III. PARTIAL WAVE DECOMPOSITION AT HIGH ENERGIES.

We saw in Section II that by using an optimized expansion it might be possible to decrease the number of parameters that need to be determined at a given energy to characterize the partial wave structure of a particle reaction. Even so, however, the number of parameters will increase as we go to higher and higher energies, and soon the number of parameters, whether optimized or not, becomes excessive in terms of the conventional methods of determination.

At the same time, the need for a partial wave decomposition at higher energies is no less than at lower energies. One of the central questions in particle physics is whether the resonances that dominate strong interactions up to 2-3 GeV, continue to higher energies or not. One of the very few practical ways of answering this question is through partial wave analyses. In the past, scarcity of data and the lack of analytical techniques precluded answering the question, but this is likely to change during the next decade.

The method I will now discuss, which permits, at least in principle, the partial wave decomposition (or, for that matter, the parametrization in terms of optimized coefficients) of particle reactions at arbitrary high energies is called phase-band analysis[1,2]. I said "in principle", since the method has so far been applied only in one real case, and that was at not too high energies. This one application yielded very favorable results, but only more extensive uses at higher energies will definitely establish whether one can in fact make a partial wave decomposition at arbitrary energies. It is therefore hoped that as the set of data become more complete and more extensive at higher energies, new analyses will be made.

The phase band analysis is based on a statistical view of angular momentum parameters at high energies. At those energies there will be many such parameters, and the assumption is therefore made that the very complex

network of dynamic correlations among these parameters can be approximated very well by suggesting that in fact each angular momentum parameter can be determined by itself from the data, independently of the other parameters.

It will now be undoubtedly objected that this assumption is obviously wrong, since we just showed in Section II, when we discussed optimized expansions, that each optimized parameter in fact contributes to many partial wave parameters and hence analyticity does establish dynamical correlations between partial wave amplitudes.

To clear up this apparent paradox, let me discuss the similar situation of making a Fourier expansion of a given function. The various Fourier coefficients in this expansion are highly correlated in a "dynamical way". In fact, once the function is given (i.e. the "dynamics" is fixed), all coefficients are completely determined and hence the "correlation" is absolute. At the same time, it is also true that each Fourier coefficient can be completely determined absolutely independently of the other coefficients, by simply using the well-known formula based on the completeness and orthogonality of Fourier functions.

The analogy can be carried a bit further. If the expansion of the above function were to be carried out in terms of a finite series of Fourier functions, the above procedure would not work as well, since the set of functions would not be complete. The longer finite series we use, however, the better approximation we get using the formula which determines the coefficients independently of the other coefficients.

Similarly in a partial wave expansion, if we have a large number of angular momentum states, we can in a good approximation determine the partial wave parameters one by one. As we will see, a good approximation is all one will need, since an iteration procedure will then assure a quick convergence toward the exact values.

The partial wave parameters being related to experimental observables

through bilinear relations, the actual determination is of course not in terms of a single inversion formula as it is for a Fourier expansion of a function, but that difference is not essential for understanding the basic idea of the phase band method.

The above observation forms the main basis of the phase band method. There is also a second observation that is helpful, though not indispensible, which is perhaps more debatable than the first one. It pertains to the average behavior of a partial wave amplitude as we increase the energy. The claim is that first the partial wave amplitude goes through the regular threshold behavior, then enters an energy region in which it is still primarily elastic and exhibits one or several resonances, and finally reaches the high energy range in which it is predominantly absorptive and non-resonant.

Based on these two considerations, the phase band method consists of two parts: the single-state phase band decomposition, and the iterative process utilizing it.

Let me first discuss the single stages. Treating as an example the spinless case, the scattering amplitude is written as[1]

$$f(\theta, E) = \sum_{\ell=0}^{\ell_0} (2\ell + 1)(2\,ik)^{-1}[\eta(\ell)e^{2i\delta(\ell)} - 1\,]P_\ell(\cos\theta)$$

$$+ \sum_{\ell=\ell_0+1}^{\ell_{max}} (2\ell + 1)(2\,ik)^{-1}(\eta_\ell\, e^{2i\delta_\ell} - 1)P_\ell(\cos\theta)$$

Here $P_\ell(\cos\theta)$ is the usual Legendre function, k is the center-of-mass momentum Q the scattering angle, E the energy of the incident particle, δ_ℓ and η_ℓ the usual phase shifts and absorption parameters, and $\delta(\ell)$ and $\eta(\ell)$ two functions of the angular momentum ℓ.

