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THIRD-ORDER CHARGED PARTICLE BEAM OPTICS

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THIRD-ORDER CHARGED PARTICLE BEAM OPTICS

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ABSTRACT

The motion of a charged particle through a magnetic field configuration can be described in terms of deviation from a certain ideal trajectory. One uses power series expansion of the phase-space coordinates to obtain the transfer matrices for a particular optical system.

In this thesis we present a complete third-order theory of computing transfer matrices and apply it to magnetic elements in an accelerator beam-line. A particular attention is devoted to studying particles' orbits in an extended fringing field of a dipole magnet. Analytical solutions are obtained up to the third order in the formalism of the matrix theory. They contain form factors describing the fall-off pattern of the field. These form factors are dimensionless line integrals of the field strength and its derivative. There is one such integral in the first-order solution, two in the second, and nine in the third.

An alternate way of describing charged particle optics is also presented. It is based on a Hamiltonian treatment and uses certain symplectic operators, which are defined in terms of Poisson brackets, to parametrize the transfer map of a system. We apply this approach to the fringing field problem and obtain a third-order solution. We furthermore show how to convert this solution into conventional transfer matrices by examining the connection between the non-canonical matrix theory and the Hamiltonian description.

TO MY FAMILY

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PREFACE

The Chapters in this dissertation are arranged as follows:

- Chapter 1 gives an overview of the charged particle beam optics, its methods and applications. Here, we define the curvilinear coordinate system, derive the equations of motion, and show how to obtain a general third-order transfer matrix solution for mid-plane symmetric magnetic elements. We also obtain a formula for concatenating nonlinear matrices to obtain the transport map for a system of elements.

- Chapter 2 deals with application of general methods to ideal multipole magnets. First-, second-, and third-order transfer matrices are calculated for dipole, quadrupole, sextupole, and octupole in the formalism of a computer code TRANSPORT.

- Chapter 3 is the main part of the thesis. It gives a solution to the problem of determining the effects of an extended fringe field of a dipole magnet. The method of solution is detailed and the field parametrization scheme is described. All the calculations leading to obtaining the transport matrices through the third order are carried out here.

- Chapter 4 introduces Hamiltonian methods, which are based on Lie Algebraic operators defined in terms of Poisson brackets. Relevant theorems are given and the results are applied to the dipole fringe field problem. Again, the third-order solution is obtained and a procedure to obtain TRANSPORT matrices from the Lie Algebraic solution is described.

- The lonely Appendix applies the transfer matrix methods to the case of a curved boundary.

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Chapter 1

General Theory of Charged Particle Beam Optics

1.1 Introduction

Beams of charged particles are used in nuclear and high-energy experiments to explore the structure of matter. Magnetic fields guide and focus the beam onto the target and additional magnetic fields are used to analyze the products of microscopic collisions.

The optics of charged particle beams studies the effect of magnetic fields on the paths of a group of moving charged particles. A beam of particles can be influenced in a way similar to the focusing of light by an optical lens or the dispersion into colors by a prism. It may be changed in direction, brought together to a small spot, or have its particles separated by momentum. The beam line is an optical system with different types of magnets accomplishing these different functions.

Given a magnetic element, to understand the *optics* means to relate the coordinates and momenta of a particle before and after its passage through this element, as shown schematically in Fig. 1.1. The relationship, or the *transfer map*, can be used to design an optical device with various desired properties.

The relation between the initial and final quantities can always in principle be obtained by integrating numerically the equations of motion. In general, however, numerical integration is slow if performed for a large number of rays. Moreover, ray tracing does not necessarily give much insight into the behavior of an optical system or what could be done to change it. A common procedure, first developed in [1], is to seek an expansion of the final quantities in the form of a power series in the initial quantities.

A charged particle in a magnetic field $\mathbf{B}$ experiences a Lorenz force $\mathbf{F}$

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \tag{1.1}$$

Here q is the charge of the particle, and $\mathbf{v}$ is its velocity. We can use Eq. 1.1 to formulate a differential equation of motion with time as the independent variable. Given an initial position and velocity, the equation can be solved for the particle's position as a function of time. In charged particle optics, however, one is not so much interested in finding a particle's position as a function of time as in determining the path it follows. The independent variable is then taken to be the distance along a particular one of the many trajectories passing down the beam line, the so called *reference trajectory.*

The positions and momenta of the other particles may be defined in terms of that of the reference particle. We will use the variable t to denote distance along the reference trajectory. At any point on the reference trajectory, longitudinal or t axis is defined in the direction of the reference velocity. The two transverse coordinates x and y are orthogonal to t axis, with x horizontal and y vertical. The general description of this moving coordinate system is given in Section 1.2.

Transverse distances from the reference trajectories x and y are two of the spatial coordinates of the phase space. The third coordinate is l, the difference in path length between a given trajectory and the reference trajectory. Three quantities specifying the momentum of the particle are chosen to be the two direction tangents, x' and y', and the fractional deviation from the reference momentum, δ. The direction tangents are the ratios of transverse to longitudinal momentum components,

$$x' = \frac{p_x}{p_t} \quad \text{and} \quad y' = \frac{p_y}{p_t} \tag{1.2}$$

The fractional momentum deviation is given by

$$p = p_0(1 + \delta) \tag{1.3}$$

where p_0 is the reference momentum.

In a uniform magnetic field, a charged particle whose velocity vector is perpendicular to the field moves in a circular path. The radius ρ of the circle is related to the field B, the momentum p, and the charge q, in the units of high energy physics, by

$$3.3356p[GeV/c] = B[Tesla]\rho[meters] \tag{1.4}$$

Using Eq. 1.4 for the reference momentum, one determines the field of the bending magnets in the beam line. The bending magnets, or dipoles, are those that change the direction of the reference trajectory. They are used to transmit the particles to a specific location and also to determine different ranges of momenta present in the beam.

Other common beam line components are quadrupoles and sextupoles, whose cross sections are shown in Fig. 1.2 together with that of a dipole. Both possess a magnetic axis where the field is zero and are usually placed in the beam line so that this axis is along the reference trajectory. The quadrupole field strength depends linearly on the distance from the axis. Quadrupoles are used for focusing a beam of particles. The sextupole field depends quadratically on the distance from the axis. Sextupoles are used to correct for the momentum dependence of the focusing strength of quadrupoles. Higher order multipoles are also sometimes used in the beam lines for higher order corrections, as well as "combined function" elements that have more than one magnetic multipole component.

By analogy with the light optics, the beam optics can be classified by orders. In a common procedure, the transfer map of a magnetic element is sought as a power series expansion of the final phase space coordinates $(x, x', y, y', l, \delta)$ in terms of the initial ones. In the lowest order, this approach yields the linear matrix approximation of paraxial optics. The higher (nonlinear) terms in such an expansion provide a description in terms of aberration coefficients, whose number grows rapidly with order although not all of them are independent.

We start the detailed discussion of the matrix theory by describing the basic assumptions about the beam line elements and introducing the moving coordinate system.

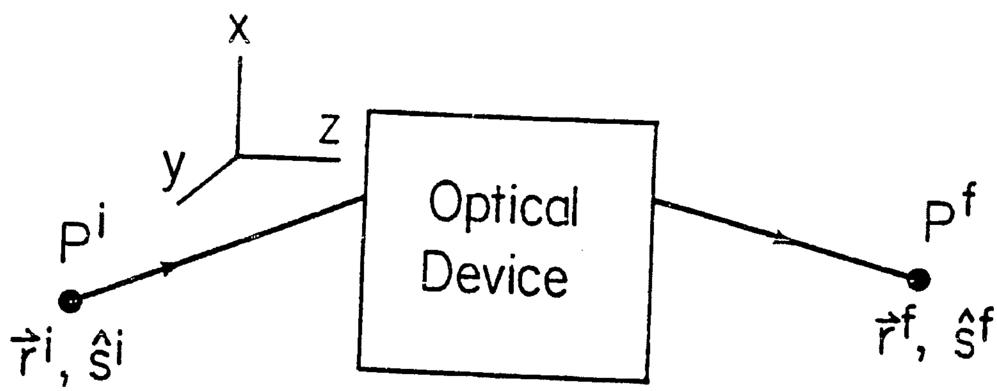

Figure 1.1: An optical system consisting of an optical device preceded and followed by simple transit. The relationship between initial and final coordinates is described by a transfer map.

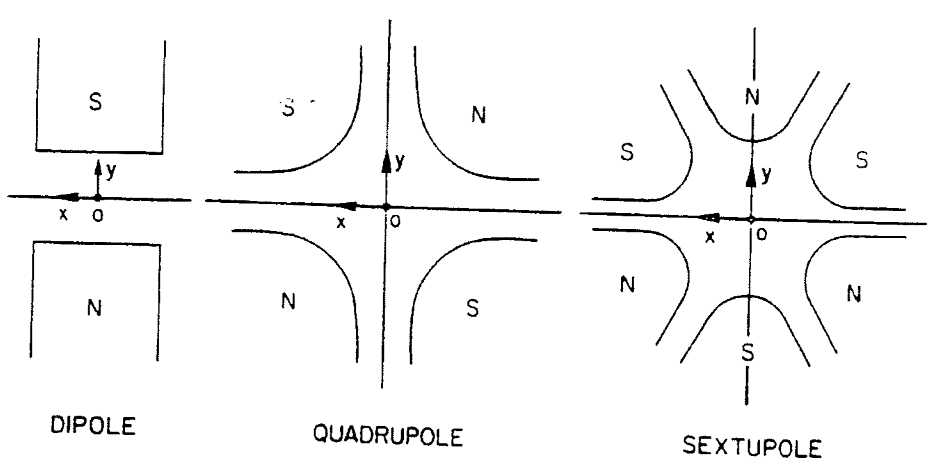

Figure 1.2: Cross sections for dipole, quadrupole, and sextupole magnets, respectively. The magnetic midplane and local coordinate system are shown.

1.2 Curvilinear Coordinate System

Designs of charged particle optical systems are invariably based on two assumptions:

1. The existence of a reference trajectory;

2. Midplane symmetry (except for solenoids).

The reference trajectory is taken to be the path of a charged particle with certain specified initial conditions. The magnetic elements are laid out along this trajectory, and a particle with the reference momentum which follows the reference trajectory initially will continue to do so through all the magnetic elements. It will pass along the axes of quadrupoles, sextupoles, and octupoles and experience a uniform magnetic field in a combined function magnet. Therefore, through a bending magnet the reference trajectory is an arc of a circle, while through all other magnetic elements it is a straight line. Thus, it is ssummed to have a piecewise constant curvature.

The midplane symmetry restriction means that relative to a plane designated as the magnetic midplane, the magnetic scalar potential ϕ is an odd function in the transverse coordinate y, i. e. $\phi(x,y,t) = -\phi(x,-y,t)$. This restriction greatly simplifies the calculations; and from experience in designing beam transport systems, it appears that for most applications there is little, if any, advantage to be gained from a more complicated field pattern. For particle accelerators or nuclear physics spectrometers, midplane symmetry is often assumed for the system in its entirety. For systems such as the SLAC linear collider, or the Fermilab beam transfer system, midplane symmetry may apply separately to parts of the entire system. At the very least, each magnetic element is taken to be midplane symmetric.

Because of the symmetry, the only non-zero component of the magnetic field on the midplane is the component perpendicular to that plane. Eq. 1.1 implies that a particle starting on the midplane will never leave it. In particular, the reference trajectory is confined to the symmetry plane of an element.

