# Kaluza-Klein Theory with Torsion

Kameshwar C. Wali

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- INTRODUCTORY REMARKS
- BASIC FORMALISM
- A SET OF MINIMAL CONSTRAINTS ON TORSION
- CONNECTION COEFFICIENTS
- RICCI TENSOR, RICCI SCALAR
- EINSTEIN EQUTIONS
- SOME RESULTS
- CONCLUDING REMARKS

- WORK DONE IN COLLABORATION WITH KARTHIK H. SHANKAR.

- WE CONSIDER A VARIANT OF 5D K-K THEORY, WITHIN THE FRAMEWORK OF EINSTEIN-CARTAN FORMALISM,BUT GENERALIZABLE TO ARBITRARY DIMENSIONS THAT INCLUDES TORSION.

- TORSION, A GEOMETRICAL PROPERTY OF SPACE-TIME, REPRESENTS SPIN DEGREES OF FREEDOM. IN ANALOGY TO EINSTEIN EQUATIONS WHERE THE ENERGY-MOMENTUM OF MATTER FIELDS IS COUPLED TO THE GEOMETRICAL RIEMANNIAN METRIC, WE CAN COUPLE ANGULAR MOMENTUM AND SPIN OF MATTER FIELDS TO THE GEMETRICAL TORSION.

- METRIC AND TORSION TWO INDEPENDENT GEOMETRICAL CHARACTERISTICS OF SPACE-TIME.

- THIS GEOMETRICAL PROPERTY OF SPACE-TIME, *TORSION*, ARISES IN THE DEFINITION OF COVARIANT DERIVATIVE OPERATOR OF VECTOR AND TENSOR FIELDS, FORMING AN ANTI-SYMMETRIC PART OF THE AFFINE CONNECTION COEFFICIENTS (ACCs).

- METRIC COMPATIBILITY LEADS TO THE DETERMINATION OF ACCs IN TERMS OF METRIC COMPONENTS AND THEIR DERIVATIVES, KNOWN AS CHRISTOFFEL SYMBOLS, WHICH ARE SYMMETRIC.

- WITH THE CHOICE OF A SET OF MINIMAL CONDITIONS ON TORSION,WE DETERMINE THE NON-VANISHING COMPONENTS OF TORSION IN TERMS OF METRIC COMPONENTS OF SPACE-TIME

- THE RESULTING ACCs ARE COMPLETELY DETERMINED FROM THE ASSUMED 5D METRIC, LEADING TO SOME REMARKABLE MODIFICATIONS OF THE CONVENTIONAL K-K THEORY

- BASIC FORMALISM; NOTATIONS:

- LET $i, j, k...$ and $A, B, ..$DENOTE COORDINATE AND INERTIAL FRAME INDICES, RESPECTIVELY.

- LET $\mathbf{e}_i = \partial_i$ AND $\theta^i = dx^i$ BE THE BASIS OF THE TANGENT AND DUAL SPACES AT EACH POINT OF SPACE-TIME. LET THE CORRESPONDING INERTIAL BASIS BE $\mathbf{e}_A = e^i_{\ A}\mathbf{e}_i$ AND $\theta^A = e^A_i \theta^i$.

- THE VIELBINES $e^A_i, e^i_{\ A}$ SATISFY THE ORTHONORMALITY CONDITIONS,

$$e^A_i e^j_{\ A} = \delta^j_i, \quad e^A_i e^i_{\ B} = \delta^A_B$$

THE METRIC,

$$\mathbf{g}_{ij} = e^A_i e^B_j \eta_{AB}, \quad \mathbf{g}^{ij} = e^i_{\ A} e^j_{\ B} \eta^{AB},$$

WHERE $\eta_{AB}$ IS THE MINKOWSKIAN METRIC IN THE INERTIAL COORDINATE SYSTEM.