The second sum represents the angular momentum contributions in the usual way from $\ell_0 + 1$ to ℓ_{max}. The ℓ_{max} is selected in the usual way, for

example by a semi-classical estimate of the largest important angular momentum. I will discuss the choice of ℓ_0 later. The first sum represents the contribution of the lower $\ell_0 + 1$ angular momentum states, but not in terms of individual partial wave parameters, but in terms of a collective description of the partial waves in that range of ℓ, using the two functions $\delta(\ell)$ and $\eta(\ell)$. These two functions, in turn, are constructed with the help of a few free parameters, to be determined by the experiments in the same way as the δ_ℓ's and η_ℓ's are determined.

Thus the total scattering amplitude is described in terms of two bands of partial waves: the lower, collective band and the higher, individual band. When such an expression is compared with experiment to obtain the optimal values of the partial wave parameters, detailed information is obtained about the parameters in the individual band, but the individual values of the various partial wave parameters in the collective band, as predicted by the functions $\delta(\ell)$ and $\eta(\ell)$ (with the optimal values of their parameters) will most likely be incorrect. The total contribution of the collective band is, however, expected to be approximately correct, and since it is assumed that the partial wave parameters in the individual band can be determined largely independently of the knowledge of the other parameters, even an approximate knowledge of the contribution of the collective band will suffice.

The advantage of the above scheme, of course, is that one can thus determine the upper partial waves using many fewer parameters than if one had carried out a full-fledged partial wave analysis. For example, consider the case with $\ell_{max} = 20$, and $\ell_0 = 1$[illegible] the conventional analysis would require 42 parameters. Using 3 parameters each to describe $\delta(\ell)$ and $\eta(\ell)$, the phase band method requires 16 parameters, a dramatic reduction.

The reason why the higher partial waves were singled out for being placed in the individual band is that our second initial assumption favors this course for several reasons. First, the higher partial waves should be

the most interesting ones if we want to study resoances. Second, the lower partial waves (in the collective band) should be further damped by their being very absorptive. Finally, the $2\ell + 1$ factor also favors the higher phases.

The choice of ℓ_0 is governed by the following considerations. The broader the individual band is, the more parameters we have to deal with, but at the same time the partial wave parameters can be expected to be determined to a better approximation that way. As a check, the values of the parameters in the individual band can be determined using several neighboring values of ℓ_0 and ascertain whether the values of the parameters depend on ℓ_0 or not.

This concludes the conceptual discussion of the single stage. Let me now turn to the discussion of the iterative process based on it. It is schematically shown in Figure 3.1. The first step there represent the single stage discussed above. In the second step the individual parameters determined in the first step are held constant, and the remaining lower partial waves are redivided into a new individual band and a collective band, on which the single stage is then repeated. The third step is a self-consistency check between steps 1-2 to see if the individual parameters change if they are all released at the same time. Such changes are expected to be minute if the procedure works at all, and therefore this step should take very little computer time. Step 4 is then similar to step 3 except for pushing the bottom of the individual band even lower. Step 5 is again a consistency check, etc. Finally, the last step is also a consistency check, releasing all phases to allow small readjustments in the values. With this step, a complete partial wave decomposition bas been accomplished.

What is the advantage of this iteraction process over performing the analysis in the traditional way?

First, the iterated phase band is likely to represent a considerable

reduction is computing time. To make a quantitative estimate of the extent of this reduction is difficult because the time used in different search techniques varies differently with the number of parameters, and because the reduction of time originates in several different factors. Thus, some examples will suffice. If, for example, the time for one search goes as the square of the number of parameters N, and one uses k bands, with the collective band being characterized, on the average, by c parameters, the time for the conventional search is N^2 while for the iterated phase band it is $\frac{N^2}{k} + c(2N + ck)$. For N = 40, k = 10, c = 4, this represents a three-fold reduction in the computing time. In addition, a large saving is achieved in random searching. If for each of the N parameters one uses m values in the random search, the conventional analysis would require N^m searches, while the phase band only $k(\frac{N}{k})^m$ searches. For k = 10 and m = 3 (which is a very modest number for m), the reduction in computing time for the random searches is a factor of 100.

