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ELLIPTIC DISTRIBUTION IN LONGITUDINAL PHASE SPACE
(Supplement to TM-1381, "The Beam and the Bucket")

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Summary

As an alternative to either the uniform distribution (which is not really physical) or the Gaussian distribution (which is not finite), a finite distribution called "elliptic" is proposed and its main properties are presented. With canonical variables ($q \equiv \phi - \phi_s, y \propto \Delta p/p$), this distribution takes the form

$$\rho(q,y) \propto \sqrt{y_B^2(q) - y^2}$$

where $y_B(q)$ defines the boundary of a finite bunch in (q,y) space. It is assumed that the boundary is determined by the condition "Hamiltonian in (q,y) = constant" so that the shape is in general not symmetric in phase. This report is intended as a supplemnt to the previous one, "The Beam and the Bucket" and, as such, it is primarily for people working in the accelerator control rooms.

I.

A certain degree of "awkwardness" exists when Gaussian distributions are assumed in the longitudinal phase space for rf bunches of protons since the bunches must be confined within finite bucket boundaries. This is especially the case when the bunch occupies a substantial fraction of the bucket area. On the other hand, a uniform distribution within a finite bunch boundary can hardly be regarded as physical. For many years, people at CERN (some of them, at least) have been advocating a distribution called "elliptic" as something not only convenient but realistic as well. For example, there is a beautiful picture of the CERN Booster bunches in an article by Frank Sacherer, Proc. of the IXth Int. Conf. on High Energy Accelerators, SLAC, 1974, p.347. When a pair of canonical variables are* $q \equiv \phi - \phi_s$ and $y \equiv (R/hc)(cp)(\Delta p/p)$, this distribution takes the form

$$\rho(q,y) \propto \sqrt{y_B^2(q) - y^2}$$

where $y_B(q)$ is the boundary of a finite bunch. One particularly appealing feature of this distribution is that the resulting local current density is proportional to $y_B^2(q)$:

$$I(q) \propto \int_{-y_B}^{y_B} dy\ \rho(y,q) \propto y_B^2(q)$$

As a consequence, the effect of the beam-induced voltage (which arises from a distributed wall inductance) can be treated in a simple and consistent manner. (See, for example, S. Ohnuma, TM-749, "EXPECTED BUNCH LENGTH AND MOMENTUM SPREAD OF THE BEAM IN THE MAIN RING WITH CEA CAVITIES", October 24, 1977.)

II.

The Hamiltonian in (q,y) space is given in TM-1381, p. 2:

$$H(q,y;t) = -\tfrac{1}{2}|A|y^2 + B\{\cos(\phi_s+q) + q\cdot\sin(\phi_s)\} \tag{1}$$

* Unless otherwise noted, all notations are identical to what I used in TM-1381.

where a constant term, $\cos(\phi_s)$, is dropped from the original expression. Two parameters A and B are

$$A \equiv (hc/R)^2(\eta/E_s) \quad \text{and} \quad B \equiv (eV/2\pi h) \tag{2}$$

Since we are considering the case below transition, η and A are negative. The corresponding bucket and an example of bunch are shown on p. 3. The bucket extends from $\phi = \phi_L$ to ϕ_R ($\equiv \pi - \phi_s$) and the beam from ϕ_1 to ϕ_2. The bucket boundary is specified by the relation

$$-\tfrac{1}{2}|A|y^2 + B\{\cos(\phi_s+q)+\Gamma\cdot q\} = B\{\cos(\phi_R)+\Gamma\cdot q_R\} \tag{3}$$

where $\Gamma \equiv \sin(\phi_s)$ and $q_R \equiv \phi_R - \phi_s$. Similarly, the bunch boundary is

$$\begin{aligned} -\tfrac{1}{2}|A|y_B^2 + B\{\cos(\phi_s+q) + \Gamma\cdot q\} &= B\{\cos(\phi_1) + \Gamma\cdot q_1\} \\ &= B\{\cos(\phi_2) + \Gamma\cdot q_2\} \end{aligned} \tag{4}$$