- COVARIANT DERIVATIVE OPERATOR AND AFFINE CONNECTION COEFFICIENTS:

$$\nabla_{\mathbf{e}_i}\mathbf{e}_j = \tilde{\Gamma}^k_{ij}\mathbf{e}_k, \quad \nabla_{\mathbf{e}_A}\mathbf{e}_B = \omega^C_{AB}\mathbf{e}_C,$$

WHERE $\tilde{\Gamma}^i_{jk}$ AND $\omega^C_{AB}$ ARE THE AFFINE AND THE RICCI ROTATION COEFFICIENTS, RESPECTIVELY. FROM THE TRANSFORMATION LAWS BETWEEN THE COORDINATE AND INERTIAL FRAMES, WE HAVE,

$$\omega^A_{BC} = e^i_B(\nabla_{\mathbf{e}_i} e^j_C)e^A_j$$

THE ANTI-SYMMETRIC PART OF THE AFFINE CONNECTION IS THE **TORSION** TENSOR,

$$T^i_{jk} = \tilde{\Gamma}^i_{jk} - \tilde{\Gamma}^i_{kj}$$

AGAIN, FROM THE TRANSFORMATION LAW BETWEEN THE COORDINATE AND THE INERTIAL FRAMES, WE HAVE THE RELATION,

$$T^i_{jk} = e^B_j e^C_k e^i_A T^A_{BC}$$

- FURTHERMORE, WITH THE STANDARD ASSUMPTION OF METRIC COMPATIBILITY, NAMELY, $\nabla_{\mathbf{e}_i} g_{jk} = 0$, WE OBTAIN,

$$\tilde{\Gamma}^i_{jk} = \hat{\Gamma}^i_{jk} + K^i_{jk}$$

WHERE

$$\hat{\Gamma}^i_{jk} = (1/2)\mathbf{g}^{im}[\partial_j \mathbf{g}_{km} + \partial_k \mathbf{g}_{jm} - \partial_m \mathbf{g}_{jk}],$$

ARE THE USUAL CHRISTOFFEL COEFFICIENTS (SYMBOLS) THAT ARE SYMMETRIC, AND

$$K^i_{jk} = (1/2)[T^i_{.jk} + T^{.i}_{j.k} + T^{.i}_{k.j}],$$

IS KNOWN AS THE CONTORSION TENSOR.

- 5D SPACE-TIME METRIC:

CONSIDER A FOLIATION OF THE 5D SPACE-TIME IN TERMS OF 4D HYPERSURFACES WITH $x^\mu, (\mu, \nu, ..)$ COORDINATE INDICES ON THE HYPERSURFACES, AND $x^5$ THE 5D COORDINATE.

$$e^{\mu}_{.a} e^{.b}_{\mu} = \delta^a_b, \quad e^{\mu}_{.a} e^{a}_{.\nu} = \delta^{\mu}_{\nu}$$

$$g_{\mu\nu} = e^{.a}_{\mu} e^{.b}_{\nu} \eta_{ab}, \quad g^{\mu\nu} = e^{\mu}_{.a} e^{\nu}_{.b} \eta^{ab},$$

WHERE $e^{\mu}_{.a}$, $e^{a}_{.\mu}$ ARE THE VIERBINES WITH $(a, b)$ AS THE TETRAD INDICES ON THE HYPERSURFACE.

- EXTENSION TO 5D

DEFINE THE 5D VIELBEINES TO BE

$$e^{i}_{A} = (e^{\mu}_{.a}, e^{\mu}_{.\dot{5}}, e^{5}_{.a}, e^{5}_{.\dot{5}}), \quad e^{.A}_{i} = (e^{.a}_{\mu}, e^{.a}_{5}, e^{.\dot{5}}_{\mu}, e^{.\dot{5}}_{5})$$

- EXTENDING THE ORTHONORMALITY RELATIONS, WE ARE LED TO THE UNIQUE CHOICE OF VIELBEINES,

$$e^{\mu}_{.\dot{5}} = 0, \quad e^{5}_{.a} = -e^{\mu}_{.a} A_{\mu} \quad e^{5}_{\dot{5}} = \Phi^{-1}$$

$$e^{\cdot a}_{5} = 0, \quad e^{\cdot\dot{5}} = A_{\mu}\Phi, \quad e^{\cdot\dot{5}}_{5} = \Phi$$