The general right-hand curvilinear coordinate system (x,y,t) is shown in Fig. 1.3. A point O on the reference trajectory is taken to be the origin. A point A is specified by the

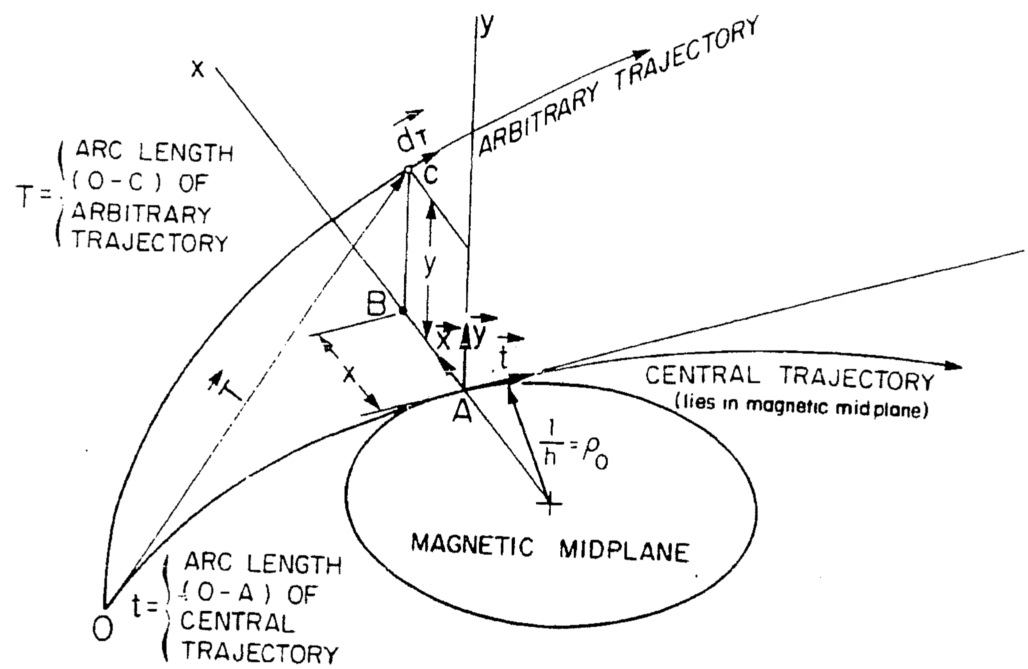

Figure 1.3: Curvilinear coordinate system. Unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{t}}$ "move" along the reference trajectory.

arc length t measured along the reference curve from O to A. To specify an arbitrary point B lying in the midplane, one constructs a line segment from B to the reference trajectory intersecting the latter perpendicularly at A: the point A provides one coordinate t, the length of the segment BA is the second coordinate x. To specify an off the midplane point C, one constructs a line segment from C to the plane, intersecting the latter perpendicularly at B: the point B provides the two coordinates t and x, the length of the segment CB is the third coordinate y.

Three mutually perpendicular unit vectors $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{t}})$ are defined as follows. At a point A, $\hat{\mathbf{t}}$ is tangent to the reference trajectory and is directed in the direction of motion of particles; $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are perpendicular to $\hat{\mathbf{t}}$, and are respectively parallel and perpendicular to the symmetry plane. The unit vectors $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{t}})$ constitute a local right-handed coordinate system and satisfy the relations:

$$\begin{aligned} \hat{\mathbf{x}} &= \hat{\mathbf{y}} \times \hat{\mathbf{t}} \\ \hat{\mathbf{y}} &= \hat{\mathbf{t}} \times \hat{\mathbf{x}} \\ \hat{\mathbf{t}} &= \hat{\mathbf{x}} \times \hat{\mathbf{y}} \end{aligned} \tag{1.5}$$

Using the prime to denote the derivative with respect to the independent variable t, we may write the equations describing the rate of change of the unit vectors:

$$\begin{aligned} \hat{\mathbf{x}}' &= h\hat{\mathbf{t}} \\ \hat{\mathbf{y}}' &= 0 \\ \hat{\mathbf{t}}' &= -h\hat{\mathbf{x}} \end{aligned} \tag{1.6}$$

where $h = h(t)$ is the curvature, equal to the reciprocal of the reference trajectory's local radius ρ.

At any specified position in the system, an arbitrary charged particle can be represented by a six-dimensional phase space vector $\mathbf{Z} \equiv (x, x', y, y', l, \delta)$.

We are now ready to derive the equations of motion in the curvilinear coordinates we have defined.

1.3 Equations of Motion

The equations of motion for transverse coordinates are derived from Eq. 1.1, equating the Lorenz force to the time rate of change of the momentum and eliminating time in favor of the distance parameter t.

Let v be the speed of the particle, p its momentum magnitude, $\mathbf{T}$ its position vector, and T the distance traversed on its own trajectory; the last two are shown in Fig. 1.3. Then, the velocity and momentum vectors are equal to $vd\mathbf{T}/dT$ and $pd\mathbf{T}/dT$ respectively, where $d\mathbf{T}/dT$ is a unit vector tangent to the particle trajectory. Since the Lorenz force is always perpendicular to the direction of motion, p is a constant of motion and the equation becomes

$$\frac{d^2\mathbf{T}}{dT^2} = \frac{q}{p}\frac{d\mathbf{T}}{dT} \times \mathbf{B} \tag{1.7}$$

Next, we transform Eq. 1.7 to the curvilinear coordinate system. To this end, we may write

$$\frac{d\mathbf{T}}{dT} = \frac{d\mathbf{T}/dt}{dT/dt} = \frac{d\mathbf{T}'}{dT'}$$

$$\frac{d^2\mathbf{T}}{dT^2} = \frac{1}{T'}\frac{d}{dt}\left(\frac{d\mathbf{T}'}{dT'}\right)$$

or

$$(T')^2\frac{d^2\mathbf{T}}{dT^2} = \mathbf{T}'' - \frac{1}{2}\frac{\mathbf{T}'}{(T')^2}\frac{d}{dt}(T')^2$$

The resulting equation of motion becomes

$$\mathbf{T}'' - \frac{1}{2}\frac{\mathbf{T}'}{(T')^2}\frac{d}{dt}(T')^2 = \frac{q}{p}T'(\mathbf{T}' \times \mathbf{B}) \tag{1.8}$$

In this coordinate system, the differential line element is given by

$$d\mathbf{T} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + (1 + hx)\hat{\mathbf{t}}dt$$

and

$$(dT)^2 = d\mathbf{T} \cdot d\mathbf{T} = dx^2 + dy^2 + (1 + hx)^2 dt^2$$

By taking d/dt of the above equations, it follows that

$$(T')^2 = (x')^2 + (y')^2 + (1 + hx)^2$$

$$\frac{1}{2}\frac{d}{dt}(T')^2 = x'x'' + y'y'' + (1+hx)(hx' + h'x)$$
$$\mathbf{T}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} + (1+hx)\hat{\mathbf{t}}$$

and

$$\mathbf{T}'' = \hat{\mathbf{x}}x'' + \hat{\mathbf{x}}'x' + \hat{\mathbf{y}}y'' + \hat{\mathbf{y}}'y' + \hat{\mathbf{t}}'(1+hx) + \hat{\mathbf{t}}(hx' + h'x)$$

Use of the relations of Eq. 1.5, reduces the expression for $\mathbf{T}''$ to

$$\mathbf{T}'' = \hat{\mathbf{x}}\left[x'' - h(1+hx)\right] + \hat{\mathbf{y}}y'' + \hat{\mathbf{t}}\left[2hx' + h'x\right]$$

The Eq. 1.8 may now be separated into its component parts,

$$\begin{aligned}
& \hat{\mathbf{x}}\left\{[x'' - h(1+hx)] - \frac{x'}{(T')^2}\left[x'x'' + y'y''(1+hx)(hx'+h'x)\right]\right\} \\
+ \;& \hat{\mathbf{y}}\left\{y'' - \frac{y'}{(T')^2}\left[x'x'' + y'y''(1+hx)(hx'+h'x)\right]\right\} \\
+ \;& \hat{\mathbf{t}}\left\{(2hx' + h'x) - \frac{(1+hx)}{(T')^2}\left[x'x'' + y'y''(1+hx)(hx'+h'x)\right]\right\} \\
= \;& \frac{q}{p}T'\left\{\hat{\mathbf{x}}\left[y'B_t - (1+hx)B_y\right] + \hat{\mathbf{y}}\left[(1+hx)B_x - x'B_t\right] + \hat{\mathbf{t}}\left[x'B_y - y'B_x\right]\right\}
\end{aligned} \tag{1.9}$$

where

$$T' = \sqrt{(x')^2 + (y')^2 + (1+hx)^2}$$

The equation of motion for the reference particle is readily obtained from Eq. 1.9 by setting x, x', y, and y' equal to zero and momentum magnitude to p_0. One obtains:

$$h(t) \equiv \frac{1}{\rho} = \frac{q}{p_0}B_y(0,0,t) \quad \text{or} \quad B\rho = \frac{p_0}{q} \tag{1.10}$$

We now discuss the "order-by-order" solution to Eq. 1.9.

1.4 Aberration Expansion and Transfer Matrices

Our phase space variables describe the *deviations* from the reference trajectory, given by Eq. 1.10. Assuming the deviations are small, we may seek the solution to Eq 1.9 as a power series in the initial conditions.

Let $\mathbf{Z}^0$ be a six-dimensional vector of initial conditions. Then, the solution vector $\mathbf{Z}$ at a point t can be written as

$$Z_i(t) = \underbrace{\sum_{j=1}^{6} R_{ij}(t) Z_j^0}_{6\ \text{terms}} + \underbrace{\sum_{j=1}^{6}\sum_{k=j}^{6} T_{ijk}(t) Z_j^0 Z_k^0}_{21\ \text{terms}} + \underbrace{\sum_{j=1}^{6}\sum_{k=j}^{6}\sum_{l=k}^{6} U_{ijkl}(t) Z_j^0 Z_k^0 Z_l^0}_{56\ \text{terms}} + \cdots \tag{1.11}$$

Eq. 1.11 can be viewed as a transfer map from $\mathbf{Z}^0$ to $\mathbf{Z}$. Aberration coefficients $R_{ij}(t)$, $T_{ijk}(t)$, and $U_{ijkl}(t)$ form first, second, and third order *transfer matrices* respectively. They determine the optical properties of a system.

To obtain the transfer matrices to a desired order, the following procedure is followed:

1. Magnetic field $\mathbf{B}(x, y, t)$ is expanded around the reference trajectory using Maxwell's equations and midplane symmetry;
2. Eq. 1.9 is expanded to the desired order;
3. Eq. 1.11 is substituted into the expanded equations;
4. Equating coefficients of the same terms in initial conditions, one obtains differential equations for the transfer matrix coefficients;
5. The n^{th} order coefficients satisfy linear second-order (in terms of the order of differentiation) equations characteristic of forced harmonic oscillators, with the "driving terms" involving coefficients of order $(n-1)$ and lower. It is, therefore, possible to obtain solutions in a systematic order-by-order way, employing Green's function integration of the equations.

In the next section, we carry out the above procedure to the third order.

1.5 Third Order Optics

1.5.1 Field Expansion

The static magnetic field in vacuum may be expressed in terms of a scalar potential ϕ by

$$\mathbf{B} = \nabla\phi \tag{1.12}$$

We will expand the scalar potential in the curvilinear coordinates about the reference trajectory.

