II. PARTIAL WAVE EXPANSIONS AND CONFORMAL MAPPING

Partial wave expansions in the description of particle reactions have constituted the single most powerful phenomenological tool in micro-scopic physics. There are basically two main reasons for this.

First, angular momentum, which is the parameter of these expansions, is a physical quantity, and is conserved, so that the terms in the expansion have physical meaning and the truncation of the expansion can be made using physical arguments in addition to mathematical ones. Furthermore, the omitted terms can be sometimes estimated, using physical arguments.

Second, empirical evidence has revealed to us that angular momentum is not merely one physical quantity of the many that characterize the structure of particle reactions, but is perhaps the most crucial one, in as much as resonances, which clearly dominate much of particle physics, have the one single common feature of occurring in a given angular momentum state. Thus, partial wave expansions are ideally tailored to describe particle reactions in a highly relevant fashion.

There are two difficulties that have been encountered over and over again in partial wave expansions. One is that once one rises above a certain low energy region (the extent of which is different from reaction to reaction), the number of partial waves one has to take into account is large, and hence the determination of the expansion parameters becomes laborious. This problem will be discussed in Chapter III.

The other problem is that, whether at high or low energies, and therefore whether there are many or few significant terms in the expansion, the expansion sometimes does not converge as fast as one would desire it. Two methods have been used to remedy this deficiency.

The first of these methods attempts to avoid truncation altogether by adding to a finite number of low angular momentum states an approximate ex-

pression for the remaining infinite number of higher angular momentum states. This expression makes use of the short range nature of strong interaction forces and of the relationship between range, the mass of exchanged particles, and angular momentum states: The high angular momentum states are approximated by the closed expression given by the exchange of the highest possible particle compatible with constraints on intrinsic quantum numbers. This term is indeed the dominant contribution to the high angular momentum states if the next lightest particle that can be exchanged is much heavier than the lightest. Thus this method of analysis, called the modified partial wave expansion, works particularly well when the lightest particle that can be exchanged is a boson, in which case generally the next lightest exchange is two of the same bosons. This ratio of 2:1 in the masses assures overwhelming dominance of the one-boson exchange and hence the success of the modified expansion. The most extensive use of this method has been in nucleon-nucleon scattering, and has been responsible for much progress in that field during the last decade. This method is by now well known and will not be discussed here further.

The second method to circumvent the relatively slow convergence of the partial wave expansion is to use another variable in which the convergence is faster. This is the topic of this chapter.

Although one can think of a number of ways to approach the problem of finding better variables to expand in, the actual formalism has been that of dispersion relations. In particular, the thrust has been in the direction of using the analyticity postulates for reaction amplitudes and therefore making a conformal mapping to a different complex plane, the variable of which would provide a faster converging parameter for an expansion.

The first published effort in this direction appears to be a around 1961. For some of the references of these early papers, see Reference 2. The systematic approach to this problem was, however, given much later by

Cutkosky and Deo[1], and Ciulli[4]. The topic is by no means closed, and many important and intriguing possibilities remain on the horizon.

Intuitively, it is evident what one wants to do: The transformation should move the singularities in the complex plane as far from the physical region as possible. At the same time, the transformation should result in a variable which also has a physical meaning, so that the terms in the new expansion could be readily interpreted.

The general solution of this problem has, as usual, been given by mathematicians a long time ago. The particular reference used by Cutkosky is a review book by Walsh, published in 1956. The solution in fact is elegant and easily comprehensible to a physicist in terms of a different (and unrelated) physical problem. The prescription goes as follows:

a) Transform the physical region in the momentum transfer (or reaction angle) plane (henceforth called the "old plane") unto any shape in an "intermediate plane".

b) Transform the cuts in the intermediate plane into the equipotential surface in the new plane which would result from placing a uniformly distributed charge on the physical region. The physical region in the new plane is identical to that of the intermediate plane.

The expansion in the variable in such a new plane will be optimally converging, that is, will converge as fast or faster than any other variable one could define by any other conformal mapping.

It should be noted that the above prescription defines an infinite class of optimal transformations, since each of the infinite number of transformations in a) has a corresponding transformation in b). In actuality, two of these infinitely many transformations have been utilized in practice: one by Cutkosky[1] and one by Ciulli[4].