It is convenient to use $\hat{y} \equiv (|A|/2B)^{\frac{1}{2}}\cdot y$ instead of y so that the bunch boundary is

$$\hat{y}_B^2 = \cos(\phi_s+q) + \Gamma\cdot q - C \tag{5}$$

with

$$C \equiv \cos(\phi_1) + \Gamma\cdot q_1 = \cos(\phi_2) + \Gamma\cdot q_2 \tag{6}$$

If the total number of particles in the bunch is n_B, the distribution is

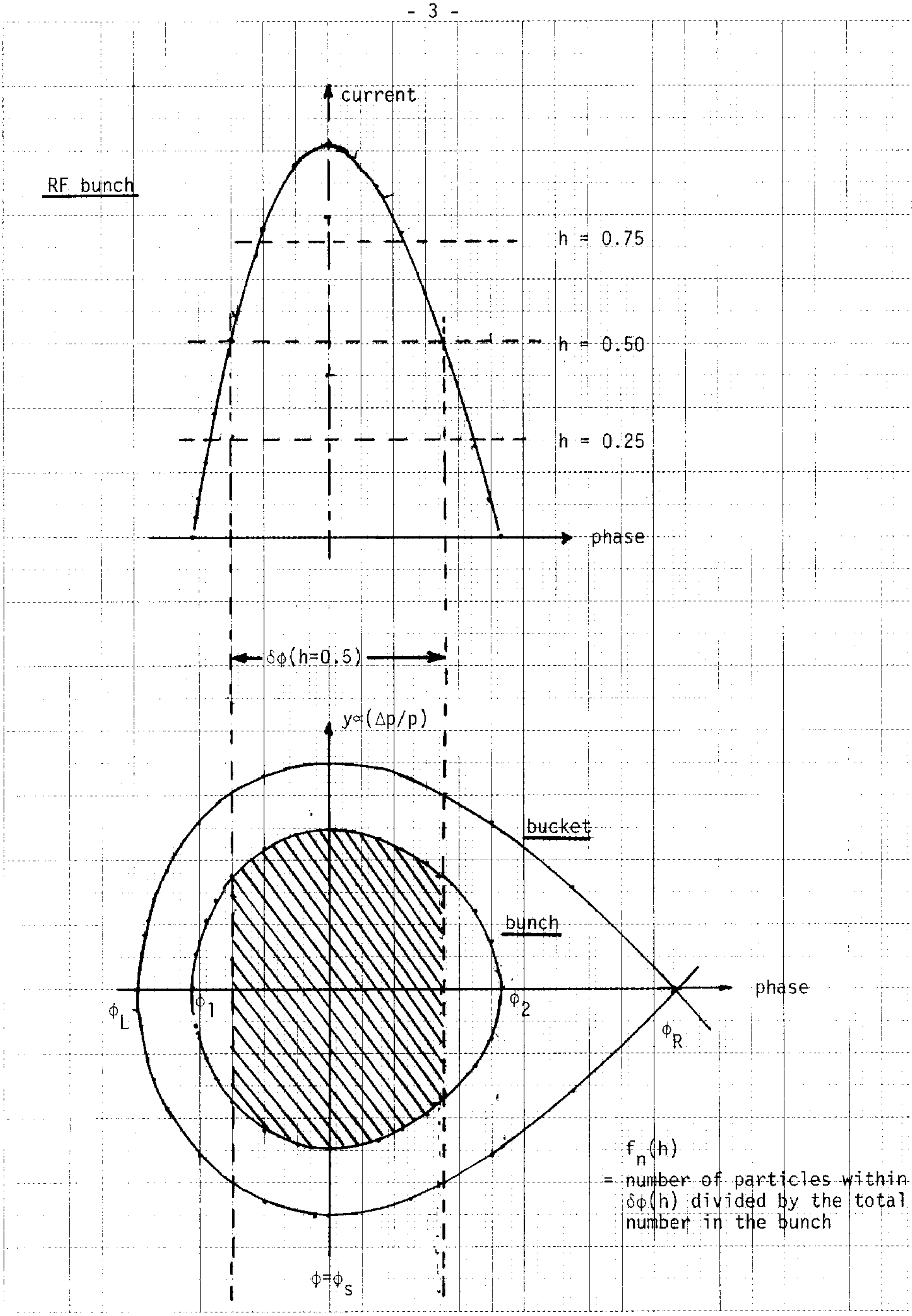
$$\rho(\hat{y},q) = n_B(2/\pi)\cdot\frac{1}{D}\{\hat{y}_B^2 - \hat{y}^2\}^{\frac{1}{2}} \tag{7}$$