The existence of the median symmetry requires that ϕ be an odd function of y,

$$\phi(x,y,t) = -\phi(x,-y,t) \tag{1.13}$$

The most general expanded form of ϕ may be expressed as follows:

$$\phi(x,y,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A_{2i+1,j}(t)\frac{x^j}{j!}\frac{y^{2i+1}}{(2i+1)!} \tag{1.14}$$

The Laplace equation, which ϕ satisfies, has the following form in the (x,y,t) coordinates,

$$\nabla^2\phi = \frac{1}{(1+hx)}\frac{\partial}{\partial x}\left[(1+hx)\frac{\partial\phi}{\partial x}\right] + \frac{\partial^2\phi}{\partial y^2} + \frac{1}{(1+hx)}\frac{\partial}{\partial t}\left[\frac{1}{(1+hx)}\frac{\partial\phi}{\partial t}\right] = 0 \tag{1.15}$$

Substitution of Eq. 1.14 into Eq. 1.15 gives the recursion formula for the coefficients,

$$\begin{aligned} -A_{2i+3,j} &= A''_{2i+1,j} + jhA''_{2i+1,j-1} - jh'A'_{2i+1,j+2} \\ &\quad (3j+1)hA_{2i+1,j+1} + j(3j-1)h^2A_{2i+1,j} + n(n-1)^2h^3A_{2i+1,j-1} \\ &\quad + 3jhA_{2i+3,j-1} + 3j(j-1)h^2A_{2i+3,j-2} + j(j-1)(j-2)h^3A_{2i+3,j-3} \end{aligned} \tag{1.16}$$

Eq. 1.16 expresses all the coefficients in terms of the midplane field

$$B_y(x,0,t) = \sum_{j}^{\infty} A_{1,j}\frac{x^j}{j!} \tag{1.17}$$

where

$$A_{1,j} = \left(\frac{\partial^j B_y}{\partial x^j}\right)\Bigg|_{x=y=0} \tag{1.18}$$

The x-direction derivatives of B_y on the reference trajectory define B_y over the entire median plane, hence the field **B** over the whole space. The field components are given by

$$\begin{aligned} B_x(x,y,t) &= \frac{\partial\phi}{\partial x} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A_{2i+1,j+1}(t)\frac{x^j}{j!}\frac{y^{2i+1}}{(2i+1)!} \\ B_y(x,y,t) &= \frac{\partial\phi}{\partial y} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A_{2i+1,j}(t)\frac{x^j}{j!}\frac{y^{2i}}{(2i)!} \\ B_t(x,y,t) &= \frac{1}{(1+hx)}\frac{\partial\phi}{\partial t} = \frac{1}{(1+hx)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A'_{2i+1,j}(t)\frac{x^j}{j!}\frac{y^{2i+1}}{(2i+1)!} \end{aligned} \tag{1.19}$$

The coefficients up to the fourth degree terms in x and y are given explicitly below from Eq. 1.16,

$$
\begin{aligned}
A_{30} &= -A''_{10} - A_{12} - hA_{11} \\
A_{31} &= -A''_{11} + 2hA''_{10} + h'A'_{10} - A_{13} - hA_{12} + h^2 A_{11}
\end{aligned}
\tag{1.20}
$$

If the field expansion is terminated with the third order terms, we obtain from Eq. 1.19

$$
\begin{aligned}
B_x &= A_{11}y + A_{12}xy + \frac{1}{2}A_{13}x^2y + \frac{1}{6}A_{31}y^3 + \cdots \\
B_y &= A_{10} + A_{11}x + \frac{1}{2}A_{12}x^2 + \frac{1}{2}A_{30}y^2 + \frac{1}{6}A_{13}x^3 + \frac{1}{2}A_{31}xy^2 + \cdots \\
B_t &= A'_{10}y + A'_{11}xy - A'_{10}hxy + \frac{1}{2}A'_{12}x^2 - A'_{11}hx^2y + A'_{10}h^2x^2y + \frac{1}{6}A'_{30}y^3 + \cdots
\end{aligned}
\tag{1.21}
$$

It is evident that B_x, B_y, and B_t are all expressed in terms of A_{10}, A_{11}, A_{12}, A_{13} and their derivatives with respect to t. Consider B_y on the midplane only,

$$
\begin{aligned}
B_y(x,0,t) &= A_{10} + A_{11}x + \frac{1}{2}x^2A_{12} + \frac{1}{6}x^3A_{13} + \cdots \\
&= \underbrace{B_y\big|_{x=y=0}}_{\text{dipole}} + \underbrace{\frac{\partial B_y}{\partial x}\bigg|_{x=y=0} x}_{\text{quadrupole}} + \underbrace{\frac{1}{2}\frac{\partial^2 B_y}{\partial x^2}\bigg|_{x=y=0} x^2}_{\text{sextupole}} + \underbrace{\frac{1}{6}\frac{\partial^3 B_y}{\partial x^3}\bigg|_{x=y=0} x^3}_{\text{octupole}} + \underbrace{\cdots}_{\text{etc.}}
\end{aligned}
$$

The successive derivatives identify the terms as being dipole, quadrupole, sextupole, octupole, etc. in the field expansion. It is useful to express the midplane field in terms of dimensionless "multipole strength" coefficients $k_1(t)$, $k_2(t)$, $k_3(t)$,

$$
B_y(x,0,t) = B_y(0,0,t)\left[1 - k_1hx + k_2h^2x^2 + k_3h^3x^3 + \cdots\right]
\tag{1.22}
$$

with

$$
\begin{aligned}
k_1 &= -\left[\frac{1}{hB_y}\frac{\partial B_y}{\partial x}\right]\bigg|_{x=y=0} \\
k_2 &= \left[\frac{1}{2h^2B_y}\frac{\partial^2 B_y}{\partial x^2}\right]\bigg|_{x=y=0} \\
k_3 &= \left[\frac{1}{6h^3B_y}\frac{\partial^3 B_y}{\partial x^3}\right]\bigg|_{x=y=0}
\end{aligned}
\tag{1.23}
$$

We now make use of Eq. 1.10; to the third order the expansions for the magnetic field components become

$$
\begin{aligned}
B_x(x,y,t) &= \frac{p_0}{q}\Big[-h^2k_1y + 2h^3k_2xy + 3h^4k_3x^2y \\
&\quad + (-h^4k_3 - \frac{1}{3}h^4k_2 + \frac{2}{3}hh'k_1' + \frac{1}{6}h^2k_1'' \\
&\quad + \frac{1}{3}(h')^2k_1 + \frac{1}{3}hh''k_1 - \frac{1}{6}h^4k_1 + \frac{1}{6}(h')^2 + \frac{1}{3}hh'')y^3 + \cdots\Big] \\
B_y(x,y,t) &= \frac{p_0}{q}\Big[h - h^2k_1x + h^3k_2x^2 + h^4k_3x^3 + (-h^3k_2 + \frac{1}{2}h^3k_1 - \frac{1}{2}h'')y^2 \\
&\quad + (-3h^4k_3 - h^4k_2 + 2hh'k_1' + \frac{1}{2}h^2k_1'' \\
&\quad + (h')^2k_1 + hh''k_1 - \frac{1}{2}h^4k_1 + \frac{1}{2}(h')^2 + hh'')xy^2 + \cdots\Big] \\
B_t(x,y,t) &= \frac{p_0}{q}\Big[h'y + (-h^2k_1' - 2hh'k_1 - hh')xy \\
&\quad + (-\frac{1}{3}h^3k_2' - h^2h'k_2 + \frac{1}{6}h^3k_1' + \frac{1}{2}h^2h'k_1)y^3 \\
&\quad + (h^3k_2' + 3h^2h'k_2 + h^3k_1' + 2h^2h'k_1 + h^2h')x^2y + \cdots\Big]
\end{aligned}
\tag{1.24}
$$

1.5.2 Equations Expanded to Third Order

If we retain only terms through third order in x and y and their derivatives in Eq. 1.9, the x and y components of the equation of motion become

$$
\begin{aligned}
x'' - h(1+hx) - x'(hx' + h'x) + hh'x^2x' + h^2x(x')^2 & \\
- (1-2hx)(x''(x')^2 + y''x'y') &= \frac{q}{p}T'\left[y'B_t - (1+hx)B_y\right] \\
y'' - y'(hx' + h'x) + y'(hh'x^2 + h^2xx') & \\
- (1-2hx)(x''x'y' + y''(y')^2) &= \frac{q}{p}T'\left[-x'B_t + (1+hx)B_x\right]
\end{aligned}
\tag{1.25}
$$

Solving for x'' and y'' to the third order and noting that

$$
\frac{1}{T'} = 1 - hx + h^2x^2 - \frac{1}{2}\left((x')^2 + (y')^2\right) - h^3x^3 + \frac{3}{2}hx\left((x')^2 + (y')^2\right) + \cdots
$$

we get

$$
\begin{aligned}
x'' &= h(1+hx) + x'(h'x + 2hx') - hxx'(h'x + 2hx') \\
&\quad + \frac{q}{p}\left[B_x x'y' - B_y\left(1 + 2hx + h^2x^2 + \frac{3}{2}(x')^2 + \frac{1}{2}(y')^2\right)\right. \\
&\quad \left. + B_t\left(y'(1+hx) + \frac{1}{2}(x')^2y' + \frac{1}{2}(y')^3\right)\right] \\
y'' &= y'(h'x + 2hx') - hx'y'(h'x + 2hx') \\
&\quad + \frac{q}{p}\left[B_x\left(1 + 2hx + h^2x^2 + \frac{1}{2}(x')^2 + \frac{2}{3}(y')^2\right) - B_y x'y'\right. \\
&\quad \left. - B_t\left(x'(1+hx) + \frac{1}{2}(x')^3 + \frac{1}{2}(x')^2y'\right)\right]
\end{aligned}
\tag{1.26}
$$

Combining Eq. 1.26 with the expanded field components of Eq. 1.24 and letting

$$
\frac{p_0}{p} = \frac{p_0}{p_0(1+\delta)} = 1 - \delta + \delta^2 - \delta^3 + \cdots
$$

we can finally express the differential equations for x and y to the third order as follows,

$$
\begin{aligned}
x'' + (1-k_1)h^2x &= h\delta - (1 - 2k_1 + k_2)h^3x^2 + h'xx' + (2-k_1)h^2x\delta \\
&\quad + \frac{1}{2}h(x')^2 + \frac{1}{2}(h'' - h^3k_1 + h^3k_2)y^2 + h'yy' - \frac{1}{2}h(y')^2 - h\delta^2 \\
&\quad + (k_1 - 2k_2 - k_3)h^4x^3 - hh'x^2x' + (1 - 2k_1 + k_2)h^3x^2\delta \\
&\quad - (2 - \frac{3}{2}k_1)h^2x(x')^2 - \left[\frac{1}{2}(h')^2 + k_1\left(\frac{1}{2}h^4 + hh'' + (h')^2\right)\right. \\
&\quad \left. +2hh'k_1' + \frac{1}{2}h^2k_1'' - 3h^4k_2 - 3h^4k_3\right] xy^2 - (hk_1' + 2h'k_1)hxyy' \\
&\quad + \frac{1}{2}h^2k_1x(y')^2 - (2-k_1)h^2x\delta^2 + \frac{3}{2}h(x')^2\delta - h^2k_1x'yy' \\
&\quad - \frac{1}{2}(h'' - h^3k_1 + 2h^3k_2)y^2\delta - h'yy'\delta + \frac{1}{2}h(y')^2\delta + h\delta^3 \\
y'' + h^2k_1y &= 2(-k_1 + k_2)h^3xy + h'xy' - h'x'y + hx'y' + h^2k_1y\delta \\
&\quad - (k_1 - 4k_2 - 3k_3)h^4x^2y - hh'x^2y' + (2h'k_1 + hk_1')hxx'y \\
&\quad - (2-k_1)h^2xx'y' + 2(k_1 - k_2)h^3xy\delta - \frac{1}{2}h^2k_1(x')^2y + h'x'y\delta
\end{aligned}
\tag{1.27}
$$

$$+hx'y'\delta+\left[\frac{1}{3}hh''+(h')^2-\left(\frac{1}{6}h^4-\frac{1}{3}hh''-\frac{1}{3}(h')^2\right)k_1\right.$$
$$\left.+\frac{2}{3}hh'k_1'+\frac{1}{6}h^2k_1''-\frac{1}{3}h^4k_2-h^4k_3\right]y^3-\frac{3}{2}h^2k_1y(y')^2-h^2k_1y\delta^2$$

1.5.3 Linear Equations and Characteristic Rays

Before considering the full third-order solution to Eq. 1.27, let us look at its linear part. Keeping terms to the first order only, we obtain from Eq. 1.27,

$$\begin{aligned} x''+(1-k_1)h^2x &= h\delta \\ y''+h^2k_1y &= 0 \end{aligned} \tag{1.28}$$

The equation for x is inhomogeneous, and its general solution is the sum of a particular solution and the general solution of the homogeneous equation. There are two linearly independent solutions for both x and y homogeneous equations. The most general solution of a homogeneous linear second order differential equations is a linear sum of the two independent solutions with arbitrary coefficients.