- THE METRIC IN THE 5D SPACE-TIME

$$\mathbf{g}_{ij} = e^{\cdot A}_{i} e^{\cdot B}_{j} \eta_{AB}, \quad \mathbf{g}^{ij} = e^{i}_{.A} e^{j}_{.B} \eta^{AB}$$

$$\mathbf{g}_{\mu\nu} = g_{\mu\nu} + A_{\mu}A_{\nu}\Phi^{2}, \quad \mathbf{g}_{\mu 5} = A_{\mu}\Phi^{2}, \quad \mathbf{g}_{55} = \Phi^{2}$$

$$\mathbf{g}^{\mu\nu} = g^{\mu\nu}, \quad \mathbf{g}^{\mu 5} = -A^{\mu}, \quad \mathbf{g}^{55} = A^{\lambda}A_{\lambda} + \Phi^{-2}$$

- A SET OF CONDITIONS ON THE TORSION:
- CONDITION **1**

$$T^{a}_{BC} = 0$$

THIS IMPLIES WITH THE GIVEN METRIC, $T^{\mu}_{ij} = 0$;THE ONLY NON-VANISHING COMPONENTS OF TORSION ARE $T^{5}_{ij}$ .

- CONDITION **2**

$$\omega^{\dot{5}}_{.BC} = \omega^{A}_{.B\dot{5}} = 0,$$

- WHICH IMPLY,

$$\tilde{\Gamma}^{\mu}_{.i5} = 0, \quad \tilde{\Gamma}^{5}_{.i5} = -e^{.\dot{5}}_{5}\partial_i e^{5}_{.\dot{5}}$$

- THE ABOVE TWO CONDITIONS ARE SUFFICIENT TO DETERMINE ALL THE COMPONENTS OF TORSION AND HENCE CONTORSION $K^{i}_{jk}$:

$$T^{\mu}_{.ij} = T^{5}_{.55} = 0$$

$$T^{5}_{\mu\nu} = -T^{5}_{\nu\mu} = F_{\mu\nu} + S_{\mu\nu}$$

$$T^{5}_{.\mu 5} = -T^{5}_{.5\mu} = -\Phi^{-2} g_{\mu\sigma}\Gamma^{\sigma}_{.55},$$

WHERE

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad S_{\mu\nu} = A_\mu J_\nu - A_\nu J\mu; \quad J_\mu = \Phi^{-1}\partial\mu\Phi$$

- THE IMPOSED CONDITIONS NOT ONLY DETERMINE THE TORSION IN TERMS OF THE METRIC, BUT MOST IMPORTANTLY, REQUIRE

$$\partial_5 g_{\mu\nu} = 0$$

**ALL THE HYPERSURFACES IN THE FOLIATING FAMILY HAVE THE SAME 4D METRIC! ALTERNATELY, ONE MIGHT SAY, GRAVITY IS CONFINED TO 4D!!**

- NOW, FROM THE GENERAL RELATION,

$$\tilde{\Gamma}^i_{jk} = \hat{\Gamma}^i_{jk} + K^i_{jk},$$

WE FIND, DUE TO REMARKABLE CANCELATIONS,THE RESULTING AFFINE CONNECTION COEFFICIENTS,

$$\tilde{\Gamma}^\lambda_{55} = \tilde{\Gamma}^\lambda_{\nu 5} = \tilde{\Gamma}^\lambda_{5\nu} = 0,$$

$$\tilde{\Gamma}^5_{\mu\nu} = \nabla_\mu A_\nu + J_\mu A_\nu,$$

$$\tilde{\Gamma}^5_{5\mu} = \partial_5 A_\mu + J_5 A_\mu,$$

$$\tilde{\Gamma}^5_{\mu 5} = J_\mu, \quad \tilde{\Gamma}^5_{55} = J_5, \quad \tilde{\Gamma}^\lambda_{\mu\nu} = \gamma^\lambda_{\mu\nu},$$

WHERE $\gamma^\lambda_{\mu\nu}$ IS DERIVED FROM 4D SPACE-TIME WITH METRIC $g^\lambda_{\mu\nu}$; NOTE $\gamma^\lambda_{\mu\nu} \neq \hat{\Gamma}^\lambda_{\mu\nu}$

- SUBSTITUTING THE ABOVE AFFINE CONNECTION COEFFICIENTS IN THE RICCI TENSOR,

$$\widehat{R}_{ik} = \partial_k \tilde{\Gamma}^{j}_{.ji} - \partial_j \tilde{\Gamma}^{j}_{.ki} + \tilde{\Gamma}^{j}_{.km} \tilde{\Gamma}^{m}_{.ji} - \tilde{\Gamma}^{j}_{.jm} \tilde{\Gamma}^{m}_{.ki}$$

WE FIND,

$$\tilde{R}_{\mu\nu} = R_{\mu\nu}, \quad \tilde{R}_{\mu 5} = \tilde{R}_{5\mu} = \tilde{R}_{55} = 0,$$

WHERE $R_{\mu\nu}$ REPRESENTS THE RICCI TENSOR ON THE TORSION-FREE 4D SPACE-TIME. NOTE ALSO RICCI SCALAR IN 5D $\tilde{R} = R$, THE 4D TORSION-FREE RICCI SCALAR.