with

$$D \equiv \sin(\phi_s+q) + \tfrac{1}{2}\Gamma q^2 - Cq\Big|_{q_1}^{q_2} \tag{8}$$

The local current density is

$$I(q) \propto (n_B/D)\hat{y}_B^2(q) = (n_B/D)\{\cos(\phi_s+q) + \Gamma\cdot q - C\} \tag{9}$$


current
RF bunch
h = 0.75
h = 0.50
h = 0.25
phase
δφ(h=0.5)
y∝(Δp/p)
bucket
bunch
phase
φL
φ1
φ2
φR
φ=φs
fn(h)
= number of particles within δφ(h) divided by the total number in the bunch

In particular, the maximum (local) current is*

$$I(q=0) \propto (n_B/D)\{\cos(\phi_s) - C\} \tag{10}$$

III.

We start with the assumption that the synchronous phase ϕ_s is known so that the bucket length $(\phi_R - \phi_L)$ is also known. The upper part of p. 3 is meant to be a typical picture of a bunch one sees on a scope. If we can tell where the baseline is (which is not easy most of the time!), we can find the bunch length as well as various length corresponding to the relative current density h. Remember that the current density (so that the parameter h) is proportional to y_B^2. It is easy to show that when the bunch length $(\phi_2 - \phi_1)$ is much less than the bucket length $(\phi_R - \phi_L)$,

$$\delta\phi(h) \simeq (1 - h)^{\frac{1}{2}} \times (\text{bunch length}), \tag{11}$$

and the number of particles within the phase distance $\delta\phi(h)$ relative to the total number n_B in the bunch is

$$f_n(h) \simeq (1 + h/2)(1 - h)^{\frac{1}{2}} \tag{12}$$

As the bunch length increases, these relations must be modified by a factor $(1+k)$. Fig. 1 shows k as a function of (bunch length)/(bucket length) when h = 0.5, solid curves for $\delta\phi(h=0.5)$, Eq.(11), and dashed curves for $f_n(h=0.5)$, Eq.(12). One sees that the approximate relation for f_n, Eq.(12),

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* The maximum longitudinal charge density (which is proportional to the maximum current density) is an important quantity in the discussion of space-charge detuning by the self field. It is interesting to note that, when the bunch shape is symmetric (ϕ_s=0), the maximum density of the Gaussian distribution $(n_B/\sqrt{2\pi}\sigma_z)$ is very close to the maximum density of the elliptic distribution if σ_z is interpreted to be one-quarter of the total bunch length. For the Gaussian distribution, 95% of the beam is contained within $\pm 2\sigma_z$.

is good with less than 5% error for $\Gamma = \sin(\phi_s) > 0.1$ and (bunch length) $<$ 0.9 $\times$ (bucket length).

IV.