Let me discuss now briefly Cutkosky's particular transformation. Let us denote by $\underline{x}$ the cosine of the scattering angle, so the "old" plane will

be the $\underline{x}$ plane. In this plane the cuts run from $-\infty$ to some $-x_-$ and from x_+ to $+\infty$. We now use the transformation

$$w = \frac{x - x_0}{1 - xx_0}$$

to go into the intermediate plane which here is the $\underline{w}$ plane. The cuts are symmetrized by this transformation, now going from $-\infty$ to $-W$, and from W to ∞, where

$$W = \frac{x_+X_- + x_-X_+}{X_+ + X_-}$$

with

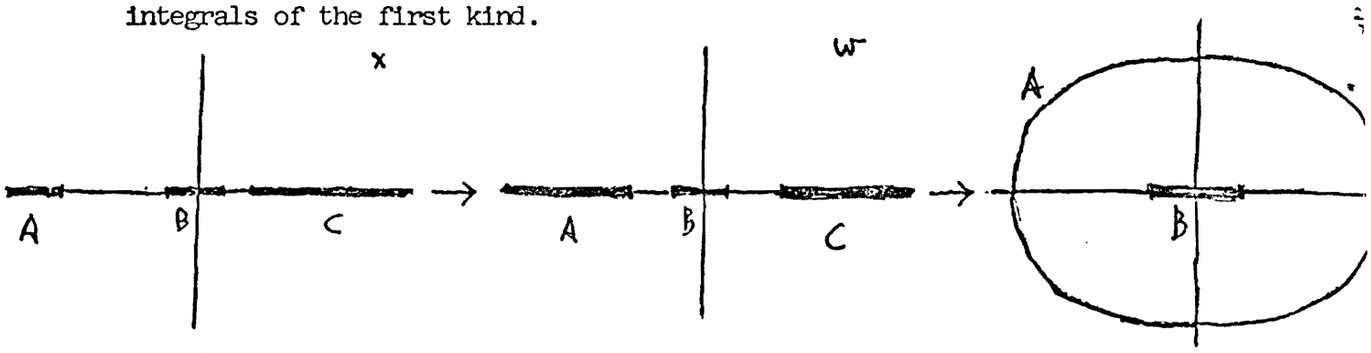
$$X_{\pm} = \sqrt{x_{\pm}^2 - 1}$$

As far as the physical region is concerned, it runs from $+1$ to -1 in the $\underline{x}$ plane as well as in the $\underline{w}$ plane.

Now we go from the intermediate $\underline{w}$ plane to the new $\underline{z}$ plane by the transformation

$$z = \sin \Phi(w, k_0), \quad k_0 = \frac{1}{W}, \quad \Phi(w, k) = \frac{\pi F(\sin^{-1} w, k)}{2K(k)}$$

with $F(\psi, k)$ and $K(k) \equiv F(\frac{1}{2}\pi, k)$ being the incomplete and complete elliptic integrals of the first kind.

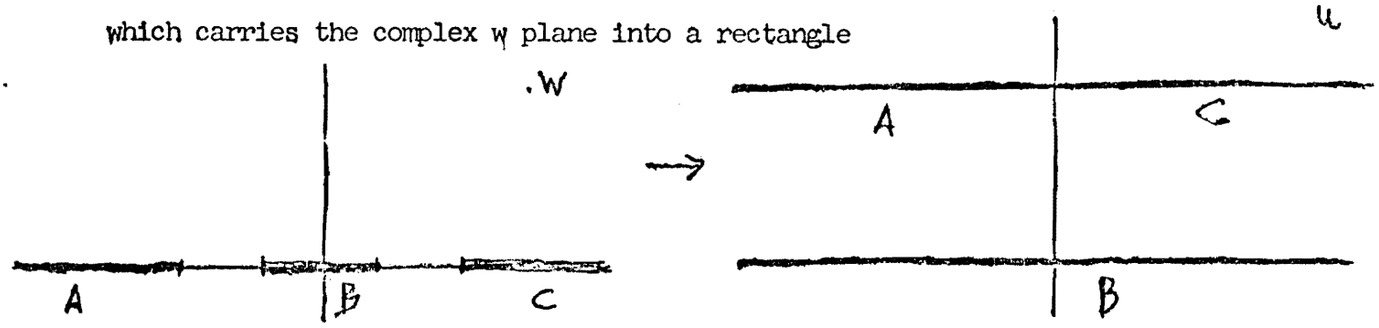


The transformation we want maps the physical region into what it was in the w plane, and the cuts in the w plane into an ellipse in the z plane, since ellipses are the equipotential surfaces for a uniformly charged finite line from $z = -1$ to $+1$. It can be ascertained that the above transformation indeed satisfies these requirements.

