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THE BEAM AND THE BUCKET

- A Handbook for the Analysis of Longitudinal Motion -

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This handbook is intended primarily for people working in the accelerator control rooms. "Convenience" is the only criterion observed in compiling this note. All materials are available in various sources but not everything is in one place; hence this enterprise. Errors of any sort you find should be promptly communicated to me.

List of Frequently Occuring Symbols

ϕ and $E = m_oc^2\gamma$ = rf phase and energy of a particle*

ϕ_s and $E_s = m_oc^2\gamma_s$ = rf phase and energy of the synchronous particle

$q \equiv \phi - \phi_s$; $\Delta E \equiv E - E_s$

$\omega_{rf} = 2\pi f_{rf} = 2\pi h f_{rev}$ (f_{rev} = revolution frequency); h = harmonic number†

$y \equiv \Delta E/\omega_{rf} = (R/hc)(cp)(\Delta p/p)$; $2\pi R$ = machine circumference†

p = particle momentum

V = peak rf accelerating voltage (per turn)

γ_t = transition gamma** 5.446 Booster (design value)

18.7 - 18.75 Main Ring and Tevatron

$\eta \equiv (1/\gamma_t^2 - 1/\gamma_s^2)$ ***

f_s = synchrotron oscillation frequency

Δ = <u>total</u> bunch length in radians

$\Gamma = \sin(\phi_s)$

(ϕ_1 and ϕ_2) = two limiting values of ϕ for a rf bucket

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* At $\phi = 0$, the rf voltage is zero and rising. The convention used for linacs is different; $\phi = 0$ when the rf voltage is at its maximum value.

**

weak focusing machine	$\gamma > \gamma_t$	
linac	$\gamma < \gamma_t$	(since $\gamma_t \to \infty$)
sector-focused cyclotron (γ_t varies with γ)	$\gamma = \gamma_t$	(hence the name "isochronous")

***

Below transition	$\eta < 0$,	$\cos(\phi_s) > 0$	$\eta\cdot\cos(\phi_s) < 0$
Above transition	$\eta > 0$,	$\cos(\phi_s) < 0$	

†

	Booster	Main Ring	Tevatron
h =	84	1113	1113
$R(m)$ =	$1000\times\frac{84}{1113}$	1000	1000

1. Hamiltonian and equations of motion

$$H(q,y;t) = \frac{1}{2} Ay^2 + B\{\cos(\phi_s+q) + q\cdot\sin(\phi_s) - \cos(\phi_s)\}$$

The last term, $\cos(\phi_s)$, is added to make $H = 0$ at the origin, $q=y=0$.

Note: $(\Delta p/p) = (hc/R)\frac{1}{cp}\cdot y$ (y is in eV-s, cp is in eV)

$$A \equiv (hc/R)^2(\eta/E_s) \quad \text{and} \quad B \equiv (eV/2\pi h) \tag{1}$$

$$dq/dt = A\cdot y , \qquad dy/dt = B\{\sin(\phi_s+q) - \sin(\phi_s)\}$$

2. Stationary Bucket ($\Gamma = \sin(\phi_s) = 0$; $\phi_s = 0$ or π)

bucket area = $16\cdot(B/|A|)^{\frac{1}{2}}$ in (q,y) phase space, eV-s (2)

bucket height (=max. y) = $\pm\ 2\cdot(B/|A|)^{\frac{1}{2}}$, eV-s (3)

max. $\Delta p/p = \pm\ 2\cdot(B/|A|)^{\frac{1}{2}}(hc/R)\frac{1}{cp}$ (3a)

From now on, we consider the case below transition only. Above transition, all phases should be regarded as $(\pi - \phi)$. Instead of η, we use $|\eta|$. With this convention,

two limiting phases of bucket: $\phi_1 < \phi_s < \phi_2$; $\phi_2 \equiv \pi - \phi_s$

3. Moving Bucket ($\Gamma = \sin(\phi_s) \neq 0$)

bucket area in (q,y) phase space $\equiv$(stationary bucket area)$\times$ $\alpha(\Gamma)$

$= 16\cdot(B/|A|)^{\frac{1}{2}} \times \alpha(\Gamma)$, eV-s (4)

bucket height $\equiv$ (stationary bucket height)$\times$ $\beta(\Gamma)$

max. $y = \pm\ 2\cdot(B/|A|)^{\frac{1}{2}} \times \beta(\Gamma)$, in eV-s (5)

max. $\Delta p/p = \pm\ 2\cdot(B/|A|)^{\frac{1}{2}}(hc/R)\frac{1}{cp} \times \beta(\Gamma)$ (5a)

$\alpha(\Gamma)$ and $\beta(\Gamma)$

If you need better-than-1% accuracy, see

C. Bovet, et al., A SELECTION OF FORMULAE AND DATA USEFUL FOR THE DESIGN OF A.G. SYNCHROTRONS, CERN/MPS-SI/Int. DL/70/4, 23 April, 1970

$0 \leqslant \Gamma \leqslant 0.3$ $\qquad \alpha(\Gamma) = (1.-\Gamma)(1.- 1.1695\Gamma + 1.3865\Gamma^2)$

$.3 \leqslant \Gamma \leqslant .6$ $\qquad \alpha(\Gamma) = (1.-\Gamma)(1.- 0.8644\Gamma + 0.3831\Gamma^2)$

$.6 \leqslant \Gamma \leqslant .85$ $\qquad \alpha(\Gamma) = (1.-\Gamma)(1.- 0.6328\Gamma - 0.0010\Gamma^2)$

Γ =	.86	.87	.88	.89	.90	.92	.94	.96	.98
$\alpha(\Gamma)$=	.0627	.0570	.0515	.0461	.0409	.0308	.0214	.0129	.00539

$0 \leqslant \Gamma \leqslant .65$ $\qquad \beta(\Gamma) = 1. - 0.7703\Gamma - 0.1227\Gamma^2$

$.65 \leqslant \Gamma \leqslant .85$ $\qquad \beta(\Gamma) = 1. - 0.6940\Gamma - 0.2406\Gamma^2$

Γ =	.86	.87	.88	.89	.90	.92	.94	.96	.98
$\beta(\Gamma)$=	.223	.211	.199	.186	.173	.146	.118	.0869	.0517

ϕ_1 in degrees (Remember $\phi_2 = \pi - \phi_s$)

$0 \leqslant \Gamma \leqslant .45$ $\qquad \phi_1 = \sqrt{\phi_s}\,(25.809 - 3.351\Gamma + 7.050\Gamma^2) - 180^\circ$

$.45 \leqslant \Gamma \leqslant .9$ $\qquad \phi_1 = \sqrt{\phi_s}\,(25.761 - 1.784\Gamma + 3.717\Gamma^2) - 180^\circ$

Note! ϕ_s in degrees! $\qquad \Gamma = \sin(\phi_s)$

$.9 \leqslant \Gamma$ $\qquad \phi_1 = 2\cdot\phi_s\cdot(1. - 0.0657\Gamma + 0.0677\Gamma^2) - 90^\circ$

(Of course, for $\Gamma = 1$, $\phi_1 = \phi_s = \phi_2 = 90^\circ$)

4. Beam in a Bucket; Stationary

Beam Area $\simeq$ (Stationary Bucket Area)$\times(\frac{\pi}{64})\Delta^2(1. - \frac{5}{384}\Delta^2)$ (6)

$\Delta \equiv$ total bunch length in radians $\lesssim$ 4 radians

For large Δ, see Figs. 1 & 4.

Beam Height = (Stationary Bucket Height)$\times$ $\sin(\Delta/4)$ (7)

This relation is exact for all values of Δ.

For $\Delta \ll 1$, $\quad q^2 + (|A|/B)\cdot y^2 = (\Delta/2)^2$ (8)

$$\text{max. } y = \pm\frac{1}{2}\Delta\,(B/|A|)^{\frac{1}{2}}, \qquad \text{eV-s} \tag{9}$$

$$\text{max. } \Delta p/p = \pm\frac{1}{2}\Delta(B/|A|)^{\frac{1}{2}}(hc/R)\frac{1}{cp} \tag{9a}$$

synchrotron oscillation frequency (Γ = 0 only!)

$$f_s = (1/2\pi)(|A|B)^{\frac{1}{2}}\frac{1}{(2/\pi)K(\Delta)} \qquad \text{--- This is exact for any } \Delta! \tag{10}$$

where $K(\Delta)$ is the complete elliptic integral of the first kind,

$$K(\Delta) = \int_0^{\pi/2}(1. - m\sin^2\theta)^{-\frac{1}{2}}\,d\theta\ ; \quad m = \sin^2(\Delta/4) \tag{11}$$

For $\Delta \ll 1$, $\quad (2/\pi)K(\Delta) \simeq 1 + \frac{1}{64}\Delta^2$ (Remember, Δ= total bunch length)

so that

$$f_s \simeq (1/2\pi)(|A|\,B)^{\frac{1}{2}}\,(1. - \frac{1}{64}\Delta^2) \tag{12}$$

Actually, this relation is surprisingly good up to $\Delta \sim 300^0$. See Fig. 3.

5. Beam in a Moving Bucket

For $\Delta \ll 1$,

$$q^2 + \left(\frac{|A|}{B\cos(\phi_s)} \right) \cdot y^2 = (\Delta/2)^2 \tag{13}$$

$$\text{max. } y = \pm\frac{1}{2}\,\Delta \left(\frac{B\cos(\phi_s)}{|A|} \right)^{\frac{1}{2}}, \qquad \text{eV-s} \tag{14}$$

$$\text{max. } \Delta p/p = \pm\frac{1}{2}\,\Delta \left(\frac{B\cos(\phi_s)}{|A|} \right)^{\frac{1}{2}} (hc/R) \frac{1}{cp} \tag{14a}$$

For Δ not too small, use Fig. 1 for area and Fig. 2 for height:

$$\text{Beam Area} \equiv (\underline{\text{Stationary}}\text{ Bucket Area}) \times \left(\frac{\pi}{64}\right) \sqrt{\cos(\phi_s)} \times \Delta^2 \times (1. - K_A) \tag{15}$$

$$\text{Beam Height} \equiv (\underline{\text{Stationary}}\text{ Bucket Height}) \times \frac{\Delta}{4} \sqrt{\cos(\phi_s)} \times (1. - K_H) \tag{16}$$

(Δ in radians!)

synchrotron oscillation frequency

$$f_s \equiv (1/2\pi)\{|A|\ B\cos(\phi_s)\}^{\frac{1}{2}} (1. - K\,\Delta^2) \text{ --- } \Delta \text{ in radians!} \tag{17}$$

The parameter K is shown in Fig. 3. We have already stated that $K \simeq 1/64$ for $\Gamma = 0$, see Eq. (12).