- REMARK:THE RESULTS ARE NOT SPECIFIC TO 4 AND 5 DIMENSIONS. IT IS STRAIGHT FORWARD TO EXTEND THEM TO ARBITRARY D AND D+1 DIMENSIONS.

- DISCUSSION OF SOME RESULTS:

- GEODESIC EQUATIONS

$$\ddot{x}^5 + \tilde{\Gamma}^5_{\cdot\,\mu\nu}\dot{x}^\mu\dot{x}^\nu + \left(\tilde{\Gamma}^5_{\cdot\,\mu5} + \tilde{\Gamma}^5_{\cdot\,5\mu}\right)\dot{x}^\mu\dot{x}^5 + \tilde{\Gamma}^5_{\cdot\,55}\left(\dot{x}^5\right)^2 = 0,$$

$$\ddot{x}^\lambda + \Gamma^\lambda_{\cdot\,\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0.$$

THE GEODESIC EQUATIONS ALONG HYPER-SURFACE ARE EXACTLY THE SAME AS THE GEODESIC EQUATIONS IN THE TORSION FREE 4D SPACE-TIME. THIS IS IN CONTRAST WITH THE CONVENTIONAL KALUZA-KLEIN THEORY.THE CONTRIBUTIONS OF VECTOR AND SCALAR FIELDS ARE COMPLETELY NULLIFIED BY THE CONTRIBUTIONS FROM TORSION TO THE 4D GEODESIC EQUATIONS.

## EINSTEIN'S EQUATIONS:

TO OBTAIN THE FIELD EQUATIONS, ONE NEEDS TO VARY THE ACTION WITH RESPECT TO INDEPENDENT DYNAMICAL VARIABLES. IN EINSTEIN-CARTAN THEORY, THE INDEPENDENT VARIABLES ARE THE METRIC AND THE TORSION. IMPOSING THE GEOMETRICAL CONSTRAINTS USING CARTAN STRUCTURE EQUATIONS, WE HAVE EXPRESSED TORSION IN TERMS OF 5D METRIC. HENCE,TAKING THE LAGRANGIAN DENSITY TO BE THE RICCI SCALAR, VARYING THE ACTION WITH RESPECT TO 5D METRIC, WE OBTAIN THE FOLLOWING MODIFIED EINSTEIN EQUATIONS:

$$R_{\mu}^{\nu} - (1/2)R\delta_{\mu}^{\nu} + H_{\mu}^{\nu} = \Sigma_{\mu}^{\nu},$$

$$-A^{\alpha}R_{\mu\alpha} - A^{\alpha}H_{\mu\alpha} = \Sigma_{\mu}^{5},$$

$$-1/2R = \Sigma_{5}^{5},$$

Where

$$\mathrm{H}_{\mu\nu} = \nabla_{(\mu}\mathrm{B}_{\nu)} - (\nabla\cdot\mathrm{B})g_{\mu\nu} + \mathrm{J}_{(\mu}\mathrm{B}_{\nu)} - (\mathrm{J}\cdot\mathrm{B})g_{\mu\nu}$$

and $\mathrm{B}_{\mu} = \mathrm{T}^{5}_{\cdot\mu 5}$ is a vector in the 4D torsion free space-time.