Now let us turn to Ciulli's transformation. He proceeds is three steps. The first step is identical with that of Cutkosky and simply symmetrizes the cuts. From this w plane, he then goes to a u plane by the transformation

$$u(w) = \int_0^w \frac{dt}{\sqrt{(1 - t^2)(1 - t^2/w^2)}}$$

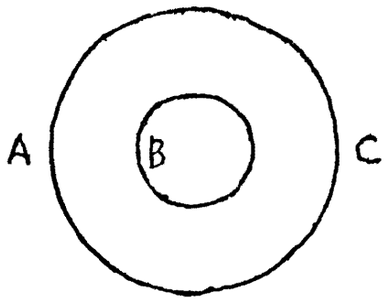
which carries the complex w plane into a rectangle



Finally, this rectangle is made into a ring by a transformation

$$v(u) = \tfrac{1}{4} \exp\left[- \frac{i\pi u}{2\mathcal{U}} \right]$$

where $\mathcal{U}$ is u(1). Thus now we have in the v plane a figure like the following



and the expansion is now between the two circles.

Cutkosky's and Ciulli's transformations are of course equivalent in the sense that one can give a transformation that can carry one into the other. This in fact is

$$z = \frac{1}{2}\left(v + \frac{1}{v}\right) \quad \text{or} \quad v = z \pm \sqrt{z^2 - 1}$$

One can now use any of these new variables to expand the scattering amplitudes in. In doing so, it is advantageous to incorporate explicitely into this expansion the contribution of poles (i.e. one-particle exchanges)

as discussed earlier in this lecture, because then the expansion itself will converge faster, just as it does in the case of the conventional partial wave expansion.

At this point a remark is in order. When the pole term is explicitely included in a partial wave expansion, the low partial wave contributions of the pole terms must be explicitely subtracted so as to avoid double counting if the physical meaning of phase parameters and the constraints of unitarity are to be preserved.

Such explicit subtraction is not necessary if the expansion is done in terms of any of the new variables, because the expansion can be said to refer only to the difference between the whole amplitude and the pole contribution (or pole contributions). This simplifies somewhat the mathematical procedure, but at the same time it also demonstrates the basic defect of present applications of the new expansion, namely that the terms in it have no physical meaning, and therefore physical constraints like unitarity cannot be applied to them directly.

The lack of direct physical meaning of the terms in the expansion is not very bothersome if one aims at the reconstructions of the M-matrix amplitudes of the reaction. One the other hand, if the aim is to give a partial wave description of the reaction, (which is of interest particularly it resonances can be expected), then the new expansion must be reconverted into a partial wave expansion. In the case of the two specific expansions so far explored, there are one-to-infinity type relations between the coefficients of the partial wave expansion and those of the new expansion. This means, in turn, that the reconversion can be made, in practice, only approximately in as much as only an unknown portion of each partial wave amplitude will be obtained, even though some portion of each of the infinitely many amplitudes can be calculated.

In practice, of course, only a finite number of them are actually calculated, and the resulting amplitudes are non-unitary. One can of course "unitarize" them just as one can unitarize Veneziano models and any of the other non-unitary schemes recently proposed in particle theory. The process of unitarization, however, is ambiguous. One can arbitrarily choose "minimal" unitarization by shifting the partial wave amplitude to that point of the unitarity circle which is the closest to the original amplitude. There is no physical reason for this choice. Furthermore, the correction thus made, even if relatively small, might make a considerable difference, since the difference between various expansions is small anyway. Finally, such corrections are even more unjustified if the reaction is inelastic (which is almost always the case when optimized expansions promise to be very advantageous in the first place), because in that case the true amplitude is somewhere inside the unitarity circle. All this points at the urgent necessity to find an optimized type of expansion which at the same time has a physical meaning and hence is subject to physical constraints and interpretation.

Let me turn now to the actual applications to optimize expansions. In doing so, I have to discuss briefly a modified goodness- of-fit criterion introduced by Cutkosky[3]. His aim is to take into account not only the statistical error due to the experimental uncertainties, but also the uncertainty caused by the neglect of the higher terms of the expansion, beyond the truncation. These neglected terms are of course not known, but in Cutkosky's scheme it is assumed that their coefficients decrease as R^{-n}, where R is roughly the distance to the nearest singularity from the physical region. Since for the optimized expansion R is larger than for the usual partial wave expansion, the taking into account of this uncertainty favors the new expansion. In most cases investigated by Cutkosky and coworkers, this uncertainty is in fact larger than the usual statistical error.