6. Beam in a Moving Bucket: Alternative Way

Some people may prefer this alternative way of estimating the beam area and the beam height. The reference is the corresponding moving bucket (instead of the stationary bucket used in Figs. 1 and 2).

$$\text{Beam Area} \equiv (\text{Moving Bucket Area}) \times \left(\frac{\text{total bunch length}}{\text{total bucket length}}\right)^2 \times C_A \tag{18}$$

$$\text{Beam Height} \equiv (\text{Moving Bucket Height}) \times \left(\frac{\text{total bunch length}}{\text{total bucket length}}\right) \times C_H \tag{19}$$

C_A : Fig. 4 $\qquad$ C_H: Fig. 5

(This alternative has been suggested to me by Jim Crisp.)

7. Matching from one ring to the next

In transfering the bunch from Ring 1 to Ring 2, we should have

$$\left| \frac{V \cos(\phi_s)}{h\ \eta} \right|_1 = \left| \frac{V \cos(\phi_s)}{h\ \eta} \right|_2 \tag{20}$$

Appendix I : Longitudinal Phase Space and the Particle Distribution*

There is no unique choice of two canonical variables to describe the phase space. The most commonly used ones are:

1. $q = \phi - \phi_s$ and $y = (E - E_s)/\omega_{rf}$

   The unit of phase space area is then eV-s which is also the unit for y since q is dimensionless.

2. q and $(\Delta p/m_0 c) = (\gamma\beta)(\Delta p/p)$

   All quantities are dimensionless. CERN people favored this but they may prefere 1. above now.

The phase space areas defined in two ways are of course related to each other but the relation is dependent on machine parameters,

$$\text{area in } (q, y) = 3.13 \text{ eV-s} \times \frac{R(m)}{h} \times \text{area in } q, (\gamma\beta)(\Delta p/p)$$

$$= 2.81 \text{ eV-s} \times (\text{area})_2 \quad \text{for Booster, Main Ring, Tevatron}$$

For electron beams, people use bi-Gaussian distribution in two canonical variables (q,y), or more generally,

$$\rho(q,y) \propto e^{-kH(q,y)}$$

with the Hamiltonian $H(q,y)$ of the motion. For proton bunches, it is more common to use finite distributions. One such distribution called "elliptic" is

$$\rho(q,y) \propto \sqrt{y_B^2(q) - y^2(q)}$$

where $y = y_B(q)$ defines the boundary of the finite bunch in (q,y) space. The local current density is

* This is essentially the same as Appendix B of EXP-111, November 28, 1983.

$$I(q) \propto \int_{-y_B}^{y_B} dy \;\; \rho(q,y) \;\; \propto \;\; y_B^2(q)$$

An appealing feature of this distribution is discussed in TM-749 in connection with the longitudinal instabilities with Landau cavities. The simplest specification of the bunch shape to be used for the distribution is

$$y_B(q) = \pm \, (\max.y)\{1 - \frac{q^2}{(\Delta/2)^2}\}^{\frac{1}{2}} \; ; \quad \Delta = \text{total bunch length}$$

The normalized (to unity) distribution is

$$\rho(q,y) = \frac{3}{2}\frac{1}{Sy_m} \; \{y_B^2(q) - y^2(q)\}^{\frac{1}{2}}$$

where $y_m = \max.y$ and the emittance $S = \pi y_m(\Delta/2)$.

Appendix II: Higher-Order Effects

On page 1, it is stated that the equation of motion for y is simply $dq/dt = A\cdot y$. This is not exact. One should write

$$dq/dt = -h \; (\; \omega - \omega_s)$$

where ω is the angular frequency of a particle and ω_s that of the synchronous particle. If the right-hand-side is expanded in $(\Delta E/E)$ and only the lowest-order term is retained, we get Ay. Note that A is proportional to η so that it vanishes at the transition. One must consider the next term in the expansion near the transition where $|\eta|$ is very small.[1] For this, it is convenient to use the parameter introduced by Johnsen,[2]

$$L(p) = L_o \;\; \{1 + \alpha_1(\Delta p/p)(1 + \alpha_2\frac{\Delta p}{2\,p})\}$$

where $L(p)$ is the path length of a particle with the momentum p and $L_o = 2\pi R$.

From the definition of transition energy, $\alpha_1 = 1/\gamma_t^2$ where, strictly speaking, γ_t is for the synchronous particle. Johnsen pointed out that the proper time of transition for a particle with momentum p different from p_s is

$$t_p \simeq -\frac{\gamma_s(t=0)}{\dot{\gamma}_s(t=0)} \left(\frac{3}{2} + \alpha_2\right) \frac{\Delta p}{p}$$

if the transition for the synchronous particle is at $t = 0$. The acceleration rate is assumed to be $dE_s/dt = m_o c^2 \dot{\gamma}_s$. The "ideal" machine should have $\alpha_2 = -1.5$ so that all particles cross the transition at the same time. Since $|\Delta p/p|_{max}$ is believed to be around $(3\sim4)\times10^{-3}$ at transition,

$$\begin{array}{lll} \text{Booster} & |t_p|_{max} & \simeq (.04 \sim .05)\text{ms}\times(1.5+\alpha_2), \\ \text{Main Ring} & & (.6 \sim .8)\text{ms}\times(1.5+\alpha_2) \end{array}$$

If α_2 is different from -1.5, one must add a higher-order term in the Hamiltonian H(q,y;t),

$$\Delta H = (hc/R)^3 \frac{1}{\beta_s E_s^2} F\, y^3 \cdot \left(\frac{1}{3}\right)$$

$$F = \frac{3}{2}\left(\frac{\beta_s}{\gamma_s}\right)^2 + \frac{\alpha_2}{\gamma_t^2} - \frac{\eta}{\gamma_t^2}$$

Since the higher-order term is important only near the transition where $\eta \simeq 0$, the last term in F can be dropped.

There is no easy way to calculate α_2 for any given machine. One must know the sextupole field since α_2 represents the second-order effect. Even in the absence of nonlinear field, it is necessary to compute the off-momentum closed orbit beyond the customary first-order approximation in $(\Delta p/p)$. The value of α_2 has been calculated by W. W. Lee for the booster with and without sextupole component.[3]

He used the second-order TRANSPORT to find

$$\alpha_2 = 1.63 \quad \text{linear booster,}$$
$$= 0.843 \quad \text{with the design value of sextupole component}^{4}$$

The calculation has been repeated with the step-by-step numerical integration of the orbit in linear magnets and the result is in good agreement with Lee's value,

$$\alpha_2 \text{ (numerical integration, linear)} = 1.619$$

The calculation with sextupole field is not so straightforward. There are indications from various measurements that booster magnets are different from the designed ones. Furthermore, correction sextupoles may not be entirely negligible near transition. For these reasons, α_2 with sextupole field has been estimated using an approximate relation[5]

$$\alpha_2 \simeq -1 - 2\xi_0 - \Delta\xi, \tag{A.1}$$

ξ_0 = horizontal chromaticity(CERN style) of the linear machine

= $(\Delta\nu_H/\nu_H)$ divided by $(\Delta p/p)$,

$\Delta\xi$ = change in ξ due to sextupole field.

This relation is derived by retaining only the average terms in the Fourier expansion of relevant quantities. The step-by-step integration of orbit yields

$$\xi_0 = -1.3683 \quad \text{(Booster, linear)}$$

so that $-1 - 2\xi_0 = 1.737$ which should be compared with the exact value 1.619. The chromaticity of the booster near the transition with the standard setting of correction sextupole has been measured by C. Hojvat:

$$\xi = \xi_0 + \Delta\xi = 0.502$$

so that $\Delta\xi = 1.87$. From (A.1),

$$\alpha_2 \simeq -1 - 2(-1.3683) - 1.87 = -0.13 .$$

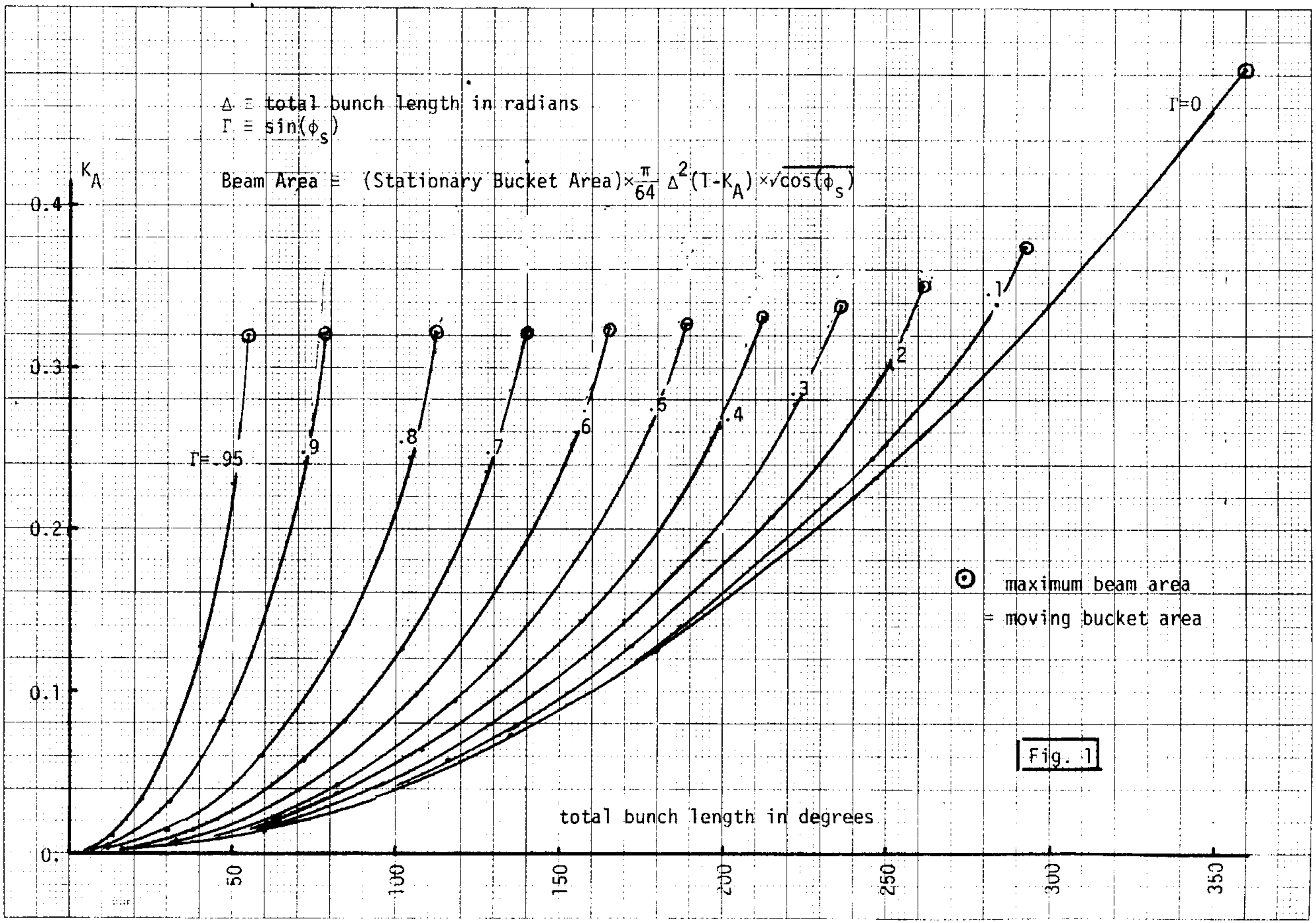
On the other hand, one might interpret (A.1) to mean $\alpha_2 = \alpha_2(\text{linear}) - \Delta\xi$. According to this interpretation, we find

$$\alpha_2 \simeq 1.619 - 1.87 = -0.25.$$

In either way, the value of α_2 for the booster seems to be far from the ideal value, -1.5. To the best of my knowledge, α_2 of the main ring is unknown.

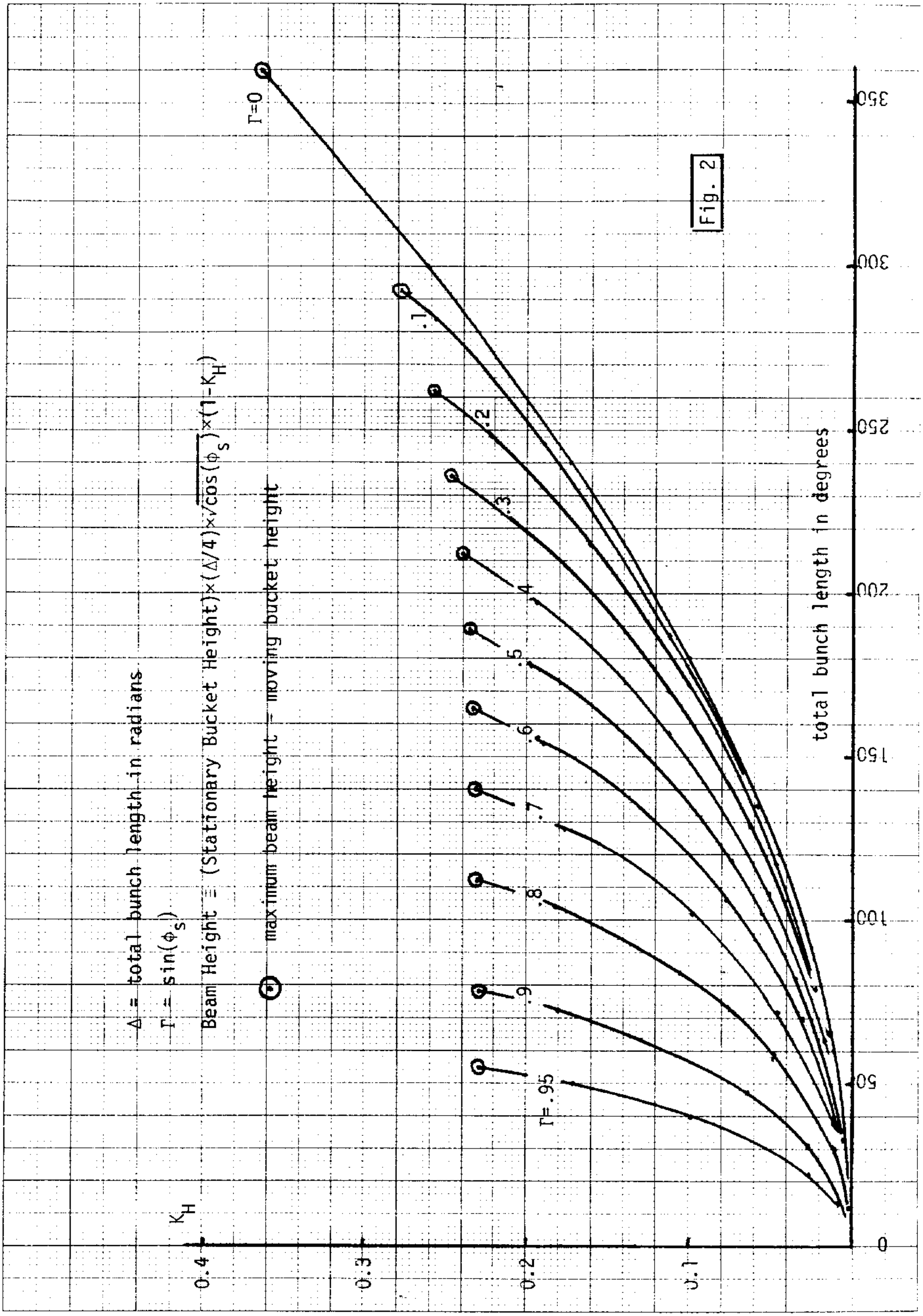
references

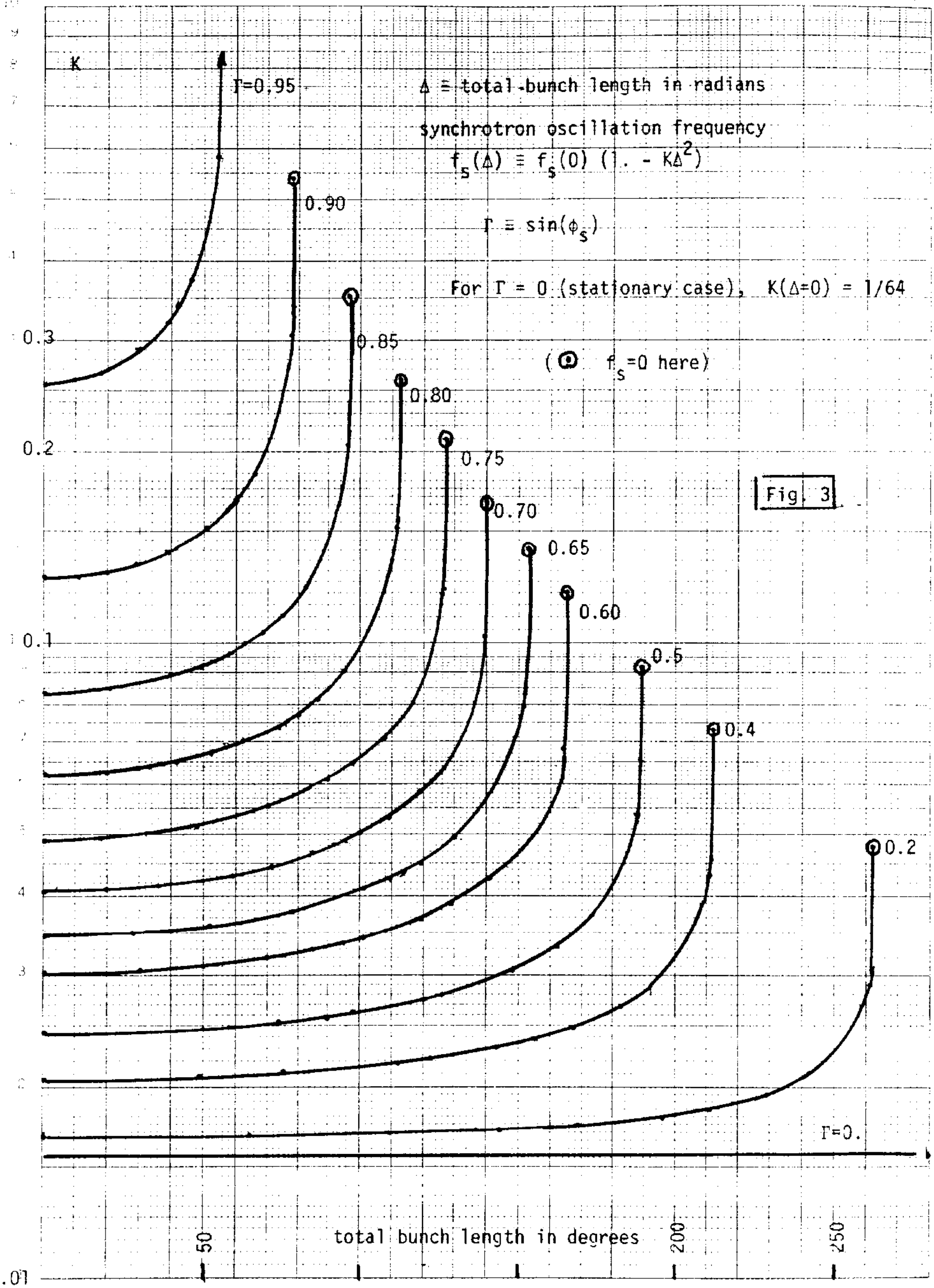
1. K.R. Symon and A.M. Sessler, Proceedings of the CERN Symposium on the High-Energy Accelerators and Pion Physics, 1956, vol. 1, p.46.
2. K. Johnsen, ibid., p.106.
3. W.W. Lee, TM-333, December 1, 1971.
4. S.C. Snowdon, TM-156, March 1969.
5. P.E. Faugeras, A. Faugier and J. Gareyte, SPS Improvement Report No.130, CERN, 24 May 1978.