Second, the phase band method is likely to reduce the uncertainty of our knowledge of the partial wave parameters, since at each stage of the process we deal with many fewer parameters, and given a data pool of a given size, one can determine from it fewer parameters with a greater accuracy.

Third, the phase band method might reduce the ambiguities of partial wave solutions, since those solutions of a conventional search which could not be obtained by a stable successive iteration of the single stage of the phase band method would not appear. It is possible that such spurious solutions would appear at certain stages of the procedure, but they would soon be eliminated because they would not show an approximate self-consistency in steps 3, 5 etc.

Finally, the phase band method allows us to select certain "interesting" parameters and determine only those without an overall partial wave decom-

position which would yield also much "uninteresting" information. For example, if we are interested only in the top quarter of the partial wave parameters, for $\ell_{max} = 12$ and $\delta(\ell)$ and $\eta(\ell)$ each containing three parameters, we can reduce the computing time to one quarter of what it would be in the conventional analysis (if the previously discussed dependence on N is quadratic).

In summary, the phase band method allows one to take a huge problem of perhaps completely unmanageable size, and divide it into a sequence of small problems, the sizes of which can be almost arbitrarily small. Thus, in some practical cases, the use of the phase band method might mean the difference between what can and what cannot be done.

But are all these sanguine expectations actually fulfilled in the de facto applications of the phase band method? One cannot answer this question fully at this time, since there have been only two applications of the phase band method so far: One to a hypothetical set of data artificially generated,[1] and one to pion-nucleon scattering[2] at 2.5 and 2.75 GeV/c.

The first of these[1] used $\ell_{max} = 20$ and $\ell_0 = 15$ and tested only the single stage of the phase band method. Partial wave parameters were artificially chosen, from them amplitudes were calculated, artificial errors were assigned to them, and then these amplitudes were subjected to a phase band analysis to see if one can reobtain the partial wave parameters one started with. The following conclusions were obtained:

a) It is possible to reobtain the initial parameters to a high degree of accuracy.

b) The procedure is stable with respect to variations in the value of ℓ_0.

c) The functional form of $\delta(\ell)$ and $\eta(\ell)$ does not matter very much.

d) The presence or absence of small angle data points does not seem to have a serious effect on the accuracy of the fit.

e) The success of the scheme does depend to a large extent on placing the highest partial waves into the individual phase band.

f) As expected, the initial values of the individual partial wave parameters in the collective band do not agree well with the individual values predicted by the final functions $\delta(\ell)$ and $\eta(\ell)$ of the collective band.

The second application[2] of the phase band method was to real data, but unfortunately at energies which were too low for the phase-band method to show its full power. Thus a number of questions asked above could not be answered. We do not know whether ambiguities are reduced, whether uncertainties of the partial wave parameters are reduced, or exactly what the saving is in computer time.

We do know, however, that the application of the phase band method to these data resulted in a statistically very good set of partial wave parameters, that this solution was reached on a small computer in a very short time, and that the solution was later successfully checked against an energy dependent conventional phase shift analysis reaching up to those energies, and performed after the phase band analysis was completed. The agreement was very good indeed. This second application also performed the iterative procedure based on the single stage, and found it stable and easy to handle.

Finally, it is clear that the phase band method is by no means restricted to partial wave parameters, but can be applied to any decomposition. In particular, it could be used with the optimized parameters discussed in Section II. Indeed, since both the use of the optimized expansion and the use of the phase band method result in an economization of the parametrization and in an increase in the precision of the parameters, a combination of the two methods would indeed be a powerful phenomenological tool, which could easily result in revealing the partial wave structure of particle reactions up to tens of BeV's.

What is needed to realize such a promise of progress? We need experimental data. They should consist of as many types of experiments as possible, since, as we saw in Section I, no amount of data of a given kind (e.g. differential cross section) and of no degree of precision can make up for the absence of other types of experimental observables, which represent different bilinear combinations of the amplitudes. It is only after the various amplitudes are disentangled by the different types of experiments that expansion methods like the optimized series or the phase band scheme can take over. Even just a few, not very accurate data points at a few angles of the more difficult types of experiments can be of great value. Furthermore, a relatively large angular range should be covered by these experiments, other than just the very small and very large angles which at high energies, from our point of view, are highly uninteresting. One would expect that eventually the second generation meson factories (that is, high current accelerators at 20-30 GeV) would be helpful in collecting such data, but progress even with present day machines is quite possible.