Qualitatively the new estimate of uncertainty makes a good point by calling attention to the effect of neglecting higher terms. Quantitatively, however, it is somewhat debatable, as it will be seen presently. The new estimate adds to the usual chi-square a term denoted by Φ. The calculation of Φ assumes that the coefficients of the expansion converge as R^{-n} where R is the size of the region of convergence, and then calculates from this assumption the effect of the terms neglected in the truncated expansion. The point is, however, that especially in a fast converging expansion, the actual truncation error will be contributed by the first very few of the neglected terms, and the behavior of the individual terms depends on the specific dynamics of the particular reaction, that is, not only on the location of the singularities in the complex plane but also the discontinuities at those singularities. These discontinuities are generally unknown. Thus, while the new estimate might be able to give the order of magnitude of the uncertainties due to truncation, its actual quantitative prediction is not likely to be reliable. Another way of stating this is to say that it is not possible to gain physical insight from purely mathematical manipulations which depend on no new physical ideas. For the optimized expansions, the only physical idea used is the qualitative statement that strong interaction forces have a short range, and this one fact, combined with some general statements about analyticity, is not enough to supply quantitative estimates for the limitations of such expansions.

On the other hand, somewhat related ideas can be used to give some guidance in the optimal placement of experimental points inside a domain of analyticity if one is to acquire the maximum of information about the value of that analytic function somewhere on the boundary[9]. This very practical problem is related to extrapolations which we will discuss below.

Let me now discuss briefly some of the actual applications of optimized expansions that have been carried out so far. There are four types of

applications.

1) Determination of coupling constants by extrapolation.

2) Determination of partial wave parameters.

3) Determination of reaction amplitudes.

4) Characterization of the energy dependence of partial wave amplitudes.

In determining coupling constants by extrapolation, one parametrizes the reaction amplitude (or cross section) in the physical region and then extrapolates the resulting function to the pole of the term which contains the coupling constant. The method has been in use since the late 1950's. The actual technique used may vary. One can, for example, extrapolate $(x - x_p)^2 \frac{d\sigma}{d\Omega}\Big|_{exp} = \sum_n a_n (x - x_p)^n$ to the pole x_p (here x is the reaction angle), in which case a_0 gives the coupling constant immediately. In connection with the optimized expansions, it was suggested[1] that one use instead $(x - x_p)[(\frac{d\sigma}{d\Omega})_{exp} - (\frac{d\sigma}{d\Omega})_{Born}] = \sum_n c_n p_n(x)$, where $p_n(x)$ are some suitably chosen polynomials. The same form can also be used with a transformed variable z. This expression will have no pole at x_p if the Born term is used with the right value of the coupling constant. The advantage of the second form is that one can use form factors and electromagnetic corrections in the Born term since it appears explicitely, and this may further ease the fitting.

Results[1] for n-p scattering did not indicate much improvement over the conventional way, presumably because there the location of the pole is very favorable anyway. For K^+-p scattering, however, the new method yielded a rough value while a determination through the conventional method is claimed to be impossible.

A different application for determination of coupling constants is to use the forward amplitude dispersion relations in a conformally mapped energy

variable.[8] Apparently here the conformal mapping helps a great deal and results in a quite well determined $\Lambda K^- p$ and $\Sigma K^- p$ coupling constant, while the conventional method is plagued by some ambiguities.

Techniques for extrapolation of scattering amplitudes have been described in a number of other papers also[13-16], but no numerical applications are given there.

Other mathematical techniques can also be constructed to get information about the value of a function where physical measurements are not carried out. For example, one can find[17] the values of the imaginary part of the proton electromagnetic form factor for positive (timelike) values of by inverting the unsubtracted dispersion relation for the form factor and then making a Taylor expansion about a point. To calculate the derivatives in the Taylor expansion one approximates the experimentally measured data by a polynomial. Application to the form factors of this method as well as of the optimized expansion method give fairly similar results.

Now let me describe the applications of the optimized expansion for partial wave analyses. So far the method has been applied to K-p scattering[6,10] and to p-p scattering[7].