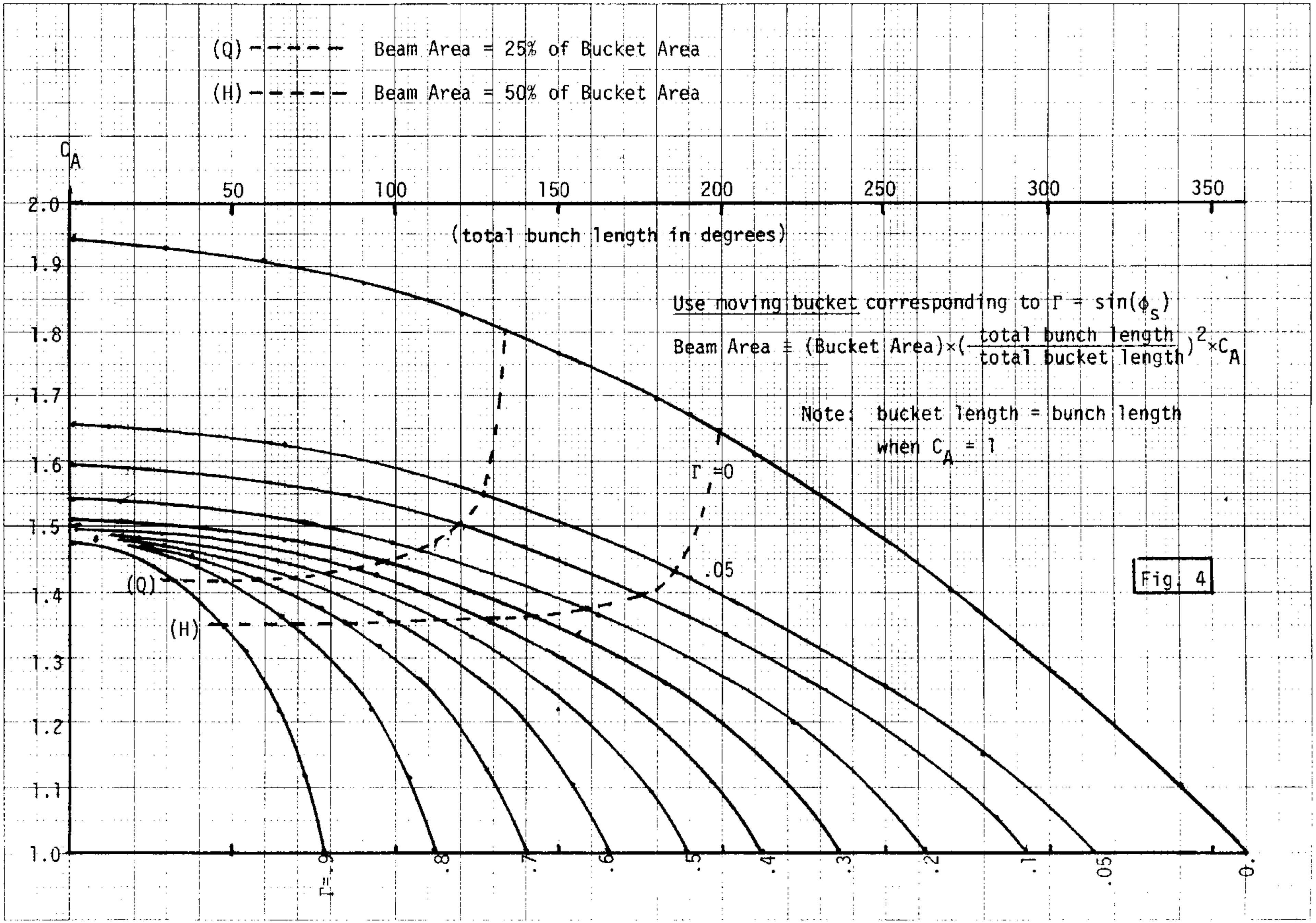


Δ ≡ total bunch length in radians
Γ ≡ sin(φ_s)
Beam Area ≡ (Stationary Bucket Area)×π/64 Δ²(1-K_A)×√cos(φ_s)
K_A
0.4
0.3
0.2
0.1
0.
50
100
150
200
250
300
350
total bunch length in degrees
Γ=0
.1
.2
.3
.4
.5
.6
.7
.8
.9
Γ=.95
⊙ maximum beam area
= moving bucket area

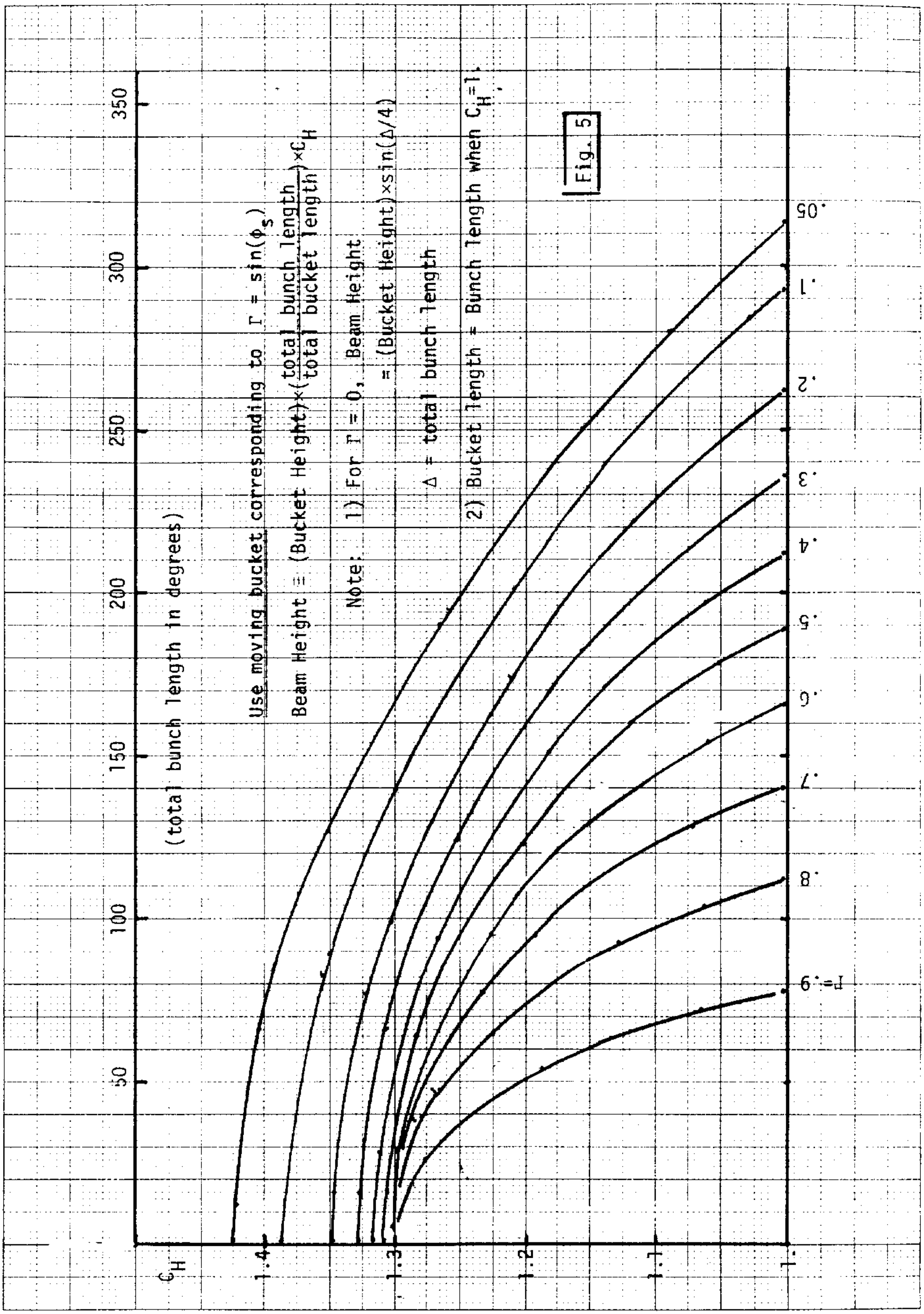
Fig. 1


Δ = total bunch length in radians
Γ = sin(φ_s)
Beam Height ≡ (Stationary Bucket Height)×(Δ/4)×√cos(φ_s)×(1-K_H)
⊙ maximum beam height - moving bucket height
Fig. 2
Γ=0
.1
.2
.3
.4
.5
.6
.7
.8
.9
Γ=.95
K_H
0.4
0.3
0.2
0.1
0
50
100
150
200
250
300
350
total bunch length in degrees


K
Γ=0.95
Δ ≡ total bunch length in radians
synchrotron oscillation frequency
$f_s(\Delta) \equiv f_s(0)\,(1. - K\Delta^2)$
$\Gamma \equiv \sin(\phi_s)$
For Γ = 0 (stationary case), K(Δ=0) = 1/64
(⊙ f_s=0 here)
0.90
0.85
0.80
0.75
0.70
0.65
0.60
0.5
0.4
0.2
Γ=0.
0.3
0.2
0.1
.01
50
200
250
total bunch length in degrees
Fig. 3



(Q) Beam Area = 25% of Bucket Area
(H) Beam Area = 50% of Bucket Area
C_A
50
100
150
200
250
300
350
2.0
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
(total bunch length in degrees)
Use moving bucket corresponding to $\Gamma = \sin(\phi_s)$
Beam Area $\doteq$ (Bucket Area)$\times\left(\frac{\text{total bunch length}}{\text{total bucket length}}\right)^2 \times C_A$
Note: bucket length = bunch length when $C_A = 1$
$\Gamma = 0$
.05
(Q)
(H)
Fig. 4
Γ=.9
.8
.7
.6
.5
.4
.3
.2
.1
.05
0.