It is often difficult to determine where the baseline is for a picture such as the one on p. 3. Fig. 1 is then not so useful in practical situations. The suggested procedures for such a case are as follows:

1) From the bunch shape, make a guess on where h is one-half. Based on this guess, find $\delta\phi(h=0.25)$, $\delta\phi(h=0.5)$ and $\delta\phi(h=0.75)$. Fig. 2 shows the ratios $\delta\phi(h=0.25)/\delta\phi(h=0.5)$ and $\delta\phi(h=0.75)/\delta\phi(h=0.5)$ as a function of $\delta\phi(h=0.5)$. If the guessed values of $\delta\phi$ for h=0.25, 0.5 and 0.75 are not consistent with these curves, try another guess until the best consistency is obtained. Since the distribution is unlikely to be exactly elliptic, this method may not always yield a unique solution.

2) Since $\delta\phi$'s for h=0.25, h=0.5 and h=0.75 are found (approximately), find the corresponding bunch length from Fig. 3. When the bunch length is comparable to the bucket length, $\delta\phi(h=0.25)$ should be used instead of $\delta\phi(h=.5)$ or $\delta\phi(h=.75)$.

3) Finally, find how many particles are contained within each phase interval $\delta\phi(h)$ using Fig. 4.

4) From the bunch length, one can estimate the phase space area of the bunch using, for example, Fig. 6 of TM-1381.