We can define the two solutions of each homogeneous equation and the particular solution of the inhomogeneous equation for x as follows,

1. The unit sine-like function $s_x(t)$ in the bend plane (the magnetic midplane) with initial conditions $s_x(0)=0$, $s_x'(0)=1$; and $\delta=0$.

2. The unit cosine-like function $c_x(t)$ in the bend plane with $c_x(0)=1$, $c_x'(0)=0$; and $\delta=0$.

3. The dispersion function $d_x(t)$ in the bend plane with $d_x(0)=0$, $d_x'(0)=0$; and $\delta=1$.

4. The unit sine-like function $s_y(t)$ in the non-bend plane with $s_y(0)=0$, $s_y'(0)=1$.

5. The unit cosine-like function $c_y(t)$ in the non-bend plane with $c_y(0)=1$, $c_y'(0)=0$.

The sine- and cosine-like functions are linearly independent solutions of the homogeneous equations. The dispersion function is a particular solution to the inhomogeneous equation for

x. The functions are named after the trigonometric functions with similar initial conditions. These five functions are defined as ***characteristic rays*** of a magnetic system. As will be shown below, they determine all the higher order aberration coefficients. Fig. 1.4 through Fig. 1.8 illustrate the characteristic rays, including their passage through a region of magnetic field.

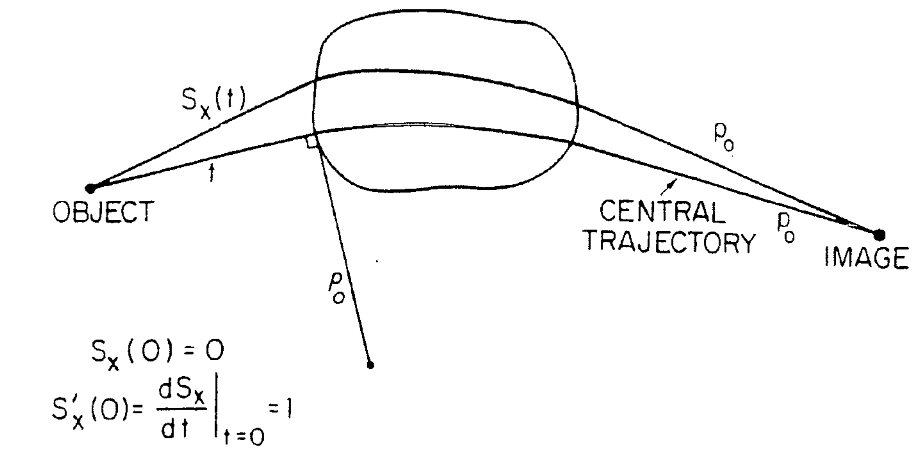

Figure 1.4: The sine-like trajectory $s_x(t)$ in the magnetic midplane.

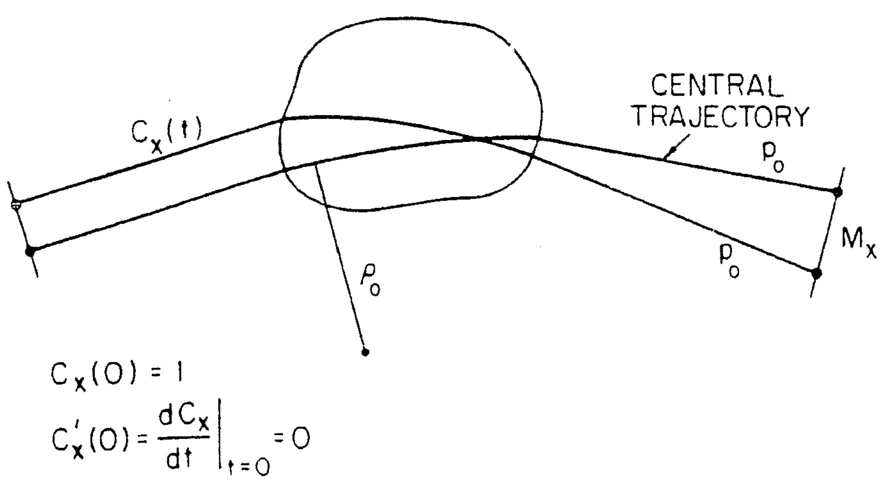

Figure 1.5: The cosine-like trajectory $c_x(t)$ in the magnetic midplane.

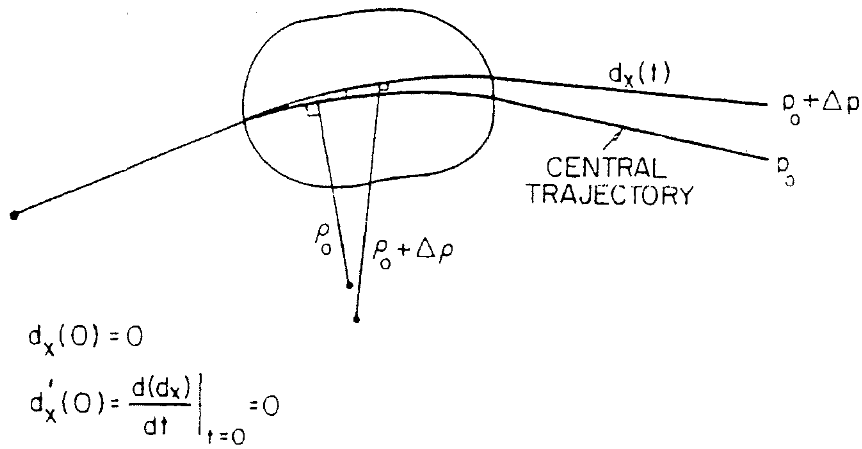

Figure 1.6: The dispersion function $d_x(t)$.

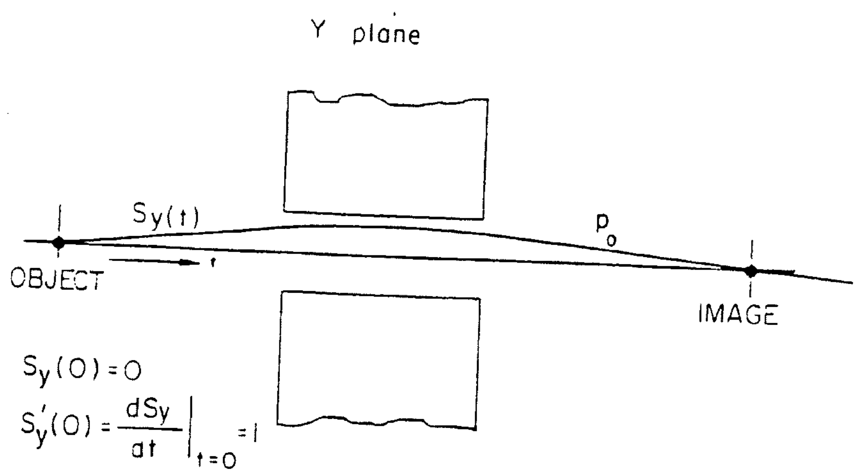

Figure 1.7: The sine-like trajectory $s_y(t)$ in the non-bend plane.

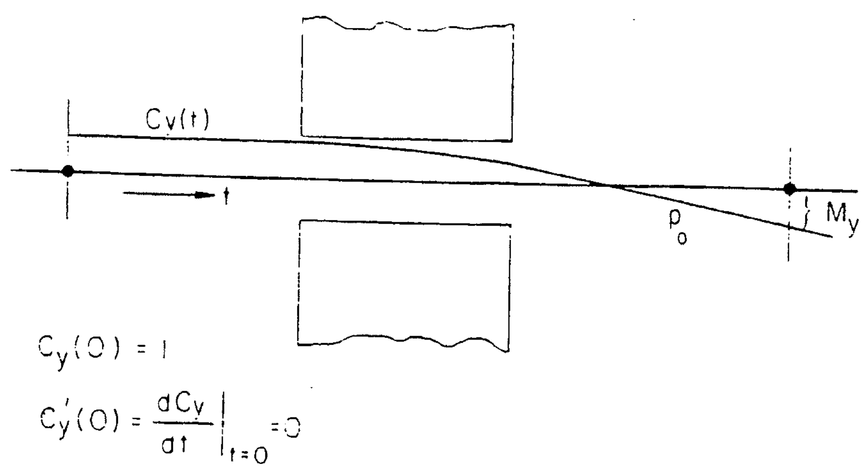

Figure 1.8: The cosine-like trajectory $c_y(t)$ in the non-bend plane.

The general solution of the linear equations for x and y can now be written as

$$
\begin{aligned}
x(t) &= c_x(t)x_0 + s_x(t)x_0' + d_x(t)\delta \\
y(t) &= c_y(t)y_0 + s_y(t)y_0'
\end{aligned}
\tag{1.29}
$$

The dispersion function can be obtained from the inhomogeneous term $h(t)\delta$ and the homogeneous solutions. A particular solution of an inhomogeneous equation of the form

$$q'' + k^2 q = f$$

can be given by an integral

$$q = \int_0^t G(t,\tau)f(\tau)d\tau$$

The Green's function $G(t,\tau)$ can be evaluated in terms of the homogeneous solutions, the sine- and the cosine-like functions, as

$$G(t,\tau) = s(t)c(\tau) - c(t)s(\tau) \tag{1.30}$$

Using this general prescription, the expression for the dispersion function can be written as

$$d_x(t) = s_x(t)\int_0^t c_x(\tau)h(\tau)d\tau - c_x(t)\int_0^t s_x(\tau)h(\tau)d\tau \tag{1.31}$$

The first order transfer matrix $\mathbf{R}_{ij}$ has the following form

$$
\mathbf{R} = \begin{pmatrix}
c_x(t) & s_x(t) & 0 & 0 & 0 & d_x \\
c_x'(t) & s_x'(t) & 0 & 0 & 0 & d_x' \\
0 & 0 & c_y(t) & s_y(t) & 0 & 0 \\
0 & 0 & c_y'(t) & s_y'(t) & 0 & 0 \\
R_{51} & R_{52} & 0 & 0 & 1 & R_{56} \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\tag{1.32}
$$

The "path-length" matrix coefficients are found from the expression for l,

$$l = \int_0^t \left\{ \left[(\frac{dx}{d\tau})^2 + (\frac{dy}{d\tau})^2 + (1+hx)^2 \right]^{1/2} - 1 \right\} d\tau \tag{1.33}$$

Expanding the square root and retaining only linear terms yields

$$l = \int_0^t h(\tau)x(\tau)d\tau \tag{1.34}$$

Using Eq. 1.29, we can express the path-length difference l in terms of the initial coordinates,

$$l = \underbrace{\int_0^t c_x(\tau)h(\tau)d\tau}_{R_{51}} x_0 + \underbrace{\int_0^t s_x(\tau)h(\tau)d\tau}_{R_{52}} x_0' + \underbrace{\int_0^t d_x(\tau)h(\tau)d\tau}_{R_{56}} \delta \qquad (1.35)$$

Next, we tackle higher order matrix elements.

