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Abstract

The coupling impedance of the vacuum chamber can be obtained from the wake potential left by a bunch of finite extent. Oftentimes, however, the impedance so obtained violates the condition that its real part must be everywhere non-negative. The proper treatment is derived and discussed.

It is well known that the Fourier transform of the *integrated* longitudinal wake potential $W(z)$ of a point charge in a vacuum chamber is the longitudinal coupling impedance $Z(\omega)$ of the vacuum chamber. However, during numerical simulation or actual experimental measurement, a finite bunch is used instead. From the wake potential of the bunch, the coupling impedance of the vacuum chamber can also be derived. However, care must be exercised in the Fourier transform so that the real part of $Z(\omega)$ is always non-negative. This necessity originates from the fact that the bunch will not gain energy from its self-field while passing through the vacuum chamber.

If the bunch has a longitudinal distribution $\rho(z)$, its wake potential $\hat{W}(z)$ is related to the point-charge wake or Green-function wake $W(z)$ by

$$\hat{W}(z) = \int dz\, W(z - z')\rho(z') \ . \tag{1}$$

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The coupling impedance $Z(\omega)$ for that length of the vacuum chamber is given by

$$W(z) = \frac{1}{2\pi} \int d\omega Z(\omega) e^{-i\omega z/c} \ , \tag{2}$$

where c is the velocity of light. The bunch distribution can be Fourier analysed as

$$\rho(z) = \int d\omega \tilde{\rho}(\omega) e^{-i\omega z/c} \ . \tag{3}$$

Substituting Eqs. (2) and (3) in Eq. (1), we obtain easily

$$\hat{W}(z) = \int d\omega Z(\omega) \tilde{\rho}(\omega) e^{-i\omega z/c} \ . \tag{4}$$

Thus, if we denote the Fourier transform of $\hat{W}(z)$ by

$$\hat{Z}(\omega) = \frac{1}{2\pi} \int d\omega \hat{W}(z) e^{i\omega z/c} \ , \tag{5}$$

the *true* coupling impedance can be derived from

$$Z(\omega) = \frac{\hat{Z}(\omega)}{2\pi \tilde{\rho}(\omega)} \ . \tag{6}$$

For a symmetric bunch, like a Gaussian, we usually place the zero of the coordinate at the center of the bunch, or

$$\rho(z) = \frac{1}{\sqrt{2\pi}\sigma_\ell} e^{-\frac{1}{2}(z/\sigma_\ell)^2} \ , \tag{7}$$

where σ_ℓ is the rms length, so that the bunch spectrum

$$\tilde{\rho}(\omega) = \frac{1}{2\pi} e^{-\frac{1}{2}(\omega\sigma_\ell/c)^2} \tag{8}$$

is totally real. In this way, $\hat{Z}(\omega)$ will have exactly the same phase as the impedance $Z(\omega)$. When the bunch is short enough, $\hat{Z}(\omega)$ can actually be used to approximate the true impedance at low frequencies ($\omega \ll c/\sigma_\ell$). For this reason, a symmetric bunch is usually used in numerical simulations. Because of the multiplication of the exponential factor from $1/\tilde{\rho}(\omega)$, Eq. (6) also tells us that there will be loss of information when ω is large. This is just the uncertainty principle which states that a shorter test bunch is necessary whenever high-frequency information is required.

In TBCI, the test bunch is a truncated Gaussian. The code starts computing the wake $\hat{W}(z)$ from the first charged particle in the bunch. In other words, the zero of the coordinate is placed at the very beginning of the bunch. If we stick to this coordinate system to obtain $\hat{Z}(\omega)$ through Eq. (5), we also need to do the same for the spectrum of the bunch. That is, instead of Eqs. (7) and (8), we have

$$\rho(z) = \frac{1}{\sqrt{2\pi}\sigma_\ell} e^{-\frac{1}{2}\left(\frac{z-\ell}{\sigma_\ell}\right)^2} \qquad 0 \le z \le 2\ell \ , \tag{9}$$

where 2ℓ is the full width of the symmetric bunch, and approximately

$$\tilde{\rho}(\omega) = \frac{1}{2\pi} e^{-\frac{1}{2}(\omega\sigma_\ell/c)^2} e^{i\omega\ell} \ , \tag{10}$$

which is no longer real. Therefore, $Z(\omega)$ and $\hat{Z}(\omega)$ differ by a phase. In other words, the real part of $\hat{Z}(\omega)$ will not be positive semi-definite. This implies that $\hat{Z}(\omega)$ can no longer be an approximate impedance even when the bunch length is extremely short. An easier way is to shift the zero of the coordinate of the wake potential $\hat{W}(z)$ to the center of the bunch by the replacement

$$\hat{W}(z) \to \hat{W}(z+\ell) \tag{11}$$

in Eq. (5) and use Eq. (8) for the bunch spectrum. Again, this will guarantee $\hat{Z}(\omega)$ and $Z(\omega)$ to have the same phase. In practice, $\mathcal{R}e\,\hat{Z}(\omega)$ obtained this way may still not be positive semi-definite at all frequencies. These occasional negative excursions arise from the truncation of the wake potential in the actual numerical calculation or experimental measurement.

Sometimes, the test bunch is not exactly symmetric, especially during experiment measurement. In this case, there will not be a coordinate system in which the bunch spectrum is real for all frequencies. Theoretically, we can choose any point as the zero of the coordinate and carry out the Fourier transforms for the wake and the bunch density. However, the rapid phase oscillations such as the one in Eq. (10) can contribute to excessive error when the impedance is computed via Eq. (6). To minimize the error, we can assign the zero of the coordinate to a point near the center of the bunch so that the phase fluctuation of the spectrum will become much slower. In the event that we are interested in the impedance of a particular small range of frequency such as a sharp resonance, we can move the zero-coordinate point around until $\tilde{\rho}(\omega)$ is mostly real in that particular frequency range.