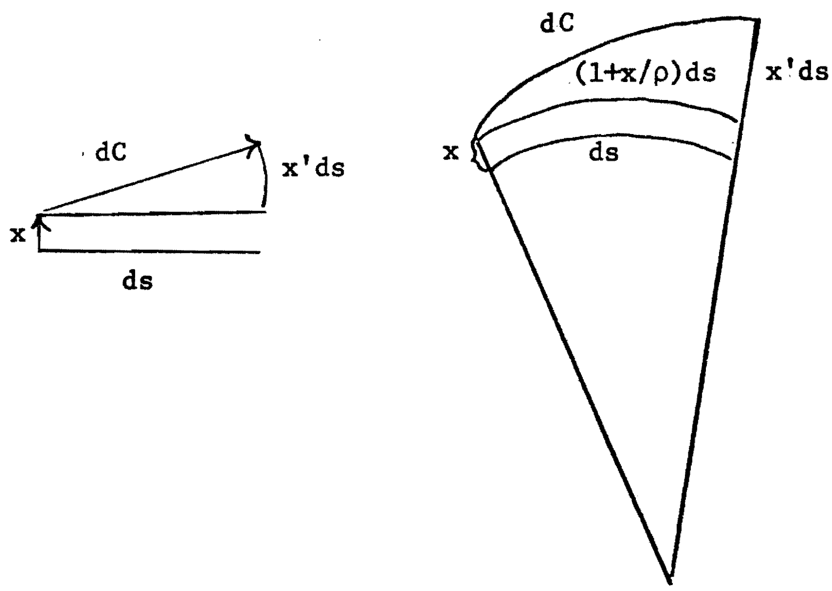
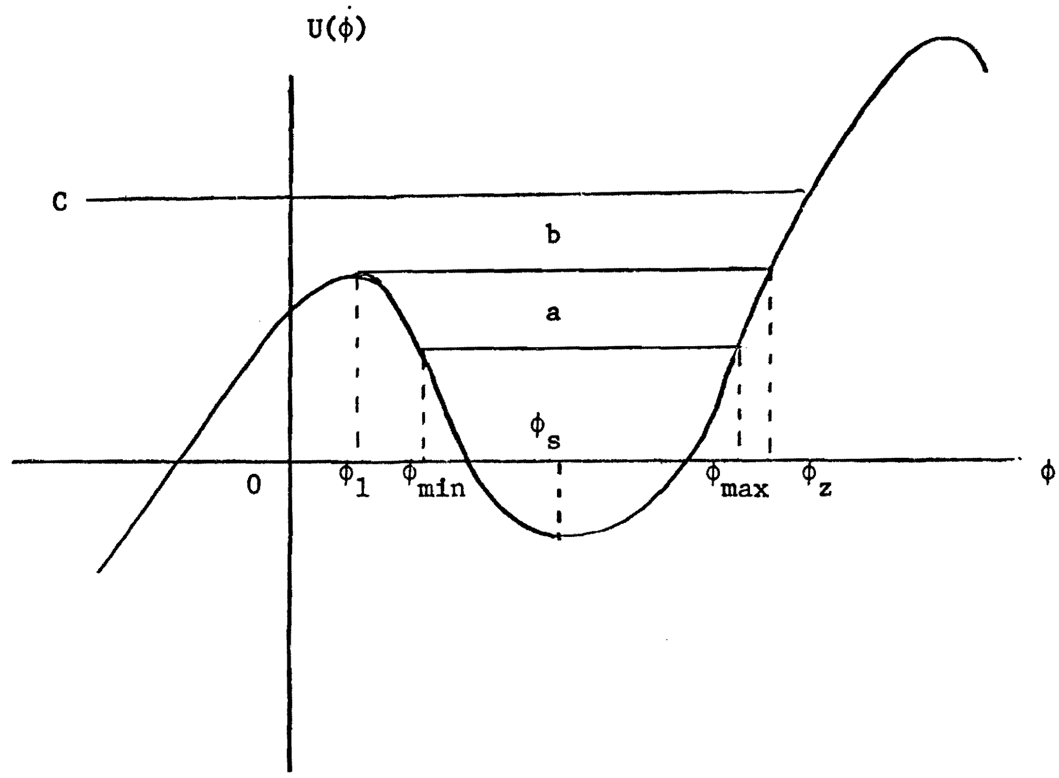


Fig. 10. Path length difference.

a) Potential



b) Phase space trajectory

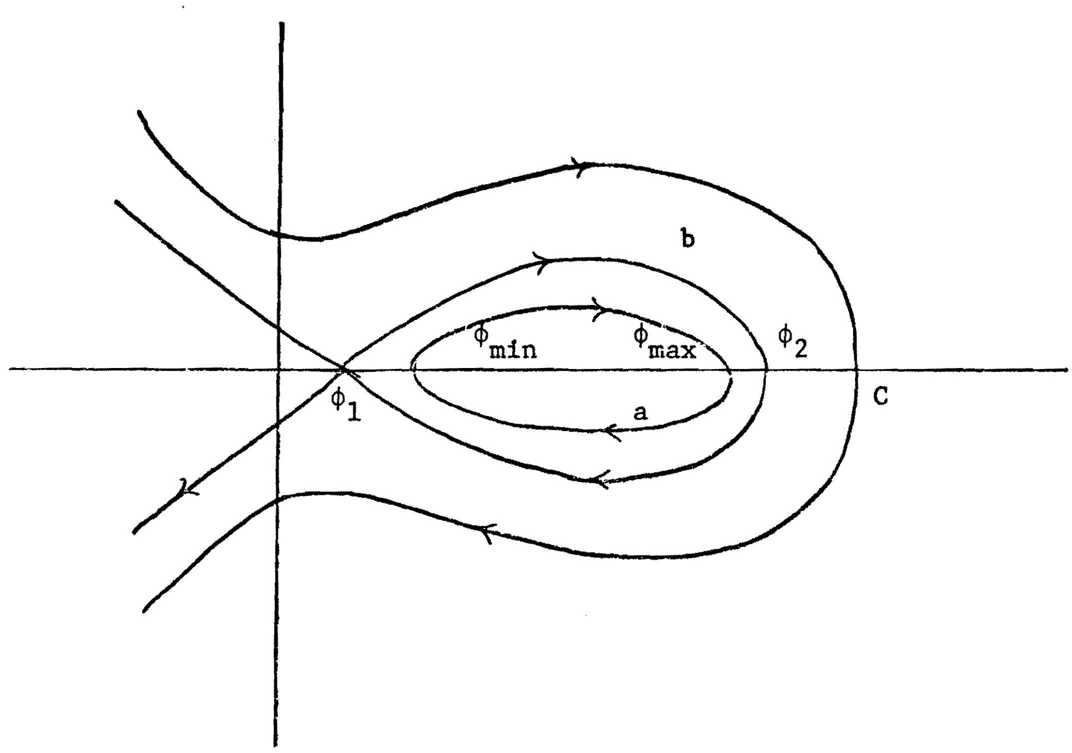


Fig. 11. Potential and phase space trajectory