(total bunch length in degrees)
50
100
150
200
250
300
350
Use moving bucket corresponding to Γ = sin(φ_s)
Beam Height ≡ (Bucket Height)×(total bunch length / total bucket length)×C_H
Note: 1) For Γ = 0, Beam Height
= (Bucket Height)×sin(Δ/4)
Δ = total bunch length
2) Bucket length = Bunch length when C_H=1.
Fig. 5
C_H
1.4
1.3
1.2
1.1
1.
Γ=.9
.8
.7
.6
.5
.4
.3
.2
.1
.05

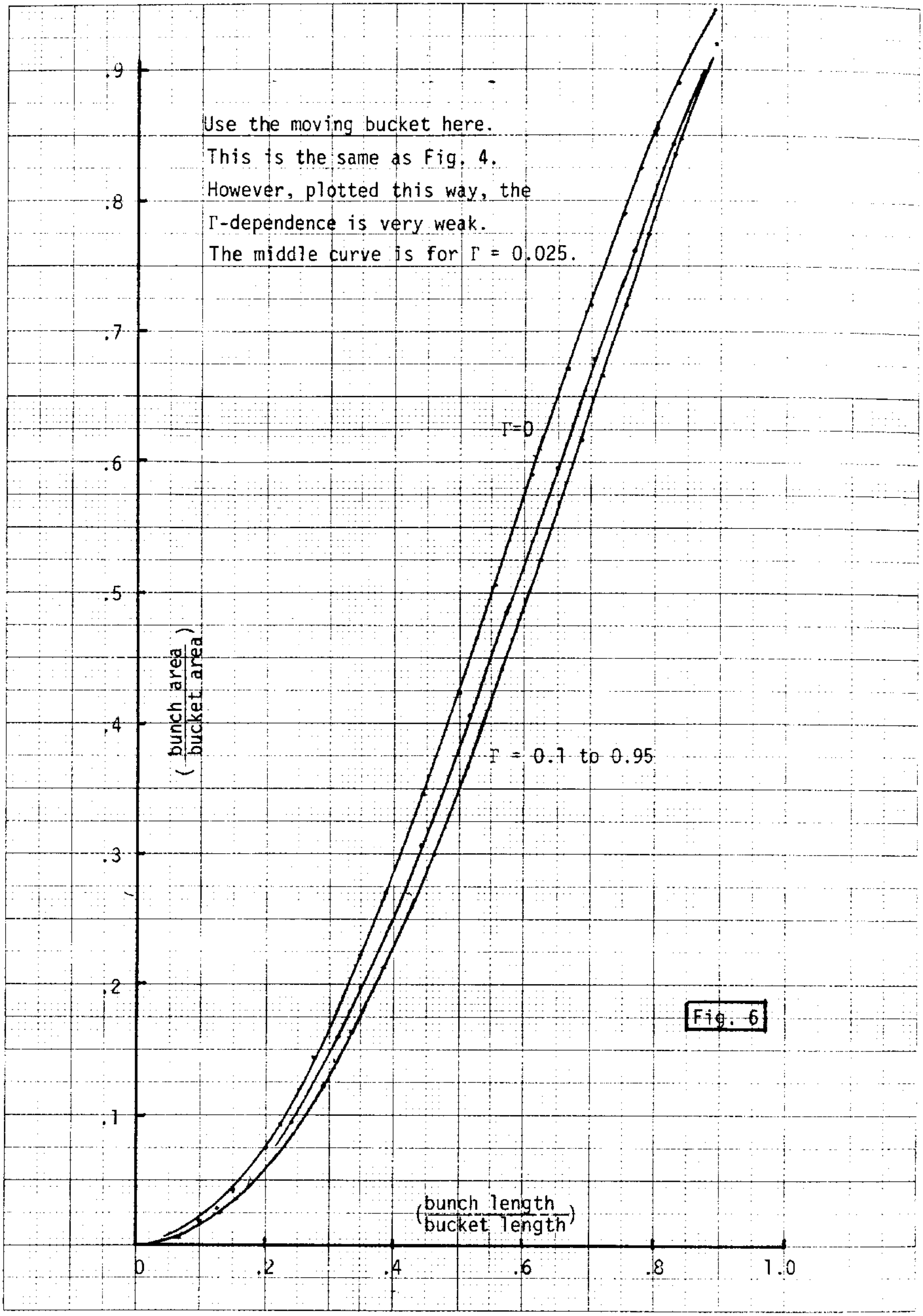
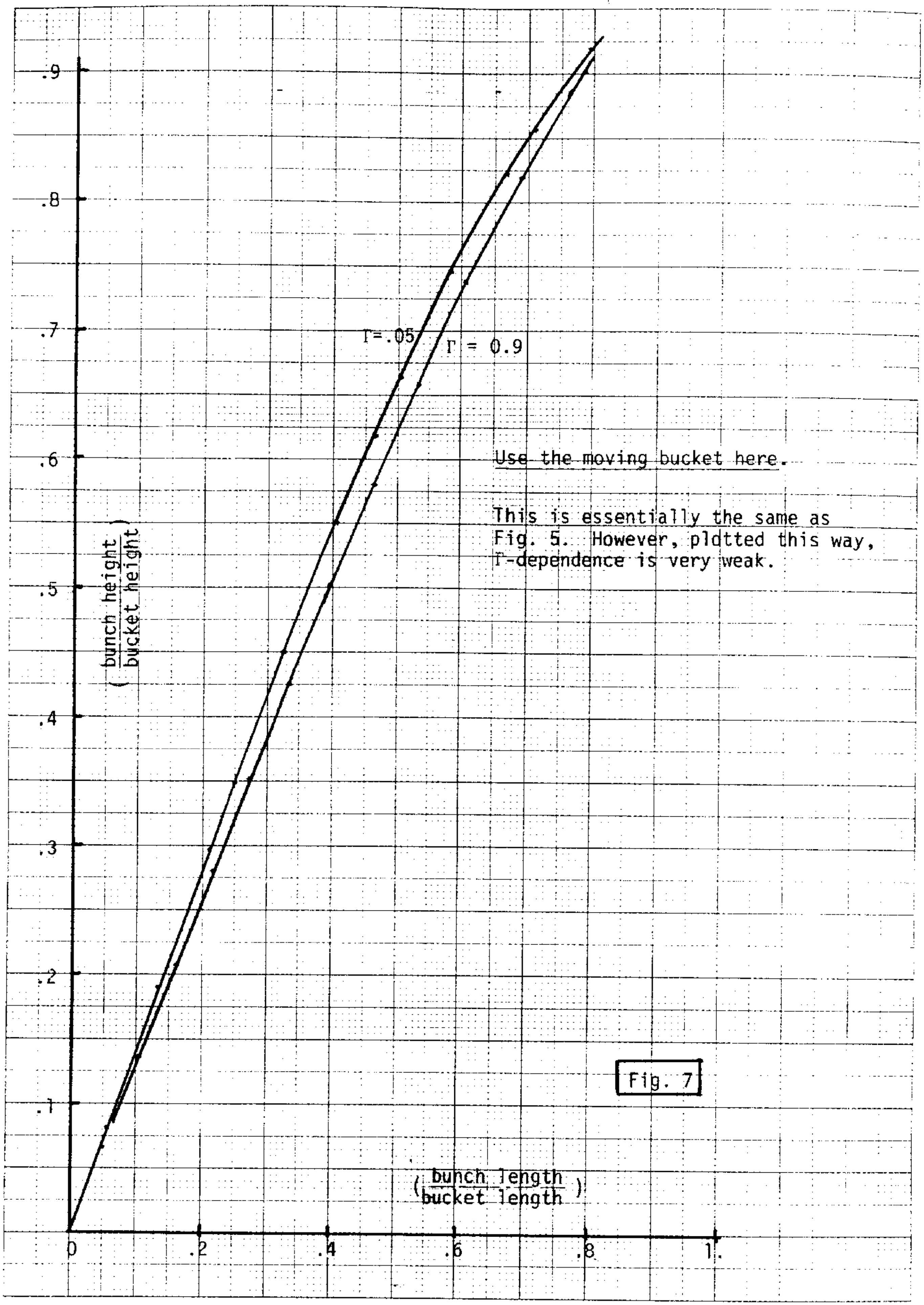

Use the moving bucket here.
This is the same as Fig. 4.
However, plotted this way, the
Γ-dependence is very weak.
The middle curve is for Γ = 0.025.
Γ=0
Γ = 0.1 to 0.95
(bunch area / bucket area)
(bunch length / bucket length)
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
.2
.4
.6
.8
1.0

Fig. 6


.9
.8
.7
.6
.5
.4
.3
.2
.1
(bunch height / bucket height)
Γ=.05
Γ = 0.9
Use the moving bucket here.
This is essentially the same as Fig. 5. However, plotted this way, Γ-dependence is very weak.
Fig. 7
(bunch length / bucket length)
0
.2
.4
.6
.8
1.