1.5.4 Equations for Aberration Coefficients

Substituting expansions of Eq. 1.11 into Eq. 1.27, we derive a differential equation for each of the first, second, and third order aberration coefficients: $R_{ij}(t)$, $T_{ijk}(t)$, and $U_{ijkl}(t)$. A systematic pattern evolves:

$$\begin{aligned} c_x'' + k_x^2 c_x &= 0 \qquad c_y'' + k_y^2 c_y = 0 \\ s_x'' + k_x^2 s_x &= 0 \qquad s_y'' + k_y^2 s_y = 0 \\ q_x'' + k_x^2 q_x &= f_x \qquad q_y'' + k_y^2 q_y = f_y \end{aligned} \tag{1.36}$$

where $k_x^2 = (1-k_1)h^2$ and $k_y^2 = k_1 h^2$ for the x and y motions, respectively. The first two of these equations represent the equations of motion for the characteristic rays discussed in the previous subsection. The third differential equation for q represents the equation of motion for the first-order dispersion d_x and for any one of the higher order aberration coefficients, where the driving term f has a characteristic form for each of these coefficients. For example, the driving term for d_x is $h(t)$. The coefficients q satisfy the following boundary conditions,

$$q(0) = q'(0) = 0$$

The driving terms f_i can be found as follows [5]. The two differential equations in Eq. 1.27 can be schematically represented by the generic equation

$$z_i'' + k_i z_i = \sum_j D_{ij} z_j + \sum_j \sum_k E_{ijk} z_j z_k + \sum_j \sum_k \sum_l F_{ijkl} z_j z_k z_l \tag{1.37}$$

where the components z_i denote the six phase-space variables x, x', y, y', l, δ. Putting Eq. 1.11 into the above series, we get the driving terms for T_{ijk}

$$f_{ijk}^T(\tau) = \sum_m \sum_n E_{imn} R_{mj}(\tau) R_{nk}(\tau) \tag{1.38}$$

The third-order driving terms for U_{ijkl} then depend on matrices E_{ijk} and F_{ijkl} as well as the second-order terms T_{ijk},

$$\begin{aligned} f_{ijkl}^U &= \sum_m \sum_n E_{imn} R_{mj}(\tau) T_{nkl}(\tau) + \sum_m \sum_n E_{imn} T_{mjk}(\tau) R_{nl}(\tau) \\ &+ \sum_m \sum_n \sum_p F_{imnp} R_{nj}(\tau) R_{mk}(\tau) R_{pl}(\tau) \end{aligned} \tag{1.39}$$

Tables 1.1 and 1.2 give the driving terms for each of the possible second-order and third-order matrix elements, respectively. The terms not listed are those not allowed by considerations of midplane symmetry; they are identically equal to zero.

The second and third order aberration coefficients are evaluated by the same method used to derive the first-order dispersion function, namely, the Green's function integral,

$$q = \int_0^t G(t,\tau) f(\tau) d\tau$$

with $G(t,\tau)$ given by Eq. 1.30.

The "angle" matrix elements are found by differentiating the "position" matrix elements:

$$\begin{aligned} T_{2jk} &= T'_{1jk} \\ T_{4jk} &= T'_{3jk} \\ &\text{and} \\ U_{2jkl} &= U'_{2jkl} \\ U_{4jkl} &= U'_{3jkl} \end{aligned} \tag{1.40}$$

The matrix elements pertaining to the longitudinal coordinate l are found from the general expression for the path-length difference given in Eq. 1.33, which is reproduced below,

$$l = \int_0^t \left\{ \left[(\frac{dx}{d\tau})^2 + (\frac{dy}{d\tau})^2 + (1 + hx)^2 \right]^{1/2} - 1 \right\} d\tau \tag{1.41}$$

Now, when expanding the square root, we retain terms up to third order. The result is

$$l = \int_0^t \left\{ hx(\tau) + \frac{1}{2} \left[(\frac{dx}{d\tau})^2 + (\frac{dy}{d\tau})^2 \right] - \frac{1}{2} hx(\tau) \left[(\frac{dx}{d\tau})^2 + (\frac{dy}{d\tau})^2 \right] \right\} d\tau \tag{1.42}$$

Substituting expanded expressions for $x(\tau)$, $dx/d\tau$, and $dy/d\tau$ into the above equation, we derive explicit expressions for each of the longitudinal matrix elements. An element q_l is given by

$$q_l = \int_0^t f_l(\tau) d\tau \tag{1.43}$$

Non-zero second- and third-order path-length matrix elements q_l and the corresponding integrands f_l are listed in Tables 1.3 and 1.4, respectively.

The driving terms and the Green's function integrals complete the solution of the general third-order theory. What remains is to find explicit solutions for specific magnetic elements. In the next chapter, we specialize to cases where the parameters describing the magnetic field configuration are independent of t. The only regions where h is typically not constant are the fringe fields at the boundaries of dipole magnets. They require special treatment and will be considered in later chapters.

Table 1.1: Driving Terms for Second-Order Matrix Elements

Matrix Element	Driving Term
q_x	f_x
T_{111}	$(2k_1-1-k_2)h^3c_x^2+h'c_xc_x'+\frac{1}{2}h(c_x')^2$
T_{112}	$2(2k_1-1-k_2)h^3c_xs_x+h'(c_xs_x'+c_x's_x)+hc_x's_x'$
T_{116}	$(2-k_1)h^2s_x+2(2k_1-1-k_2)h^3c_xd_x+h'(c_xd_x'+c_x'd_x)+hc_x'd_x'$
T_{122}	$(2k_1-1-k_2)h^3s_x^2+h's_xs_x'+\frac{1}{2}h(s_x')^2$
T_{126}	$(2-k_1)h^2s_x+2(2k_1-1-k_2)h^3s_xd_x+h'(s_xd_x'+s_x'd_x)+hs_x'd_x'$
T_{133}	$\frac{1}{2}(h''-k_1h^3+2k_2h^3)c_y^2+h'c_yc_y'-\frac{1}{2}h(c_y')^2$
T_{134}	$(h''-k_1h^3+2k_2h^3)s_yc_y+h'(c_ys_y'+c_y's_y)-hs_y'c_y'$
T_{144}	$\frac{1}{2}(h''-k_1h^3+2k_2h^3)s_y^2+h's_ys_y'-\frac{1}{2}h(s_y')^2$
T_{166}	$-h+(2-k_1)h^2d_x+(2k_1-1-k_2)h^3d_x^2+h'd_xd_x'+\frac{1}{2}h(d_x')^2$
q_y	f_y
T_{313}	$2(k_2-k_1)h^3c_xc_y+h'(c_xc_y'-c_x'c_y)+hc_x'c_y'$
T_{314}	$2(k_2-k_1)h^3c_xs_y+h'(c_xs_y'-c_x's_y)+hc_x's_y'$
T_{323}	$2(k_2-k_1)h^3s_xc_y+h'(s_xc_y'-s_x'c_y)+hs_x'c_y'$
T_{324}	$2(k_2-k_1)h^3s_xs_y+h'(s_xs_y'-s_x's_y)+hs_x's_y'$
T_{336}	$nh^2c_y+2(k_2-k_1)h^3c_yd_x-h'(c_yd_x'-c_y'd_x)+hc_y'd_x'$
T_{346}	$nh^2s_y+2(k_2-k_1)h^3s_yd_x-h'(s_yd_x'-s_y'd_x)+hs_y'd_x'$