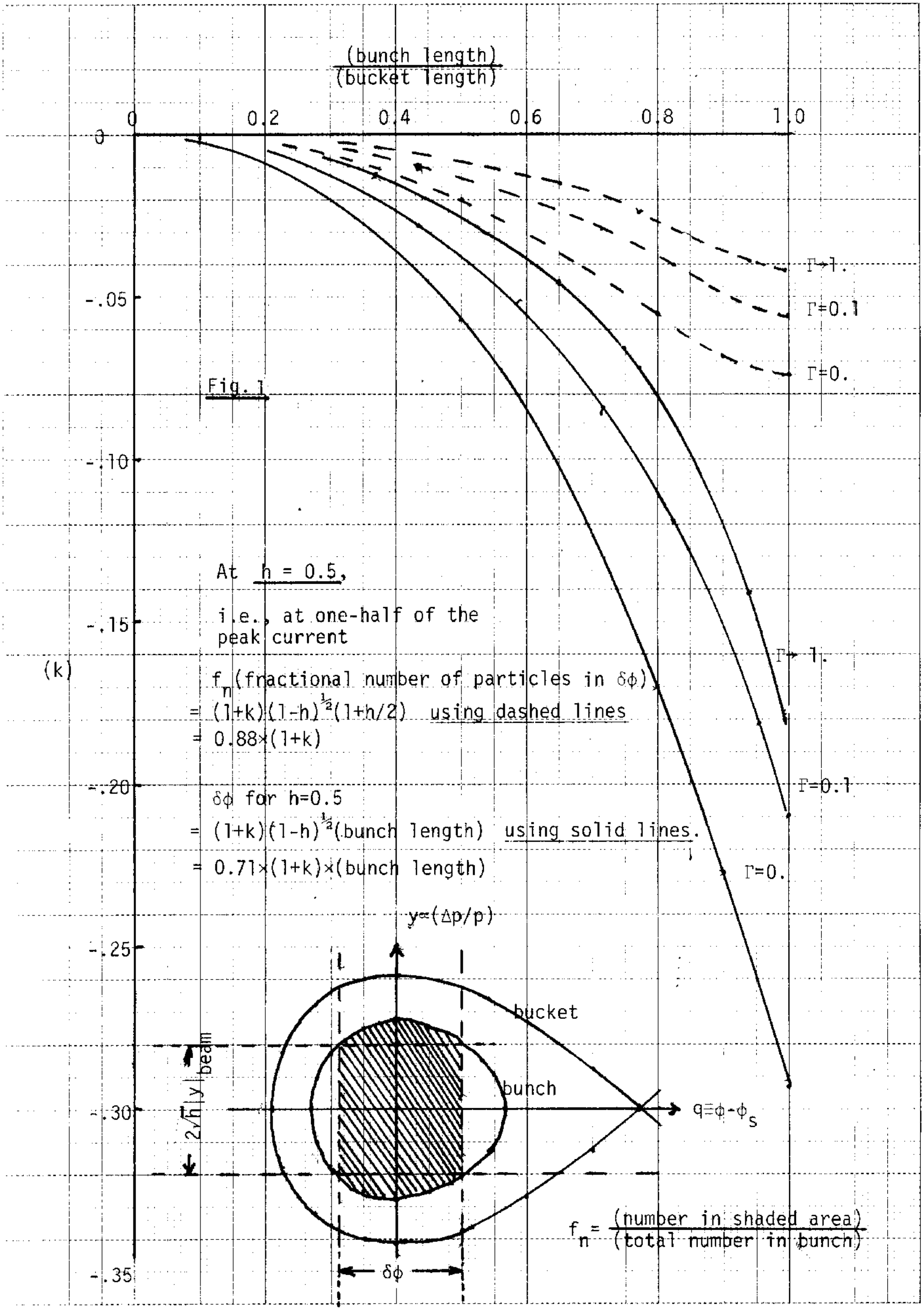

(bunch length)/(bucket length)
0
0.2
0.4
0.6
0.8
1.0
0
-.05
-.10
-.15
-.20
-.25
-.30
-.35
(k)
Γ→1.
Γ=0.1
Γ=0.
Γ→1.
Γ=0.1
Γ=0.
At h = 0.5,
i.e., at one-half of the peak current
f_n (fractional number of particles in $\delta\phi$)
$= (1+k)(1-h)^{\frac{1}{2}}(1+h/2)$ using dashed lines
$= 0.88\times(1+k)$
$\delta\phi$ for h=0.5
$= (1+k)(1-h)^{\frac{1}{2}}$(bunch length) using solid lines.
$= 0.71\times(1+k)\times$(bunch length)
$y\equiv(\Delta p/p)$
bucket
bunch
$q\equiv\phi-\phi_s$
$2\sqrt{h}|y|_{beam}$
$\delta\phi$
$f_n = \frac{\text{(number in shaded area)}}{\text{(total number in bunch)}}$