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ELLIPTIC DISTRIBUTION IN LONGITUDINAL PHASE SPACE
(Supplement to TM-1381, "The Beam and the Bucket")

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April 1986

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ELLIPTIC DISTRIBUTION IN LONGITUDINAL PHASE SPACE
(Supplement to TM-1381,"THE BEAM AND THE BUCKET")

Summary

As an alternative to either the uniform distribution (which is not really physical) or the Gaussian distribution (which is not finite), a finite distribution called "elliptic" is proposed and its main properties are presented. With canonical variables ($q \equiv \phi - \phi_s, y \propto \Delta p/p$), this distribution takes the form

$$\rho(q,y) \propto \sqrt{y_B^2(q) - y^2}$$

where $y_B(q)$ defines the boundary of a finite bunch in (q,y) space. It is assumed that the boundary is determined by the condition "Hamiltonian in (q,y) = constant" so that the shape is in general not symmetric in phase. This report is intended as a supplemnt to the previous one, "The Beam and the Bucket" and, as such, it is primarily for people working in the accelerator control rooms.

I.

A certain degree of "awkwardness" exists when Gaussian distributions are assumed in the longitudinal phase space for rf bunches of protons since the bunches must be confined within finite bucket boundaries. This is especially the case when the bunch occupies a substantial fraction of the bucket area. On the other hand, a uniform distribution within a finite bunch boundary can hardly be regarded as physical. For many years, people at CERN (some of them, at least) have been advocating a distribution called "elliptic" as something not only convenient but realistic as well. For example, there is a beautiful picture of the CERN Booster bunches in an article by Frank Sacherer, Proc. of the IXth Int. Conf. on High Energy Accelerators, SLAC, 1974, p.347. When a pair of canonical variables are* $q \equiv \phi - \phi_s$ and $y \equiv (R/hc)(cp)(\Delta p/p)$, this distribution takes the form

$$\rho(q,y) \propto \sqrt{y_B^2(q) - y^2}$$

where $y_B(q)$ is the boundary of a finite bunch. One particularly appealing feature of this distribution is that the resulting local current density is proportional to $y_B^2(q)$:

$$I(q) \propto \int_{-y_B}^{y_B} dy\ \rho(y,q) \propto y_B^2(q)$$

As a consequence, the effect of the beam-induced voltage (which arises from a distributed wall inductance) can be treated in a simple and consistent manner. (See, for example, S. Ohnuma, TM-749, "EXPECTED BUNCH LENGTH AND MOMENTUM SPREAD OF THE BEAM IN THE MAIN RING WITH CEA CAVITIES", October 24, 1977.)

II.

The Hamiltonian in (q,y) space is given in TM-1381, p. 2:

$$H(q,y;t) = -\tfrac{1}{2}|A|y^2 + B\{\cos(\phi_s+q) + q\cdot\sin(\phi_s)\} \qquad (1)$$

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* Unless otherwise noted, all notations are identical to what I used in TM-1381.

where a constant term, $\cos(\phi_s)$, is dropped from the original expression. Two parameters A and B are

$$A \equiv (hc/R)^2(\eta/E_s) \quad \text{and} \quad B \equiv (eV/2\pi h) \tag{2}$$

Since we are considering the case below transition, η and A are negative. The corresponding bucket and an example of bunch are shown on p. 3. The bucket extends from $\phi = \phi_L$ to ϕ_R $(\equiv\pi-\phi_s)$ and the beam from ϕ_1 to ϕ_2. The bucket boundary is specified by the relation

$$-\tfrac{1}{2}|A|y^2 + B\{\cos(\phi_s+q)+\Gamma\cdot q\} = B\{\cos(\phi_R)+\Gamma\cdot q_R\} \tag{3}$$

where $\Gamma \equiv \sin(\phi_s)$ and $q_R \equiv \phi_R - \phi_s$. Similarly, the bunch boundary is

$$\begin{aligned} -\tfrac{1}{2}|A|y_B^2 + B\{\cos(\phi_s+q) + \Gamma\cdot q\} &= B\{\cos(\phi_1) + \Gamma\cdot q_1\} \\ &= B\{\cos(\phi_2) + \Gamma\cdot q_2\} \end{aligned} \tag{4}$$

It is convenient to use $\hat{y} \equiv (|A|/2B)^{\frac{1}{2}}\cdot y$ instead of y so that the bunch boundary is

$$\hat{y}_B^2 = \cos(\phi_s+q) + \Gamma\cdot q - C \tag{5}$$

with

$$C \equiv \cos(\phi_1) + \Gamma\cdot q_1 = \cos(\phi_2) + \Gamma\cdot q_2 \tag{6}$$

If the total number of particles in the bunch is n_B, the distribution is

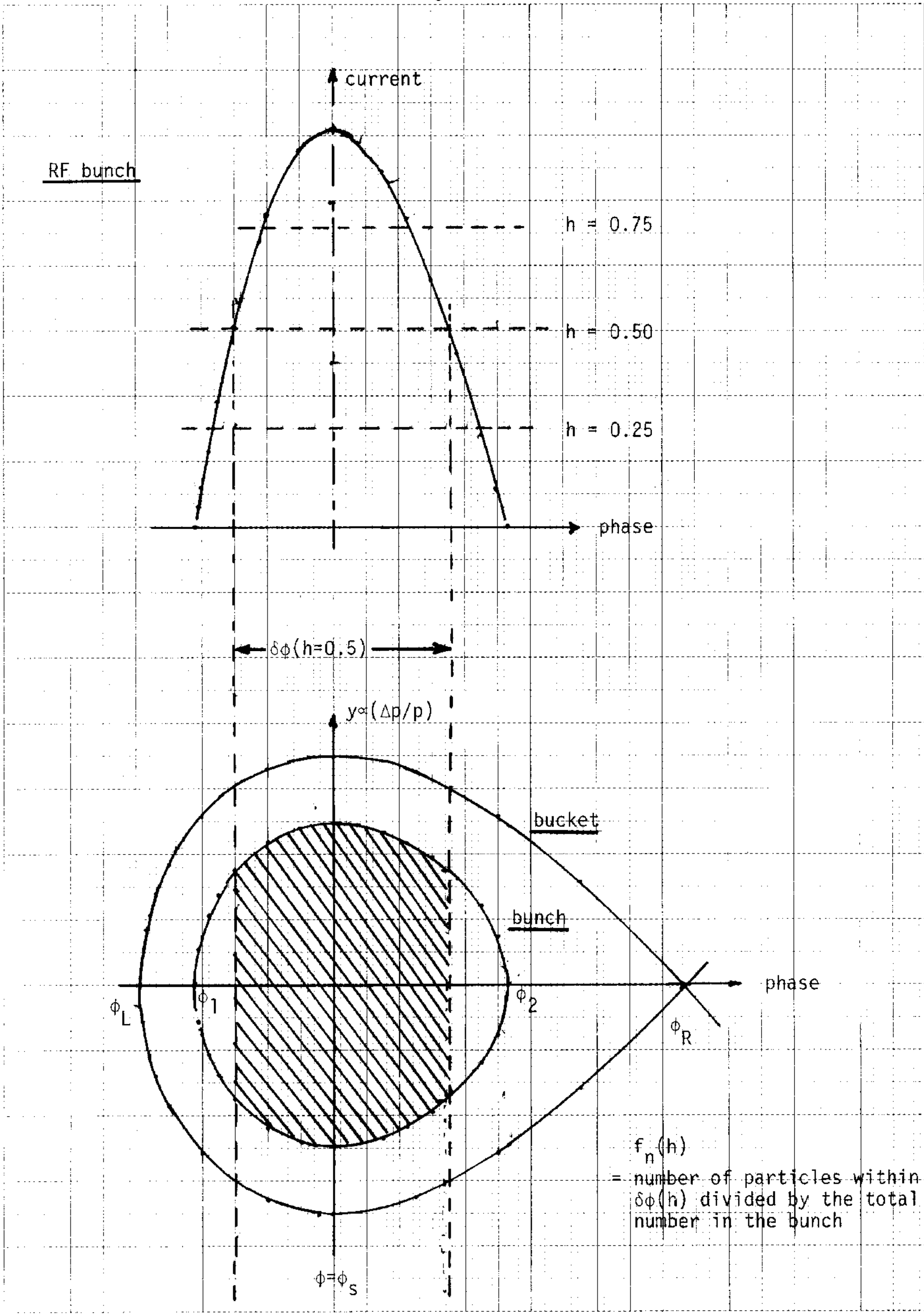
$$\rho(\hat{y},q) = n_B(2/\pi)\cdot\frac{1}{D}\{\hat{y}_B^2 - \hat{y}^2\}^{\frac{1}{2}} \tag{7}$$

with

$$D \equiv \sin(\phi_s+q) + \tfrac{1}{2}\Gamma q^2 - Cq\Big|_{q_1}^{q_2} \tag{8}$$

The local current density is

$$I(q) \propto (n_B/D)\hat{y}_B^2(q) = (n_B/D)\{\cos(\phi_s+q) + \Gamma\cdot q - C\} \tag{9}$$


RF bunch
current
h = 0.75
h = 0.50
h = 0.25
phase
δφ(h=0.5)
y∝(Δp/p)
bucket
bunch
phase
φL
φ1
φ2
φR
φ=φs
fn(h)
= number of particles within δφ(h) divided by the total number in the bunch

In particular, the maximum (local) current is*

$$I(q=0) \propto (n_B/D)\{\cos(\phi_s) - C\} \tag{10}$$

III.