Table 1.2: Driving Terms for Third-Order Matrix Elements

q_x	f_x
U_{1111}	$-(2c_x^3h^4k_3 + (4c_x^3h^4 + 4T_{111}c_xh^3)k_2 + (-2c_x^3h^4 - 8T_{111}c_xh^3 - 3c_x(c_x')^2h^2)k_1$ $+(2c_x^2c_x'h - 2T_{111}c_x' - 2T_{111}'c_x)h' + 4T_{111}c_xh^3 + 4c_x(c_x')^2h^2 - 2T_{111}'c_x'h)/2$
U_{1112}	$((6c_xc_x'h^2k_1 + (-2c_x^2h + 2T_{111})h' - 8c_xc_x'h^2 + 2T_{111}'h)s_x' + (-6c_x^2h^4k_3$ $+(-12c_x^2h^4 - 4T_{111}h^3)k_2 + (6c_x^2h^4 + 8T_{111}h^3 + 3(c_x')^2h^2)k_1 + (-4c_xc_x'h$ $+2T_{111}')h' - 4T_{111}h^3 - 4(c_x')^2h^2)s_x - 4T_{112}c_xh^3k_2 + 8T_{112}c_xh^3k_1 + (2T_{112}c_x'$ $+2T_{112}'c_x)h' - 4T_{112}c_xh^3 + 2T_{112}'c_x'h)/2$
U_{1116}	$-(6c_x^2d_xh^4k_3 + (12c_x^2d_xh^4 + (4T_{111}d_x - 2c_x^2 + 4T_{116}c_x)h^3)k_2 + (-6c_x^2d_xh^4 + (-8T_{111}d_x$ $+4c_x^2 - 8T_{116}c_x)h^3 + (-6c_xc_x'd_x' - 3(c_x')^2d_x + 2T_{111})h^2)k_1 + ((2c_x^2d_x' + 4c_xc_x'd_x)h$ $-2T_{111}d_x' - 2T_{111}'d_x - 2T_{116}c_x' - 2T_{116}'c_x)h' + (4T_{111}d_x - 2c_x^2 + 4T_{116}c_x)h^3$ $+(8c_xc_x'd_x' + 4(c_x')^2d_x - 4T_{111})h^2 + (-2T_{111}'d_x' - 3(c_x')^2 - 2T_{116}'c_x')h)/2$
U_{1122}	$((3c_xh^2k_1 - 4c_xh^2)(s_x')^2 + ((6c_x'h^2k_1 - 4c_xhh' - 8c_x'h^2)s_x + 2T_{112}h'$ $+2T_{112}'h)s_x' + (-6c_xh^4k_3 - 12c_xh^4k_2 + 6c_xh^4k_1 - 2c_x'hh')s_x^2 + (-4T_{112}h^3k_2$ $+8T_{112}h^3k_1 + 2T_{112}'h' - 4T_{112}h^3)s_x - 4T_{122}c_xh^3k_2 + 8T_{122}c_xh^3k_1 + (2T_{122}c_x'$ $+2T_{122}'c_x)h' - 4T_{122}c_xh^3 + 2T_{122}'c_x'h)/2$
U_{1126}	$((3c_xd_x' + 3c_x'd_x)h^2k_1 + (-2c_xd_xh + T_{116})h' + (-4c_xd_x' - 4c_x'd_x)h^2$ $+(3c_x' + T_{116}')h)s_x' + (-6c_xd_xh^4k_3 + (-12c_xd_xh^4 + (2c_x - 2T_{116})h^3)k_2$ $+(6c_xd_xh^4 + (-4c_x + 4T_{116})h^3 + 3c_x'd_x'h^2)k_1 + ((-2c_xd_x' - 2c_x'd_x)h$ $+T_{116}')h' + (2c_x - 2T_{116})h^3 - 4c_x'd_x'h^2)s_x + (-2T_{112}d_x - 2T_{126}c_x)h^3k_2$ $+((4T_{112}d_x + 4T_{126}c_x)h^3 - T_{112}h^2)k_1 + (T_{112}d_x' + T_{112}'d_x + T_{126}c_x'$ $+T_{126}'c_x)h' + (-2T_{112}d_x - 2T_{126}c_x)h^3 + 2T_{112}h^2 + (T_{112}'d_x' + T_{126}'c_x')h$
U_{1133}	$(6c_xc_y^2h^4k_3 + (6c_xc_y^2h^4 + (4T_{313}c_y - 4T_{133}c_x)h^3)k_2 + (-4c_xc_y^2hh' - 2c_xc_yc_y'h^2)k_1'$ $-c_xc_y^2h^2k_1'' + (-2c_xc_y^2(h')^2 - 4c_xc_yc_y'hh' - 2c_xc_y^2hh'' - c_xc_y^2h^4 + (-2T_{313}c_y$ $+8T_{133}c_x)h^3 + (c_x(c_y')^2 - 2c_x'c_yc_y')h^2)k_1 - c_xc_y^2(h')^2 + (2T_{313}c_y' + 2T_{313}'c_y$ $+2T_{133}c_x' + 2T_{133}'c_x)h' + 2T_{313}c_yh'' - 4T_{133}c_xh^3 + (-2T_{313}'c_y' + 2T_{133}'c_x')h)/2$
U_{1134}	$(-c_xc_yh^2k_1' + (-2c_xc_yhh' + (c_xc_y' - c_x'c_y)h^2)k_1 + T_{313}h' - T_{313}'h)s_y'$ $+(6c_xc_yh^4k_3 + (6c_xc_yh^4 + 2T_{313}h^3)k_2 + (-4c_xc_yhh' - c_xc_y'h^2)k_1' - c_xc_yh^2k_1''$ $+(-2c_xc_y(h')^2 - 2c_xc_y'hh' - 2c_xc_yhh'' - c_xc_yh^4 - T_{313}h^3 - c_x'c_y'h^2)k_1$ $-c_xc_y(h')^2 + T_{313}'h' + T_{313}h'')s_y + (2T_{314}c_y - 2T_{134}c_x)h^3k_2 + (-T_{314}c_y$ $+4T_{134}c_x)h^3k_1 + (T_{314}c_y' + T_{314}'c_y + T_{134}c_x' + T_{134}'c_x)h' + T_{314}c_yh''$ $-2T_{134}c_xh^3 + (-T_{314}'c_y' + T_{134}'c_x')h$
U_{1144}	$(c_xh^2k_1(s_y')^2 + ((-2c_xh^2k_1' + (-4c_xhh' - 2c_x'h^2)k_1)s_y + 2T_{314}h' - 2T_{314}'h)s_y'$ $+(6c_xh^4k_3 + 6c_xh^4k_2 - 4c_xhh'k_1' - c_xh^2k_1'' + (-2c_x(h')^2 - 2c_xhh''$ $-c_xh^4)k_1 - c_x(h')^2)s_y^2 + (4T_{314}h^3k_2 - 2T_{314}h^3k_1 + 2T_{314}'h' + 2T_{314}h'')s_y$ $-4T_{144}c_xh^3k_2 + 8T_{144}c_xh^3k_1 + (2T_{144}c_x' + 2T_{144}'c_x)h' - 4T_{144}c_xh^3 + 2T_{144}'c_x'h)/2$
U_{1166}	$-(6c_xd_x^2h^4k_3 + (12c_xd_x^2h^4 + ((-4c_x + 4T_{116})d_x + 4T_{166}c_x)h^3)k_2 + (-6c_xd_x^2h^4$ $+((8c_x - 8T_{116})d_x - 8T_{166}c_x)h^3 + (-3c_x(d_x')^2 - 6c_x'd_xd_x' - 2c_x$ $+2T_{116})h^2)k_1 + ((4c_xd_xd_x' + 2c_x'd_x^2)h - 2T_{116}d_x' - 2T_{116}'d_x - 2T_{166}c_x'$ $-2T_{166}'c_x)h' + ((-4c_x + 4T_{116})d_x + 4T_{166}c_x)h^3 + (4c_x(d_x')^2 + 8c_x'd_xd_x' + 4c_x$ $-4T_{116})h^2 + ((-6c_x' - 2T_{116}')d_x' - 2T_{166}'c_x')h)/2$

q_x	f_x
U_{1222}	$((3h^2k_1 - 4h^2)s_x(s_x')^2 + (-2hh's_x^2 + 2T_{122}h' + 2T_{122}'h)s_x' + (-2h^4k_3$ $- 4h^4k_2 + 2h^4k_1)s_x^3 + (-4T_{122}h^3k_2 + 8T_{122}h^3k_1 + 2T_{122}'h' - 4T_{122}h^3)s_x)/2$
U_{1226}	$((3d_xh^2k_1 - 4d_xh^2 + 3h)(s_x')^2 + ((6d_x'h^2k_1 - 4d_xhh' - 8d_x'h^2)s_x$ $+ 2T_{126}h' + 2T_{126}'h)s_x' + (-6d_xh^4k_3 + (-12d_xh^4 + 2h^3)k_2 + (6d_xh^4$ $- 4h^3)k_1 - 2d_x'hh' + 2h^3)s_x^2 + (-4T_{126}h^3k_2 + 8T_{126}h^3k_1 + 2T_{126}'h'$ $- 4T_{126}h^3)s_x - 4T_{122}d_xh^3k_2 + (8T_{122}d_xh^3 - 2T_{122}h^2)k_1 + (2T_{122}d_x' + 2T_{122}'d_x)h'$ $- 4T_{122}d_xh^3 + 4T_{122}h^2 + 2T_{122}'d_x'h)/2$
U_{1233}	$-((2c_yc_y'h^2k_1 - 2T_{133}h' - 2T_{133}'h)s_x' + (-6c_y^2h^4k_3 + (-6c_y^2h^4 + 4T_{133}h^3)k_2$ $+ (4c_y^2hh' + 2c_yc_y'h^2)k_1' + c_y^2h^2k_1'' + (2c_y^2(h')^2 + 4c_yc_y'hh' + 2c_y^2hh''$ $+ c_y^2h^4 - 8T_{133}h^3 - (c_y')^2h^2)k_1 + c_y^2(h')^2 - 2T_{133}'h' + 4T_{133}h^3)s_x$ $- 4T_{323}c_yh^3k_2 + 2T_{323}c_yh^3k_1 + (-2T_{323}c_y' - 2T_{323}'c_y)h' - 2T_{323}c_yh'' + 2T_{323}'c_y'h)/2$
U_{1234}	$(-c_yh^2k_1s_x' + (-c_yh^2k_1' + (-2c_yhh' + c_y'h^2)k_1)s_x + T_{323}h' - T_{323}'h)s_y'$ $+ (-c_y'h^2k_1s_x' + (6c_yh^4k_3 + 6c_yh^4k_2 + (-4c_yhh' - c_y'h^2)k_1' - c_yh^2k_1''$ $+ (-2c_y(h')^2 - 2c_y'hh' - 2c_yhh'' - c_yh^4)k_1 - c_y(h')^2)s_x + 2T_{323}h^3k_2$ $- T_{323}h^3k_1 + T_{323}'h' + T_{323}h'')s_y + (T_{134}h' + T_{134}'h)s_x' + (-2T_{134}h^3k_2$ $+ 4T_{134}h^3k_1 + T_{134}'h' - 2T_{134}h^3)s_x + 2T_{324}c_yh^3k_2 - T_{324}c_yh^3k_1 + (T_{324}c_y'$ $+ T_{324}'c_y)h' + T_{324}c_yh'' - T_{324}'c_y'h$
U_{1244}	$(h^2k_1s_x(s_y')^2 + ((-2h^2k_1s_x' + (-2h^2k_1' - 4hh'k_1)s_x)s_y + 2T_{324}h' - 2T_{324}'h)s_y'$ $+ (6h^4k_3 + 6h^4k_2 - 4hh'k_1' - h^2k_1'' + (-2(h')^2 - 2hh'' - h^4)k_1$ $- (h')^2)s_xs_y^2 + (4T_{324}h^3k_2 - 2T_{324}h^3k_1 + 2T_{324}'h' + 2T_{324}h'')s_y + (2T_{144}h'$ $+ 2T_{144}'h)s_x' + (-4T_{144}h^3k_2 + 8T_{144}h^3k_1 + 2T_{144}'h' - 4T_{144}h^3)s_x)/2$
U_{1266}	$((6d_xd_x'h^2k_1 + (-2d_x^2h + 2T_{166})h' - 8d_xd_x'h^2 + (6d_x' + 2T_{166}')h)s_x'$ $+ (-6d_x^2h^4k_3 + (-12d_x^2h^4 + (4d_x - 4T_{166})h^3)k_2 + (6d_x^2h^4 + (-8d_x + 8T_{166})h^3$ $+ (3(d_x')^2 + 2)h^2)k_1 + (-4d_xd_x'h + 2T_{166}')h' + (4d_x - 4T_{166})h^3$ $+ (-4(d_x')^2 - 4)h^2)s_x - 4T_{126}d_xh^3k_2 + (8T_{126}d_xh^3 - 2T_{126}h^2)k_1 + (2T_{126}d_x'$ $+ 2T_{126}'d_x)h' - 4T_{126}d_xh^3 + 4T_{126}h^2 + 2T_{126}'d_x'h)/2$
U_{1336}	$(6c_y^2d_xh^4k_3 + (6c_y^2d_xh^4 + (-4T_{133}d_x - 2c_y^2 + 4T_{336}c_y)h^3)k_2 + (-4c_y^2d_xhh'$ $- 2c_yc_y'd_xh^2)k_1' - c_y^2d_xh^2k_1'' + (-2c_y^2d_x(h')^2 - 4c_yc_y'd_xhh' - 2c_y^2d_xhh'' - c_y^2d_xh^4$ $+ (8T_{133}d_x + c_y^2 - 2T_{336}c_y)h^3 + (-2c_yc_y'd_x' + (c_y')^2d_x - 2T_{133})h^2)k_1$ $- c_y^2d_x(h')^2 + (2T_{133}d_x' + 2T_{133}'d_x + (-2c_y + 2T_{336})c_y' + 2T_{336}'c_y)h'$ $+ (-c_y^2 + 2T_{336}c_y)h'' - 4T_{133}d_xh^3 + 4T_{133}h^2 + (2T_{133}'d_x' + (c_y')^2 - 2T_{336}'c_y')h)/2$
U_{1346}	$(-c_yd_xh^2k_1' + (-2c_yd_xhh' + (-c_yd_x' + c_y'd_x)h^2)k_1 + (-c_y + T_{336})h'$ $+ (c_y' - T_{336}')h)s_y' + (6c_yd_xh^4k_3 + (6c_yd_xh^4 + (-2c_y + 2T_{336})h^3)k_2$ $+ (-4c_yd_xhh' - c_y'd_xh^2)k_1' - c_yd_xh^2k_1'' + (-2c_yd_x(h')^2 - 2c_y'd_xhh' - 2c_yd_xhh''$ $- c_yd_xh^4 + (c_y - T_{336})h^3 - c_y'd_x'h^2)k_1 - c_yd_x(h')^2 + (-c_y'$ $+ T_{336}')h' + (-c_y + T_{336})h'')s_y + (-2T_{134}d_x + 2T_{346}c_y)h^3k_2 + ((4T_{134}d_x$ $- T_{346}c_y)h^3 - T_{134}h^2)k_1 + (T_{134}d_x' + T_{134}'d_x + T_{346}c_y' + T_{346}'c_y)h'$ $+ T_{346}c_yh'' - 2T_{134}d_xh^3 + 2T_{134}h^2 + (T_{134}'d_x' - T_{346}'c_y')h$