Fig. 1

Fig. 2

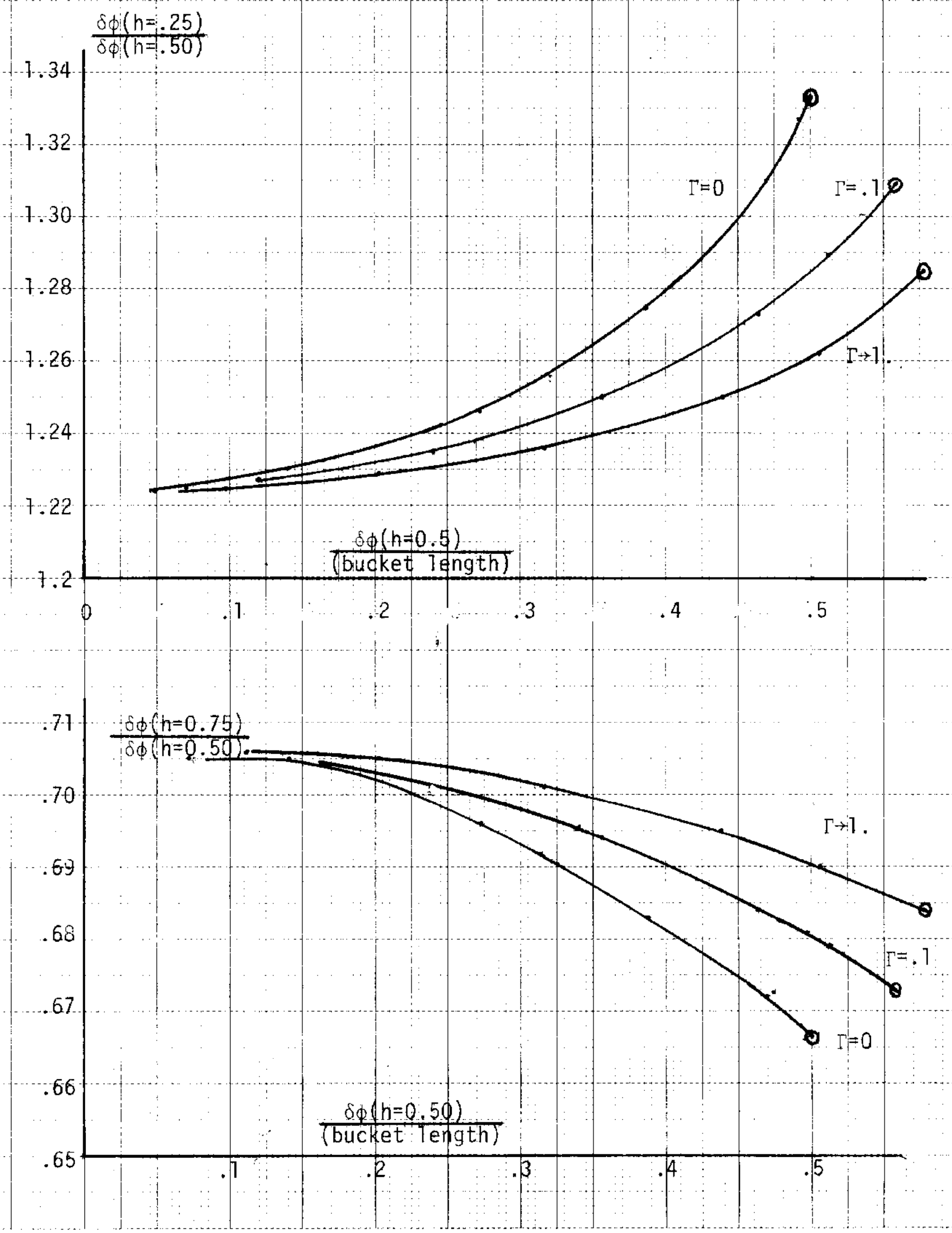