We start with the assumption that the synchronous phase ϕ_s is known so that the bucket length $(\phi_R - \phi_L)$ is also known. The upper part of p. 3 is meant to be a typical picture of a bunch one sees on a scope. If we can tell where the baseline is (which is not easy most of the time!), we can find the bunch length as well as various length corresponding to the relative current density h. Remember that the current density (so that the parameter h) is proportional to y_B^2. It is easy to show that when the bunch length $(\phi_2 - \phi_1)$ is much less than the bucket length $(\phi_R - \phi_L)$,

$$\delta\phi(h) \simeq (1 - h)^{\frac{1}{2}} \times (\text{bunch length}), \tag{11}$$

and the number of particles within the phase distance $\delta\phi(h)$ relative to the total number n_B in the bunch is

$$f_n(h) \simeq (1 + h/2)(1 - h)^{\frac{1}{2}} \tag{12}$$

As the bunch length increases, these relations must be modified by a factor (1+k). Fig. 1 shows k as a function of (bunch length)/(bucket length) when h = 0.5, solid curves for $\delta\phi(h=0.5)$, Eq.(11), and dashed curves for $f_n(h=0.5)$, Eq.(12). One sees that the approximate relation for f_n, Eq.(12),

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* The maximum longitudinal charge density (which is proportional to the maximum current density) is an important quantity in the discussion of space-charge detuning by the self field. It is interesting to note that, when the bunch shape is symmetric (ϕ_s=0), the maximum density of the Gaussian distribution ($n_B/\sqrt{2\pi}\sigma_z$) is very close to the maximum density of the elliptic distribution if σ_z is interpreted to be one-quarter of the total bunch length. For the Gaussian distribution, 95% of the beam is contained within $\pm 2\sigma_z$.

is good with less than 5% error for $\Gamma = \sin(\phi_s) > 0.1$ and (bunch length) $<$ 0.9 $\times$ (bucket length).

IV.

It is often difficult to determine where the baseline is for a picture such as the one on p. 3. Fig. 1 is then not so useful in practical situations. The suggested procedures for such a case are as follows:

1) From the bunch shape, make a guess on where h is one-half. Based on this guess, find $\delta\phi$(h=0.25), $\delta\phi$(h=0.5) and $\delta\phi$(h=0.75). Fig. 2 shows the ratios $\delta\phi$(h=0.25)/$\delta\phi$(h=0.5) and $\delta\phi$(h=0.75)/$\delta\phi$(h=0.5) as a function of $\delta\phi$(h=0.5). If the guessed values of $\delta\phi$ for h=0.25, 0.5 and 0.75 are not consistent with these curves, try another guess until the best consistency is obtained. Since the distribution is unlikely to be exactly elliptic, this method may not always yield a unique solution.

2) Since $\delta\phi$'s for h=0.25, h=0.5 and h=0.75 are found (approximately), find the corresponding bunch length from Fig. 3. When the bunch length is comparable to the bucket length, $\delta\phi$(h=0.25) should be used instead of $\delta\phi$(h=.5) or $\delta\phi$(h=.75).

3) Finally, find how many particles are contained within each phase interval $\delta\phi$(h) using Fig. 4.

4) From the bunch length, one can estimate the phase space area of the bunch using, for example, Fig. 6 of TM-1381.

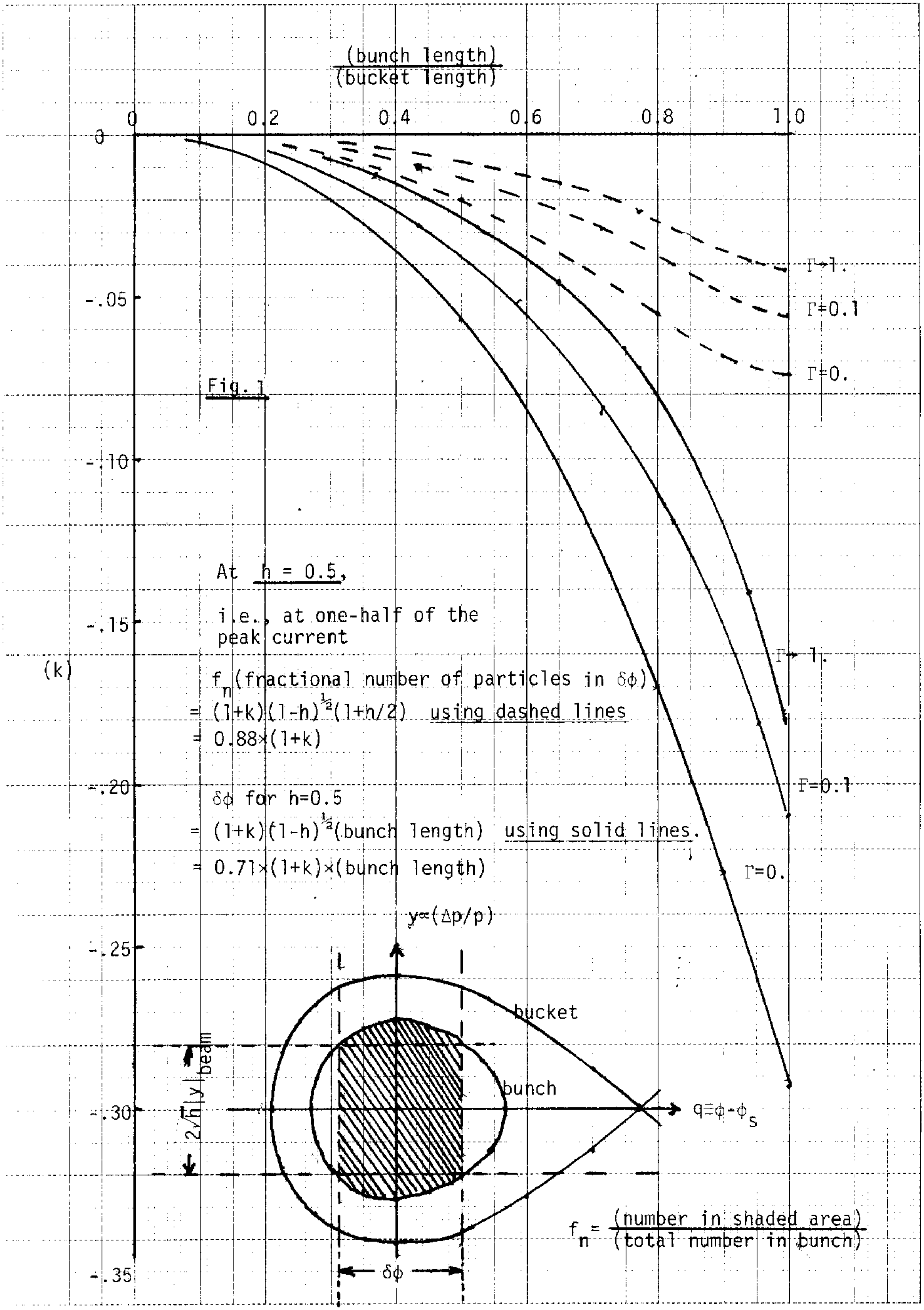
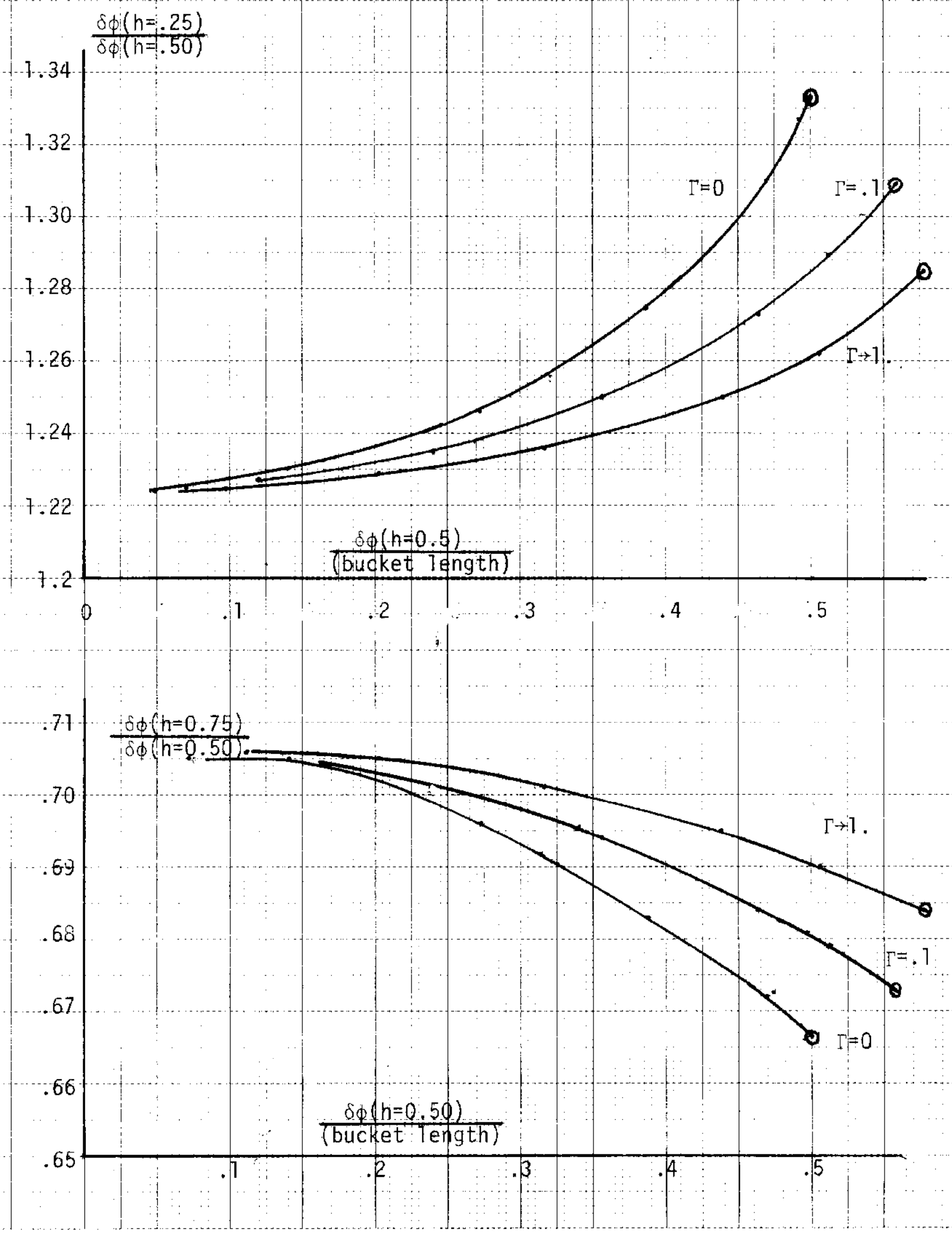