U_{1446}	$((d_xh^2k_1 + h)(s_y')^2 + ((-2d_xh^2k_1' + (-4d_xhh' - 2d_x'h^2)k_1 - 2h')s_y$ $+2T_{346}h' - 2T_{346}'h)s_y' + (6d_xh^4k_3 + (6d_xh^4 - 2h^3)k_2 - 4d_xhh'k_1'$ $-d_xh^2k_1'' + (-2d_x(h')^2 - 2d_xhh'' - d_xh^4 + h^3)k_1 - d_x(h')^2 - h'')s_y^2$ $+(4T_{346}h^3k_2 - 2T_{346}h^3k_1 + 2T_{346}'h' + 2T_{346}h'')s_y - 4T_{144}d_xh^3k_2 + (8T_{144}d_xh^3$ $-2T_{144}h^2)k_1 + (2T_{144}d_x' + 2T_{144}'d_x)h' - 4T_{144}d_xh^3 + 4T_{144}h^2 + 2T_{144}'d_x'h)/2$
U_{1666}	$-(2d_x^3h^4k_3 + (4d_x^3h^4 + (-2d_x^2 + 4T_{166}d_x)h^3)k_2 + (-2d_x^3h^4 + (4d_x^2 - 8T_{166}d_x)h^3$ $+(-3d_x(d_x')^2 - 2d_x + 2T_{166})h^2)k_1 + (2d_x^2d_x'h - 2T_{166}d_x' - 2T_{166}'d_x)h' + (-2d_x^2$ $+4T_{166}d_x)h^3 + (4d_x(d_x')^2 + 4d_x - 4T_{166})h^2 + (-3(d_x')^2 - 2T_{166}'d_x' - 2)h)/2$

q_y	f_y
U_{3113}	$(6c_x^2c_yh^4k_3 + (8c_x^2c_yh^4 + (4T_{111}c_y + 4T_{313}c_x)h^3)k_2 + 2c_xc_x'c_yh^2k_1' + (4c_xc_x'c_yhh'$ $-2c_x^2c_yh^4 + (-4T_{111}c_y - 4T_{313}c_x)h^3 + (2c_xc_x'c_y' - (c_x')^2c_y)h^2)k_1 + (-2c_x^2c_y'h$ $+2T_{111}c_y' - 2T_{111}'c_y - 2T_{313}c_x' + 2T_{313}'c_x)h' - 4c_xc_x'c_y'h^2 + (2T_{111}'c_y' + 2T_{313}'c_x')h)/2$
U_{3114}	$((2c_xc_x'h^2k_1 + (-2c_x^2h + 2T_{111})h' - 4c_xc_x'h^2 + 2T_{111}'h)s_y' + (6c_x^2h^4k_3$ $+(8c_x^2h^4 + 4T_{111}h^3)k_2 + 2c_xc_x'h^2k_1' + (4c_xc_x'hh' - 2c_x^2h^4 - 4T_{111}h^3 - (c_x')^2h^2)k_1$ $-2T_{111}'h')s_y + 4T_{314}c_xh^3k_2 - 4T_{314}c_xh^3k_1 + (-2T_{314}c_x' + 2T_{314}'c_x)h' + 2T_{314}'c_x'h)/2$
U_{3123}	$(c_xc_yh^2k_1' + (2c_xc_yhh' + (c_xc_y' - c_x'c_y)h^2)k_1 - T_{313}h' - 2c_xc_y'h^2$ $+T_{313}'h)s_x' + (6c_xc_yh^4k_3 + (8c_xc_yh^4 + 2T_{313}h^3)k_2 + c_x'c_yh^2k_1' + (2c_x'c_yhh'$ $-2c_xc_yh^4 - 2T_{313}h^3 + c_x'c_y'h^2)k_1 + (-2c_xc_y'h + T_{313}')h' - 2c_x'c_y'h^2)s_x$ $+(2T_{112}c_y + 2T_{323}c_x)h^3k_2 + (-2T_{112}c_y - 2T_{323}c_x)h^3k_1 + (T_{112}c_y' - T_{112}'c_y$ $-T_{323}c_x' + T_{323}'c_x)h' + (T_{112}'c_y' + T_{323}'c_x')h$
U_{3124}	$((c_xh^2k_1 - 2c_xh^2)s_x' + (c_x'h^2k_1 - 2c_xhh' - 2c_x'h^2)s_x + T_{112}h'$ $+T_{112}'h)s_y' + ((c_xh^2k_1' + (2c_xhh' - c_x'h^2)k_1)s_x' + (6c_xh^4k_3 + 8c_xh^4k_2$ $+c_x'h^2k_1' + (2c_x'hh' - 2c_xh^4)k_1)s_x + 2T_{112}h^3k_2 - 2T_{112}h^3k_1 - T_{112}'h')s_y$ $+(-T_{314}h' + T_{314}'h)s_x' + (2T_{314}h^3k_2 - 2T_{314}h^3k_1 + T_{314}'h')s_x + 2T_{324}c_xh^3k_2$ $-2T_{324}c_xh^3k_1 + (-T_{324}c_x' + T_{324}'c_x)h' + T_{324}'c_x'h$
U_{3136}	$6c_xc_yd_xh^4k_3 + (8c_xc_yd_xh^4 + (2T_{313}d_x + (-2c_x + 2T_{116})c_y + 2T_{336}c_x)h^3)k_2 + (c_xc_yd_x'$ $+c_x'c_yd_x)h^2k_1' + ((2c_xc_yd_x' + 2c_x'c_yd_x)hh' - 2c_xc_yd_xh^4 + (-2T_{313}d_x + (2c_x - 2T_{116})c_y$ $-2T_{336}c_x)h^3 + ((c_xc_y' - c_x'c_y)d_x' + c_x'c_y'd_x + T_{313})h^2)k_1 + (-2c_xc_y'd_xh$ $-T_{313}d_x' + T_{313}'d_x + T_{116}c_y' + (c_x' - T_{116}')c_y - T_{336}c_x'$ $+T_{336}'c_x)h' + (-2c_xc_y'd_x' - 2c_x'c_y'd_x)h^2 + (T_{313}'d_x' + (c_x' + T_{116}')c_y' + T_{336}'c_x')h$
U_{3146}	$((c_xd_x' + c_x'd_x)h^2k_1 + (-2c_xd_xh + T_{116})h' + (-2c_xd_x' - 2c_x'd_x)h^2$ $+(c_x' + T_{116}')h)s_y' + (6c_xd_xh^4k_3 + (8c_xd_xh^4 + (-2c_x + 2T_{116})h^3)k_2 + (c_xd_x'$ $+c_x'd_x)h^2k_1' + ((2c_xd_x' + 2c_x'd_x)hh' - 2c_xd_xh^4 + (2c_x - 2T_{116})h^3 - c_x'd_x'h^2)k_1$ $+(c_x' - T_{116}')h')s_y + (2T_{314}d_x + 2T_{346}c_x)h^3k_2 + ((-2T_{314}d_x - 2T_{346}c_x)h^3$ $+T_{314}h^2)k_1 + (-T_{314}d_x' + T_{314}'d_x - T_{346}c_x' + T_{346}'c_x)h' + (T_{314}'d_x' + T_{346}'c_x')h$
U_{3223}	$-(c_yh^2k_1(s_x')^2 + ((-2c_yh^2k_1' + (-4c_yhh' - 2c_y'h^2)k_1 + 4c_y'h^2)s_x + 2T_{323}h'$ $-2T_{323}'h)s_x' + (-6c_yh^4k_3 - 8c_yh^4k_2 + 2c_yh^4k_1 + 2c_y'hh')s_x^2 + (-4T_{323}h^3k_2 + 4T_{323}h^3k_1$ $-2T_{323}'h')s_x - 4T_{122}c_yh^3k_2 + 4T_{122}c_yh^3k_1 + (-2T_{122}c_y' + 2T_{122}'c_y)h' - 2T_{122}'c_y'h)/2$
U_{3224}	$(((2h^2k_1 - 4h^2)s_xs_x' - 2hh's_x^2 + 2T_{122}h' + 2T_{122}'h)s_y' + (-h^2k_1(s_x')^2$ $+(2h^2k_1' + 4hh'k_1)s_xs_x' + (6h^4k_3 + 8h^4k_2 - 2h^4k_1)s_x^2 + 4T_{122}h^3k_2 - 4T_{122}h^3k_1$ $-2T_{122}'h')s_y + (-2T_{324}h' + 2T_{324}'h)s_x' + (4T_{324}h^3k_2 - 4T_{324}h^3k_1 + 2T_{324}'h')s_x)/2$