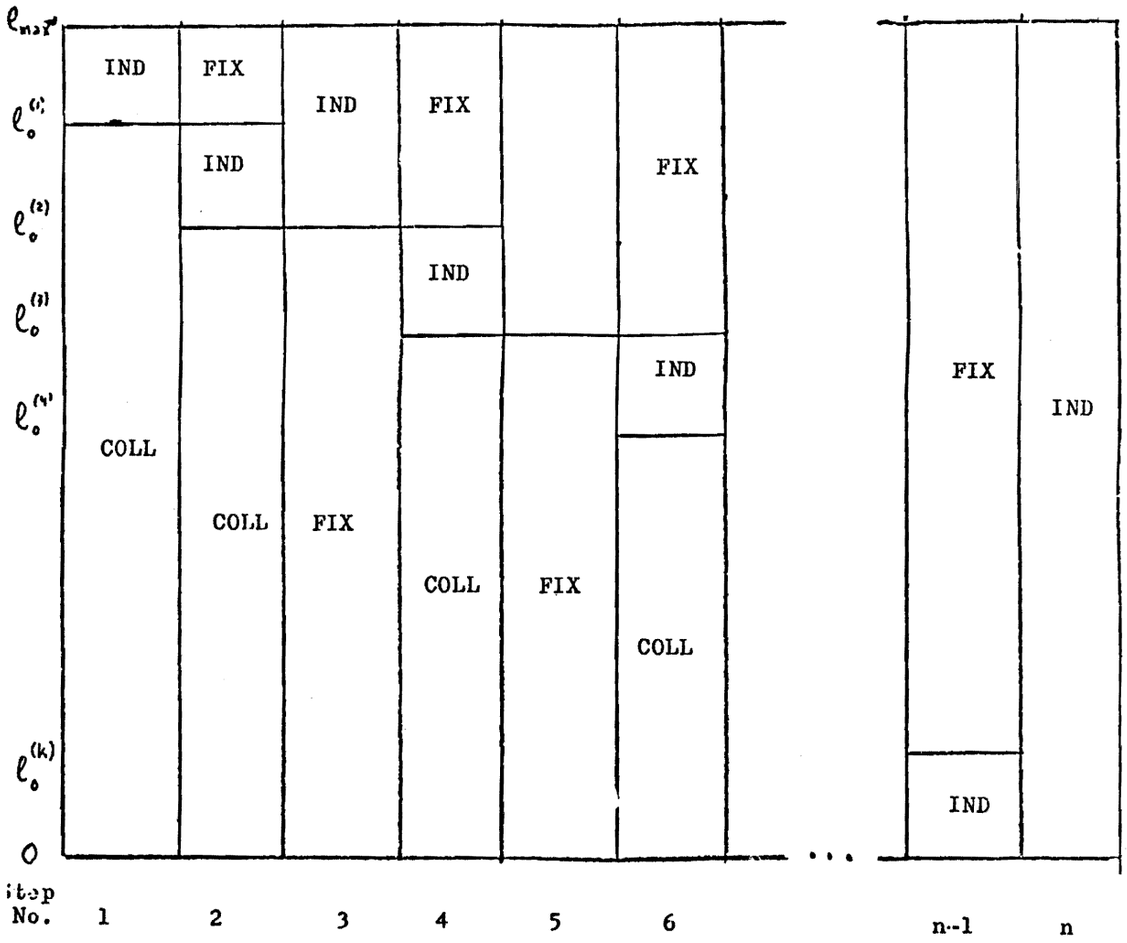
References to Section III.

1 M. J. Moravcsik, Phase Band Analysis - A Tool for Particle Reactions at High Energies Phys. Rev. 177, 2587 (1969).

2 D. Bridges, M. Moravcsik, and A. Yokosawa, First Phase Band Description: π^+ - P Scattering at 2.50 and 2.75 GeV/c
Phys. Rev. Letters 25, 770 (1970) with erratum and addenda in Phys. Rev. Letters 26, 155 (1971).

Figure 3.1

Schematic diagram of the iterated phase-band method.



Legend: IND= individual COLL= collective FIX= fixed