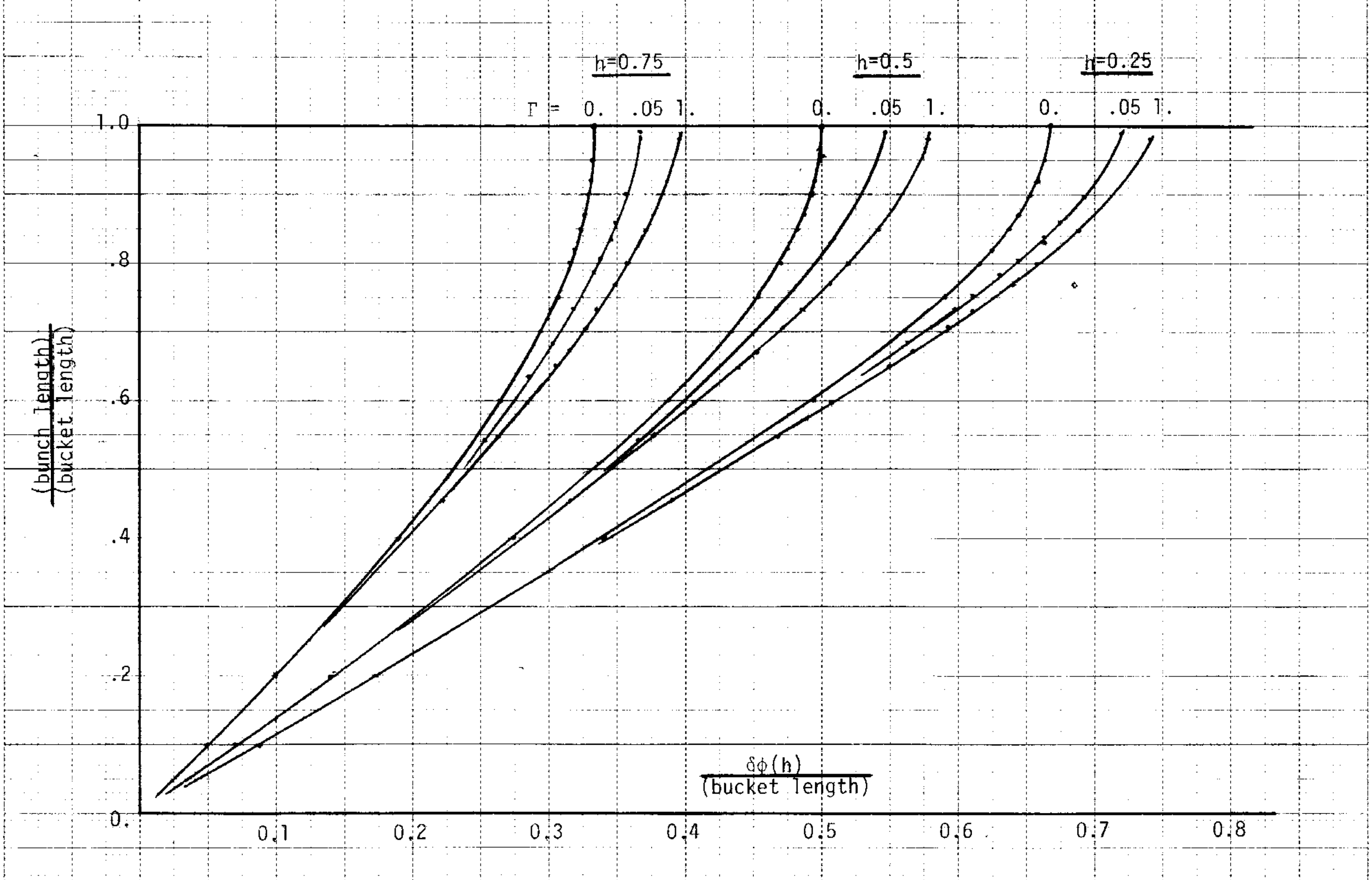