For the latter, the following four results were found:

a) Phenomenologically determined phase shifts from the Livermore analyses Nos. VII and X were used as input up to and including F waves, and an optimized expansion was then performed to predict the G and H waves in the 200-400 MeV range where these high waves are known phenomenologically. It was found that using the phases of the energy dependent solution of VII the predicted high phases agreed with experiment but using either the energy dependent solution of X or the single energy analyses of either paper did not produce agreement. The failure for single energy solutions, however, could be remedied by using second-derivative matrices to alter the input phases. In the latter case, as good a solution could be obtained with 10

parameters as a conventional single-energy analysis with 14 parameters, but whether this is due to the extra information supplied by the second derivative matrices or to the superiority of the optimized expansion is not quite clear.

b) The new criteria of convergence was applied to the conventional and the optimized analyses, and the latter was found to have smaller uncertainties, as one would expect for a faster converging expansion if the new criterion is used. Particularly marked improvements are found for some of the higher phases. Using the new criterion and the optimized expansion, the pion-nucleon coupling constant is also determined from single-energy analyses while the conventional expansion with conventional goodness-of-fit criterion could not obtain the coupling constant with as good accuracy.

c) When the new criterion is applied to n-p data, no improvement is found at 210 MeV but the uncertainties decreased at 330 MeV.

d) In all these calculations form factors were used for the Born terms. Studies showed that using instead a unitarized Born term without form factor gave similar results. It is concluded from this that form factors serve to mock up unitarity for the non-unitary Born terms.

In summary, the results of the application of optimized expansions to the two-nucleon problem has so far been somewhat ambiguous. If the new criterion is accepted at face value, definite quantitative improvements in accuracy have been achieved in some cases, none in others. A full-fledged energy-dependent optimized analysis with both the old and new goodness-of-fit criteria and with and without form factors would be helpful to make a decisive evaluation of the optimized method.

For the K-p interaction the success of the optimized expansion has been more evident[6,10], probably because for that reaction the cuts in the ordinary variables are closer to the physical region, and hence more is to be gained from transforming them away. The advantages can be expected to be more pronounced at higher energies when the conventional cuts are even closer to the

physical region, and this is borne out by the actural results. Even by the conventional goodness-of-fit tests, at the highest energy the conventional partial wave expansion gives a chi-square which is 60% higher than the chi-square for the optimized expansion with the same number of parameters. Furthermore, apparently the ambiguity of solutions is also reduced.

Let me now turn to the third type of application of optimized expansions, namely the determination of amplitudes. As mentioned earlier, the optimized expansion should be particularly successful for this purpose, since it is used here strictly as a most economical phenomenological way of combing information from various angles, and one does not have to worry about physical meaning or re-expansion into partial waves. The actual application[12] has been to pion-nucleon interaction at 6 GeV/c in the angular range from 0° to 60°. The amplitudes here were first divided by a phenomenological form factor and then expanded in the optimized sense. The real and imaginary parts of the amplitudes can then be determined in the angular region in which data exist. In this application, the optimized expansion provided a way to utilize data at different angles even though the angular range where data exist is not large enough so that partial wave amplitudes could be determined. It would in fact be interesting to compare this optimized analysis with a direct determination of the amplitudes at fixed angles. The latter analysis has not been performed yet, however.

Finally, the basic idea of conformal mapping to transform away nearby singularities can also be used in the energy plane, for example to interpolate the energy dependence of phase shifts. An application of this idea has been made[11] to two of the phase shifts for np scattering below 400 MeV. In particular, a parametrization of the $^1S_0$ and $^1P_1$ phases is attempted by making an extended effective range type expansion in the new variable. The truncated expansion is constrained by the actual values of the scattering length and effective range in the conventional effective range expansion.

Two methods were tried, one approximating a modified D function in an N/D scheme, the other directly approximating the phases. The two methods gave similar results. Unitarity is taken into account by using these methods. The claim is made that with this new approach the energy dependences of these two phases can be parametrized more economically than with the conventional methods, but no numerical evidence is presented for such a comparison. Since the parametrization of the energy dependence of partial wave amplitudes has been a bothersome point for some time in phase shift analyses, a practical and economical solution of this problem through optimized expansions would be welcome. The aim is to carry out a full-fledged partial wave analysis of, say, N-N scattering using energy dependences which incorporate the constraints of the optimized expansion.

In summary, therefore, the optimized expansions used so far have produced some good results, though there is room for further improvements. One can expect them to yield a more economical phenomenological description of particle reactions, and in some cases this has been demonstrated. The main deficiencies at the moment are the violation of unitarity and the lack of direct physical meaning of the expansion coefficients. In some applications, such as the determination of M-matrix amplitudes, these deficiencies do not count.

In conjunction with the optimized expansions but independently of it, a new goodness-of-fit criterion was also developed. Since its quantitative aspects have not been fully tested, it would be advisable for a while to continue to use the standard chi-square criterion in conjunction with the new test variable. Private communications have indicated to me that additional tests of the new goodness-of-fit criterion are in progress.

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