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SPACE-CHARGE FIELD IN A SPHEROID OF CHARGE

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I. Introduction

In the theory of electron cooling, the drag force involves a velocity-space integral that is formally identical to the integral over the space charge contained in an ellipsoid of revolution or spheroid. Below is presented a calculation of the interior space-charge field of this configuration. The method is similar to that used by Teng in the two-dimensional calculation of the fields of a beam of elliptical cross section.[1]

II. Calculation

Consider an ellipsoid of revolution populated by a uniform charge density en and bounded by the surface

$$\frac{(x - x_0)^2 + (y - y_0)^2}{a^2} + \frac{(z - z_0)^2}{b^2} = 1 \qquad (1)$$

The point at which the field is observed is taken as the origin; the center of the ellipsoid is then at the point (x_0, y_0, z_0). The field components are given by the appropriate Coulomb integrals over the charge distribution. In terms of spherical coordinates (ρ, θ, ϕ),

$$\begin{cases} E_x = \displaystyle\int_V \frac{en}{\rho^2} \sin^2\theta \cos\phi \, \rho^2 \, d\rho \, d\theta \, d\phi \\[2ex] E_y = \displaystyle\int_V \frac{en}{\rho^2} \sin^2\theta \sin\phi \, \rho^2 \, d\rho \, d\theta \, d\phi \\[2ex] E_z = \displaystyle\int_V \frac{en}{\rho^2} \cos\theta \sin\theta \, \rho^2 \, d\rho \, d\theta \, d\phi, \end{cases} \tag{2}$$

where V is the volume enclosed by the surface of Eq. (1). In spherical coordinates, this surface is given by an equation of the form

$$A\rho^2 + B\rho + C = 0, \tag{3}$$

where

$$\begin{cases} A = \dfrac{\sin^2\theta}{a^2} + \dfrac{\cos^2\theta}{b^2} \\[2ex] B = -2\left[\dfrac{x_0 \cos\phi \sin\theta + y_0 \sin\phi \sin\theta}{a^2} + \dfrac{z_0 \cos\theta}{b^2}\right] \\[2ex] C = \dfrac{x_0^2 + y_0^2}{a^2} + \dfrac{z_0^2}{b^2} - 1. \end{cases} \tag{4}$$

We can carry out the integration over ρ first, with the result $(\rho_2 - \rho_1)$, where ρ_1 and ρ_2 are the two limiting values of ρ for given θ and ϕ. Then $\rho_2(\theta, \phi) = \rho_1(\theta + \pi, \phi)$ and, from Eq. (3), we have

$$\rho_2 - \rho_1 = B/A. \tag{5}$$

Then we can integrate over ϕ, using the value of $(\rho_2 - \rho_1)$ from Eqs. (4) and (5). We have

$$
\left\{
\begin{aligned}
E_x &= \pi e n x_0 \int_0^{\pi} \frac{\sin^3\theta \, d\theta}{\sin^2\theta + \frac{a^2}{b^2}\cos^2\theta} = \pi e n x_0 I_t \\
E_y &= \pi e n y_0 \int_0^{\pi} \frac{\sin^3\theta \, d\theta}{\sin^2\theta + \frac{a^2}{b^2}\cos^2\theta} = \pi e n y_0 I_t \\
E_z &= 2\pi e n z_0 \int_0^{\pi} \frac{\sin\theta \cos^2\theta \, d\theta}{\frac{b^2}{a^2}\sin^2\theta + \cos^2\theta} = 2\pi e n z_0 I_z .
\end{aligned}
\right.
\tag{6}
$$

The integrals I_t and I_z are well-known. They are

$$
\begin{aligned}
I_t &= 2\left[\frac{a^2 b}{(a^2-b^2)^{3/2}} \tan^{-1}\frac{\sqrt{a^2-b^2}}{b} - \frac{b^2}{a^2-b^2}\right]; \quad a > b \\
&= \frac{4}{3} \qquad ; \quad a = b \\
&= 2\left[\frac{a^2 b}{(b^2-a^2)^{3/2}} \tanh^{-1}\frac{\sqrt{b^2-a^2}}{b} + \frac{b^2}{b^2-a^2}\right]; \quad a < b
\end{aligned}
\tag{7a}
$$

$$
\begin{aligned}
I_z &= 2\left[\frac{-a^2 b}{(a^2-b^2)^{3/2}} \tan^{-1}\frac{\sqrt{a^2-b^2}}{b} + \frac{a^2}{a^2-b^2}\right]; \quad a > b \\
&= \frac{2}{3} \qquad ; \quad a = b \\
&= 2\left[\frac{-a^2 b}{(b^2-a^2)^{3/2}} \tanh^{-1}\frac{\sqrt{b^2-a^2}}{b} - \frac{a^2}{b^2-a^2}\right] \quad ; \quad a < b
\end{aligned}
\tag{7b}
$$

III. Results

We can then write the result in Cartesian coordinates (with x, y, z the distances from the ellipsoidal center) as

$$\begin{cases} E_x = \pi enF_r x \\ E_y = \pi enF_r y \\ E_z = \pi enF_z z \end{cases} \tag{8}$$

or in cylindrical coordinates (r, θ, z) as

$$\begin{cases} E_r = \pi enF_r r \\ E_\theta = 0 \\ E_z = \pi enF_z z \end{cases} \tag{9}$$

A few values of F_r and F_z are tabulated below.

a > b (oblate spheroid)

$\frac{b}{a}$	F_r	F_z
0	0	4
0.2	0.49903	3.00194
0.4	0.82473	2.35054
0.6	1.04919	1.90162
0.8	1.21112	1.57775
1.0	1.33333 $\left(\frac{4}{3}\right)$	1.33333 $\left(\frac{4}{3}\right)$

a < b (prolate spheroid)

$\frac{a}{b}$	F_r	F_z
0	2	0
0.2	1.88835	0.22330
0.4	1.72971	0.54059
0.6	1.58008	0.83985
0.8	1.44802	1.10397
1.0	1.33333 $\left(\frac{4}{3}\right)$	1.33333 $\left(\frac{4}{3}\right)$

We may note that the results given in Eq. (8) or Eq. (9) identically satisfy $\nabla \cdot \vec{E} = 4\pi en$.

IV. Approximate Formulas

In the limit of a very short (oblate) spheroid ($b << a$), we can expand in powers of b/a and find

$$\begin{cases} E_r = \frac{b}{a} \cdot \pi^2 enr \\ E_z = 4\pi enz\left(1 - \frac{\pi}{2}\frac{b}{a}\right), \end{cases} \tag{10}$$

the field in a thin disc of charge.

In the opposite limit of a very long (prolate) spheroid ($a << b$), we can expand in powers of a/b and find

$$\begin{aligned} E_r &= 2\pi enr\left\{1 + \frac{a^2}{b^2}\left[\tanh^{-1}\left(\sqrt{1 - \frac{a^2}{b^2}}\right) + 1\right]\right\} \\ E_z &= -4\pi enr\,\frac{a^2}{b^2}\left(1 + \tanh^{-1}\sqrt{1 - \frac{a^2}{b^2}}\right), \end{aligned} \tag{11}$$

a further approximation beyond the usual infinitely long beam result $E_r = 2\pi enr$, $E_z = 0$.

Reference

1. L. C. Teng, Transverse Space Charge Effects, Argonne National Laboratory Accelerator Division Report ANLAD-59, Feb. 1, 1960.