δφ(h=.25)
δφ(h=.50)
1.34
1.32
1.30
1.28
1.26
1.24
1.22
1.2
Γ=0
Γ=.1
Γ→1.
δφ(h=0.5)
(bucket length)
0
.1
.2
.3
.4
.5
.71
.70
.69
.68
.67
.66
.65
δφ(h=0.75)
δφ(h=0.50)
Γ→1.
Γ=.1
Γ=0
δφ(h=0.50)
(bucket length)
.1
.2
.3
.4
.5

Fig. 3


h=0.75
h=0.5
h=0.25
Γ = 0. .05 1.
0. .05 1.
0. .05 1.
1.0
.8
.6
.4
.2
0.
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
(bunch length) / (bucket length)
δφ(h) / (bucket length)

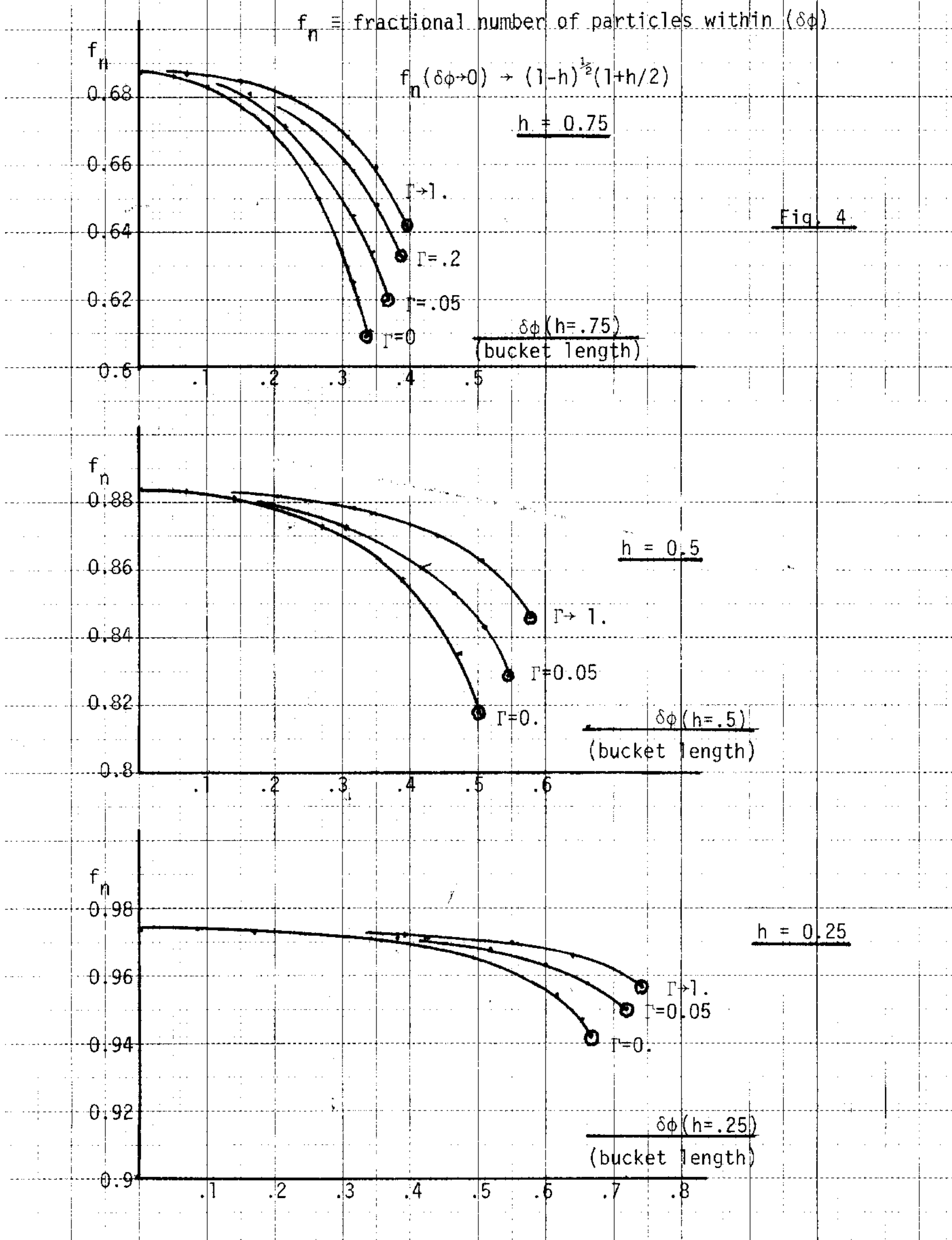

f_n ≡ fractional number of particles within ($\delta\phi$)
$f_n(\delta\phi \to 0) \to (1-h)^{1/2}(1+h/2)$
h = 0.75
f_n
0.68
0.66
0.64
0.62
0.6
.1
.2
.3
.4
.5
Γ→1.
Γ=.2
Γ=.05
Γ=0
δφ(h=.75) / (bucket length)
h = 0.5
f_n
0.88
0.86
0.84
0.82
0.8
.6
Γ→ 1.
Γ=0.05
Γ=0.
δφ(h=.5) / (bucket length)
h = 0.25
f_n
0.98
0.96
0.94
0.92
0.9
.7
.8
Γ→1.
Γ=0.05
Γ=0.
δφ(h=.25) / (bucket length)

Fig. 4