(bunch length)
(bucket length)
0
0.2
0.4
0.6
0.8
1.0
0
-.05
-.10
-.15
-.20
-.25
-.30
-.35
(k)
Γ→1.
Γ=0.1
Γ=0.
Γ→1.
Γ=0.1
Γ=0.
At h = 0.5,
i.e., at one-half of the peak current
f_n (fractional number of particles in δφ)
= (1+k)(1-h)^½(1+h/2) using dashed lines
= 0.88×(1+k)
δφ for h=0.5
= (1+k)(1-h)^½(bunch length) using solid lines.
= 0.71×(1+k)×(bunch length)
y≡(Δp/p)
bucket
bunch
q≡φ-φ_s
2√h|y|_beam
δφ
f_n = (number in shaded area) / (total number in bunch)

Fig. 1

Fig. 2


δφ(h=.25)
δφ(h=.50)
1.34
1.32
1.30
1.28
1.26
1.24
1.22
1.2
Γ=0
Γ=.1
Γ→1.
δφ(h=0.5)
(bucket length)
0
.1
.2
.3
.4
.5
.71
.70
.69
.68
.67
.66
.65
δφ(h=0.75)
δφ(h=0.50)
Γ→1.
Γ=.1
Γ=0
δφ(h=0.50)
(bucket length)
.1
.2
.3
.4
.5

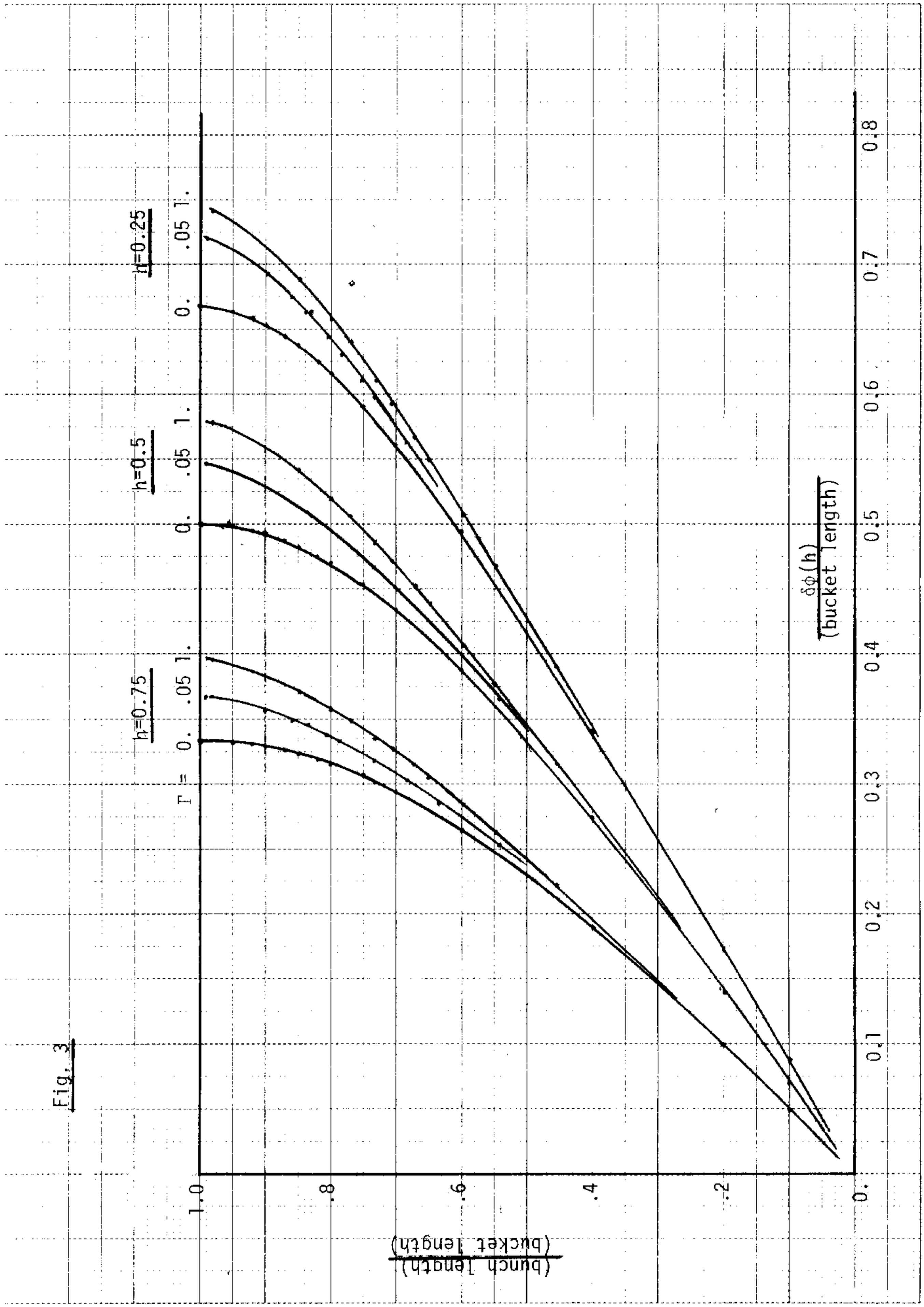
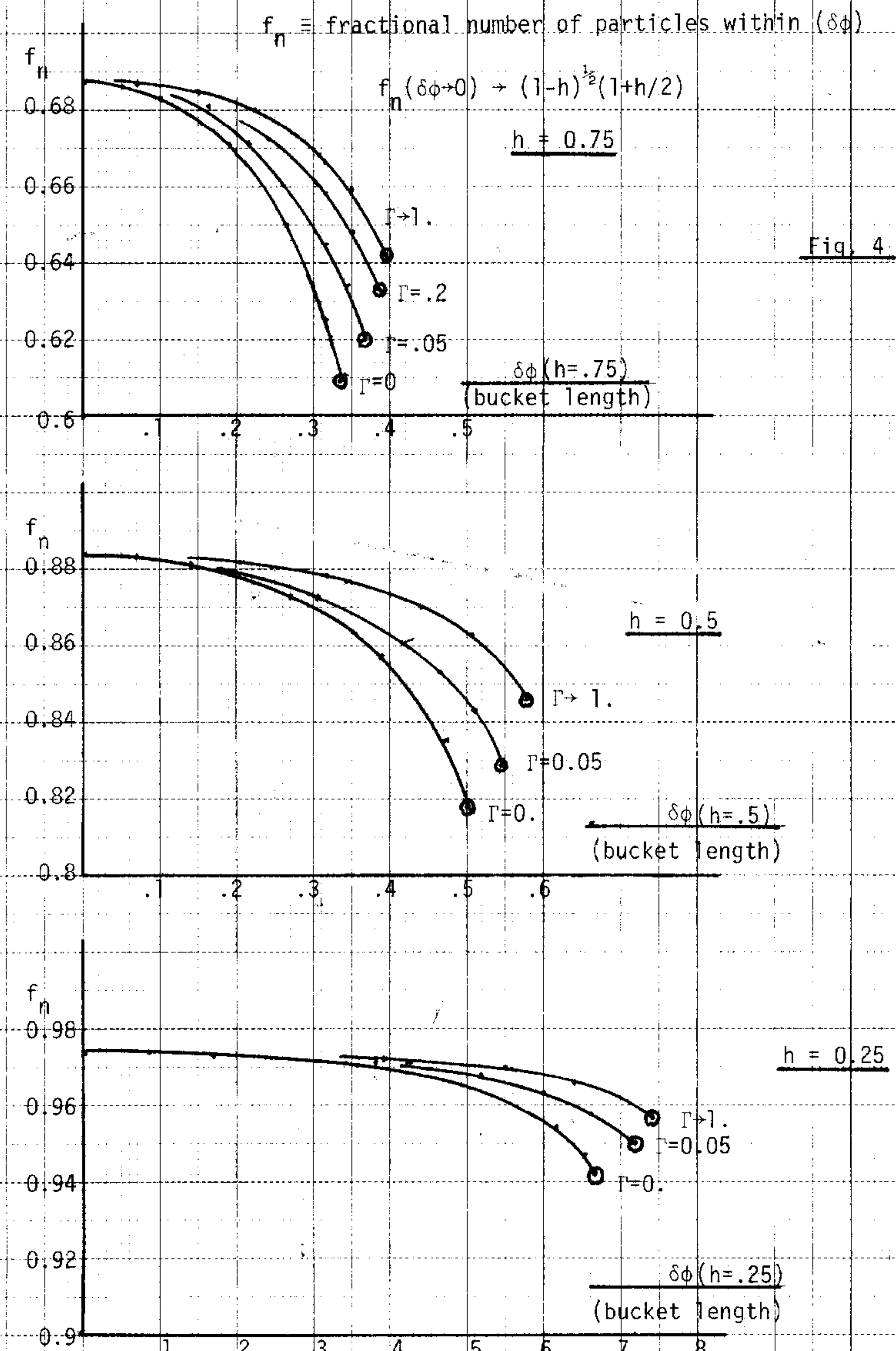

h=0.25
h=0.5
h=0.75
Γ = 0. .05 1.
$\frac{\delta\phi(h)}{(\text{bucket length})}$
$\frac{(\text{bunch length})}{(\text{bucket length})}$
0. .2 .4 .6 .8 1.0
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8

Fig. 3



$f_n \equiv$ fractional number of particles within $(\delta\phi)$
$f_n(\delta\phi \to 0) \to (1-h)^{1/2}(1+h/2)$
h = 0.75
$\Gamma \to 1.$
$\Gamma = .2$
$\Gamma = .05$
$\Gamma = 0$
$\delta\phi$(h=.75) / (bucket length)
h = 0.5
$\Gamma \to 1.$
$\Gamma = 0.05$
$\Gamma = 0.$
$\delta\phi$(h=.5) / (bucket length)
h = 0.25
$\Gamma \to 1.$
$\Gamma = 0.05$
$\Gamma = 0.$
$\delta\phi$(h=.25) / (bucket length)
f_n

Fig. 4