Conservation of 4D stress tensor requires

$$\nabla_{\nu}\mathrm{H}_{\mu}^{\ \nu} = 0.$$

## ROBERTSON-WALKER COSMOLOGY

To illustrate the implications of the above equations, consider spatially flat homogeneous and isotropic universe wit the metric,

$$\mathrm{ds}^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

With the usual assumptions regarding homogeneity and isotropy, and assuming (for simplicity) fields do not depend upon the fifth coordinate $x^5$, we are led to

$$(i)\,\dot{\Phi} = 0, \mathrm{J}_t = 0, \text{ or } \qquad (ii)\,\Phi = \dot{a}(t), \mathrm{J}_t = \ddot{a}/\dot{a}.$$

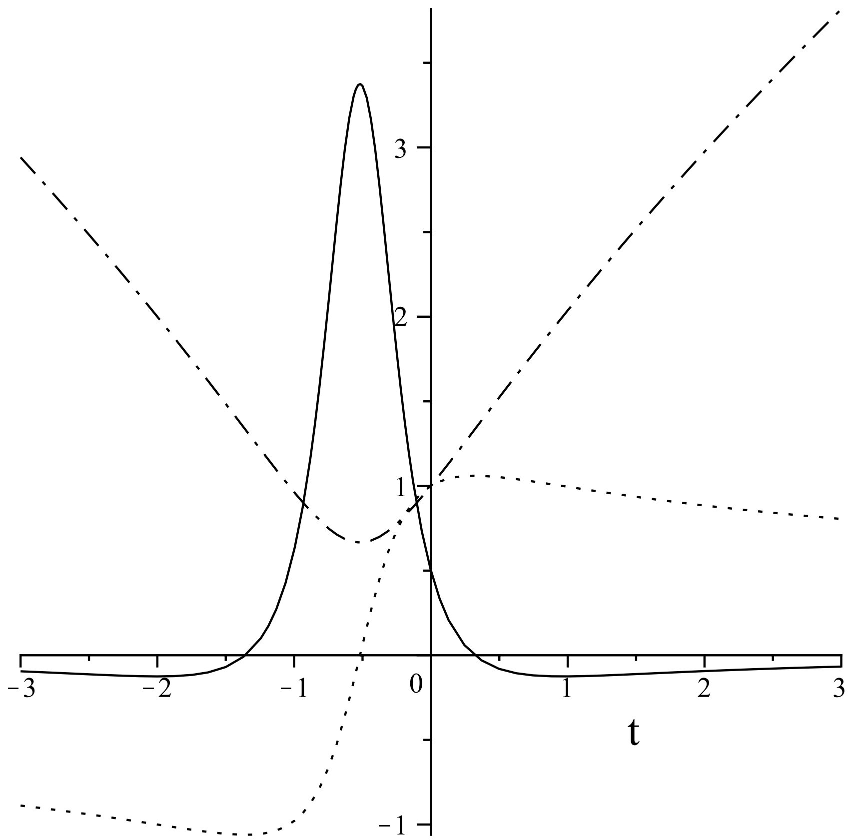
Case(i) yields, $H_{\mu\nu} = 0$ and the usual Friedman equation along with matter conservation,

$$(\dot{a}/a)^2 = \frac{8\pi}{3}\rho.$$

case(ii) yields non-vanishing $H_{\mu\nu}$ and a modified equation

$$(\dot{a}/a)^2 + \left(\ddot{a}/a\right) = \frac{8\pi}{3}\rho,$$

Matter conservation implies $\rho a^3 = \rho_o$ is a constant in a matter dominated universe.




The dashed-dot curve denotes $a(t)$, the dotted curve denotes $\dot{a}(t)$, and the solid curve denotes $\ddot{a}(t)$. The time axis is in units of $\mathrm{H}_o^{-1}$.

Initial conditions: $a = 1$, $\dot{a} = \mathrm{H}_o$ and $q_o = -0.5$,

where $\ddot{a} = -q_o\mathrm{H}_o^2$.

- CONCLUDING REMARKS:

- TORSION IS AN INTEGRAL, GEOMETRICAL PROPERTY OF SPACE-TIME,

- THE MODEL WE HAVE EXPLORED HAS A VERY GENERAL MATHEMATICAL RESULT PERTAINING TO THE USUAL METRIC AFFINE CONNECTION COEFFICIENTS AND TORSION,

- IN HIGHER DIMENSIONAL THEORIES, IT PROVIDES AN ALTERNATIVE WAY TO CONFINE GRAVITY, YET MODIFYING IT IN A SIGNIFICANT WAY TO BE RELEVANT TO ASTROPHYSICS AND COSMOLOGY,

- WE HAVE EXPLORED QUALITATIVELY SOME OF THE CONSEQUENCES. NEEDS MUCH FURTHER WORK.