q_y	f_y
U_{3236}	$(c_y d_x h^2 k_1' + (2c_y d_x hh' + (-c_y d_x' + c_y' d_x)h^2)k_1 + (c_y - T_{336})h'$ $-2c_y' d_x h^2 + (c_y' + T_{336}')h)s_x' + (6c_y d_x h^4 k_3 + (8c_y d_x h^4 + (-2c_y + 2T_{336})h^3)k_2$ $+c_y d_x' h^2 k_1' + (2c_y d_x' hh' - 2c_y d_x h^4 + (2c_y - 2T_{336})h^3 + c_y' d_x' h^2)k_1$ $+(-2c_y' d_x h + T_{336}')h' - 2c_y' d_x' h^2)s_x + (2T_{323} d_x + 2T_{126} c_y)h^3 k_2$ $+((-2T_{323} d_x - 2T_{126} c_y)h^3 + T_{323} h^2)k_1 + (-T_{323} d_x' + T_{323}' d_x$ $+T_{126} c_y' - T_{126}' c_y)h' + (T_{323}' d_x' + T_{126}' c_y')h$
U_{3246}	$((d_x h^2 k_1 - 2d_x h^2 + h)s_x' + (d_x' h^2 k_1 - 2d_x hh' - 2d_x' h^2)s_x$ $+T_{126} h' + T_{126}' h)s_y' + ((d_x h^2 k_1' + (2d_x hh' - d_x' h^2)k_1 + h')s_x'$ $+(6d_x h^4 k_3 + (8d_x h^4 - 2h^3)k_2 + d_x' h^2 k_1' + (2d_x' hh' - 2d_x h^4 + 2h^3)k_1)s_x$ $+2T_{126} h^3 k_2 + (-T_{346} h' + T_{346}' h)s_x' + (2T_{346} h^3 k_2 - 2T_{346} h^3 k_1 + T_{346}' h')s_x$ $+2T_{324} d_x h^3 k_2 + (-2T_{324} d_x h^3 + T_{324} h^2)k_1 + (-T_{324} d_x' + T_{324}' d_x)h' + T_{324}' d_x' h$
U_{3333}	$-(6c_y^3 h^4 k_3 + (2c_y^3 h^4 - 12T_{133} c_y h^3)k_2 - 4c_y^3 hh' k_1' - c_y^3 h^2 k_1'' + (-2c_y^3 (h')^2$ $-2c_y^3 hh'' + c_y^3 h^4 + 12T_{133} c_y h^3 + 9c_y (c_y')^2 h^2)k_1 - c_y^3 (h')^2$ $+(-6T_{133} c_y' + 6T_{133}' c_y)h' - 2c_y^3 hh'' - 6T_{133}' c_y' h)/6$
U_{3334}	$-((6c_y c_y' h^2 k_1 - 2T_{133} h' - 2T_{133}' h)s_y' + (6c_y^2 h^4 k_3 + (2c_y^2 h^4 - 4T_{133} h^3)k_2$ $-4c_y^2 hh' k_1' - c_y^2 h^2 k_1'' + (-2c_y^2 (h')^2 - 2c_y^2 hh'' + c_y^2 h^4 + 4T_{133} h^3$ $+3(c_y')^2 h^2)k_1 - c_y^2 (h')^2 + 2T_{133}' h' - 2c_y^2 hh'')s_y - 4T_{134} c_y h^3 k_2 + 4T_{134} c_y h^3 k_1$ $+(-2T_{134} c_y' + 2T_{134}' c_y)h' - 2T_{134}' c_y' h)/2$
U_{3344}	$-(3c_y h^2 k_1 (s_y')^2 + (6c_y' h^2 k_1 s_y - 2T_{134} h' - 2T_{134}' h)s_y' + (6c_y h^4 k_3 + 2c_y h^4 k_2$ $-4c_y hh' k_1' - c_y h^2 k_1'' + (-2c_y (h')^2 - 2c_y hh'' + c_y h^4)k_1 - c_y (h')^2$ $-2c_y hh'')s_y^2 + (-4T_{134} h^3 k_2 + 4T_{134} h^3 k_1 + 2T_{134}' h')s_y - 4T_{144} c_y h^3 k_2 + 4T_{144} c_y h^3 k_1$ $+(-2T_{144} c_y' + 2T_{144}' c_y)h' - 2T_{144}' c_y' h)/2$
U_{3366}	$(6c_y d_x^2 h^4 k_3 + (8c_y d_x^2 h^4 + ((-4c_y + 4T_{336})d_x + 4T_{166} c_y)h^3)k_2 + 2c_y d_x d_x' h^2 k_1'$ $+(4c_y d_x d_x' hh' - 2c_y d_x^2 h^4 + ((4c_y - 4T_{336})d_x - 4T_{166} c_y)h^3 + (-c_y (d_x')^2$ $+2c_y' d_x d_x' - 2c_y + 2T_{336})h^2)k_1 + (-2c_y' d_x^2 h + (2c_y - 2T_{336})d_x' + 2T_{336}' d_x$ $+2T_{166} c_y' - 2T_{166}' c_y)h' - 4c_y' d_x d_x' h^2 + ((2c_y' + 2T_{336}')d_x' + 2T_{166}' c_y')h)/2$
U_{3444}	$-(9h^2 k_1 s_y (s_y')^2 + (-6T_{144} h' - 6T_{144}' h)s_y' + (6h^4 k_3 + 2h^4 k_2 - 4hh' k_1' - h^2 k_1'' + (-2(h')^2$ $-2hh'' + h^4)k_1 - (h')^2 - 2hh'')s_y^3 + (-12T_{144} h^3 k_2 + 12T_{144} h^3 k_1 + 6T_{144}' h')s_y)/6$
U_{3466}	$((2d_x d_x' h^2 k_1 + (-2d_x^2 h + 2T_{166})h' - 4d_x d_x' h^2 + (2d_x' + 2T_{166}')h)s_y'$ $+(6d_x^2 h^4 k_3 + (8d_x^2 h^4 + (-4d_x + 4T_{166})h^3)k_2 + 2d_x d_x' h^2 k_1' + (4d_x d_x' hh' - 2d_x^2 h^4$ $+(4d_x - 4T_{166})h^3 + (-(d_x')^2 - 2)h^2)k_1 + (2d_x' - 2T_{166}')h')s_y$ $+4T_{346} d_x h^3 k_2 + (-4T_{346} d_x h^3 + 2T_{346} h^2)k_1 + (-2T_{346} d_x' + 2T_{346}' d_x)h' + 2T_{346}' d_x' h)/2$

Table 1.3: Integrands for Second-Order Path-Length Matrix Elements

Matrix Element q_l	Integrand f_l
T_{511}	$hT_{111} + \frac{1}{2}(c'_x)^2$
T_{512}	$hT_{112} + c'_x s'_x$
T_{516}	$hT_{116} + c'_x d'_x$
T_{522}	$hT_{122} + \frac{1}{2}(s'_x)^2$
T_{526}	$hT_{126} + s'_x d'_x$
T_{533}	$hT_{133} + \frac{1}{2}(c'_y)^2$
T_{534}	$hT_{134} + c'_y s'_y$
T_{544}	$hT_{144} + \frac{1}{2}(s'_y)^2$
T_{566}	$hT_{166} + \frac{1}{2}(d'_x)^2$

Table 1.4: Integrands for Third-Order Path-Length Matrix Elements

Matrix Element q_l	Integrand f_l
U_{5111}	$hU_{1111} - \frac{1}{2}c_x(c'_x)^2 h + T'_{111}c'_x$
U_{5112}	$hU_{1112} - c_x c'_x h s'_x + T'_{111}s'_x - \frac{1}{2}(c'_x)^2 h s_x + T'_{112}c'_x$
U_{5116}	$hU_{1116} - c_x c'_x d'_x h - \frac{1}{2}(c'_x)^2 d_x h + T'_{111}d'_x + T'_{116}c'_x$
U_{5122}	$hU_{1122} - \frac{1}{2}c_x h(s'_x)^2 - c'_x h s_x s'_x + T'_{112}s'_x + T'_{122}c'_x$
U_{5126}	$hU_{1126} - c_x d'_x h s'_x - c'_x d_x h s'_x + T'_{116}s'_x - c'_x d'_x h s_x + T'_{112}d'_x + T'_{126}c'_x$
U_{5133}	$hU_{1133} - \frac{1}{2}c_x(c'_y)^2 h + T'_{313}c'_y + T'_{133}c'_x$
U_{5134}	$hU_{1134} - c_x c'_y h s'_y + T'_{313}s'_y + T'_{314}c'_y + T'_{134}c'_x$
U_{5144}	$hU_{1144} - \frac{1}{2}c_x h(s'_y)^2 + T'_{314}s'_y + T'_{144}c'_x$
U_{5166}	$hU_{1166} - \frac{1}{2}c_x(d'_x)^2 h - c'_x d_x d'_x h + T'_{116}d'_x + T'_{166}c'_x$
U_{5222}	$hU_{1222} - \frac{1}{2}h s_x(s'_x)^2 + T'_{122}s'_x$
U_{5226}	$hU_{1226} - \frac{1}{2}d_x h(s'_x)^2 - d'_x h s_x s'_x + T'_{126}s'_x + T'_{122}d'_x$
U_{5233}	$hU_{1233} + T'_{133}s'_x - \frac{1}{2}(c'_y)^2 h s_x + T'_{323}c'_y$
U_{5234}	$hU_{1234} - c'_y h s_x s'_y + T'_{323}s'_y + T'_{134}s'_x + T'_{324}c'_y$
U_{5244}	$hU_{1244} - \frac{1}{2}h s_x(s'_y)^2 + T'_{324}s'_y + T'_{144}s'_x$
U_{5266}	$hU_{1266} - d_x d'_x h s'_x + T'_{166}s'_x - \frac{1}{2}(d'_x)^2 h s_x + T'_{126}d'_x$
U_{5336}	$hU_{1336} - \frac{1}{2}(c'_y)^2 d_x h + T'_{133}d'_x + T'_{336}c'_y$
U_{5346}	$hU_{1346} - c'_y d_x h s'_y + T'_{336}s'_y + T'_{134}d'_x + T'_{346}c'_y$
U_{5446}	$hU_{1446} - \frac{1}{2}d_x h(s'_y)^2 + T'_{346}s'_y + T'_{144}d'_x$
U_{5666}	$hU_{1666} - \frac{1}{2}d_x(d'_x)^2 h + T'_{166}d'_x$

Chapter 2

Third Order Optics for Ideal Magnets

2.1 Introduction

Once the equations of motion are obtained, Taylor-expanded transfer maps can be calculated for various magneto-optical elements which can the be concatenated, following well-defined rules, to obtain a resultant map that characterizes the entire system [2,3] or used to study beam properties as the beam passes through each element [4]. There exist various computer codes, e. g. [12,14,16], that use the matrix approach to perform accelerator lattice designs, optics analysis, and beam tracking.

In this chapter, we consider the optical properties of common accelerator components: a dipole, a quadrupole, a sextupole, and an octupole. We assume that the multipole fields abruptly go to zero at the boundary of the magnetic element. Thus, we limit the generality of the problem discussed in the previous chapter to the cases where h, k_1, k_2, k_3 are constant.

We also introduce so called "TRANSPORT notation", which is used in the computer program TRANSPORT [2] for designing charged particle transport systems. It simply transforms the curvilinear coordinate system (x, y, t) into the local rectangular system (x, y, z), consistent in passing from one magnetic element to the next.

Before we take up the magnetic multipoles, let us mention the simplest of the optical elements, a pure drift with zero magnetic field. The equations of motion become simply,

$$
\begin{aligned}
x'' &= 0 \\
y'' &= 0
\end{aligned}
\tag{2.1}
$$

and

$$
\begin{aligned}
s_x(t) = s_y(t) = t \\
c_x(t) = c_y(t) = 1 \\
d_x(t) = 0
\end{aligned}
\tag{2.2}
$$

The drift is a linear, non-chromatic (momentum independent) element described by the following first-order transfer matrix,

$$
\mathbf{R}^{\mathrm{drift}} = \begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\tag{2.3}
$$

where L is the length extent of the drift.

There are also two non-zero second-order matrix elements describing the path-length difference in the drift. From Eq. 1.33, we get

$$
T_{511}^{\mathrm{drift}} = T_{533}^{\mathrm{drift}} = \frac{L}{2}
\tag{2.4}
$$

2.2 Pure Dipole

2.2.1 First-Order Matrix

For a pure dipole, the field is equal to a constant,

$$
\begin{aligned}
B_x(x,y,t) &= 0 \\
B_y(x,y,t) &= B_0 \\
B_t(x,y,t) &= 0
\end{aligned}
\tag{2.5}
$$

and $h(t) = 1/\rho_0 = \text{const}$, $k_1 = k_2 = k_3 = 0$. The homogeneous solutions to Eq. 1.28 become:

$$\begin{aligned} s_x(t) &= \rho_0 \sin \frac{t}{\rho_0} \\ c_x(t) &= \cos \frac{t}{\rho_0} \\ s_y(t) &= t \\ c_y(t) &= 1 \end{aligned} \tag{2.6}$$

The dispersion function $d_x(t)$ is evaluated from Eq. 1.31:

$$\begin{aligned} d_x(t) &= \sin \frac{t}{\rho_0} \int_0^t \cos \frac{\tau}{\rho_0} d\tau - \cos \frac{t}{\rho_0} \int_0^t \sin \frac{\tau}{\rho_0} d\tau \\ &= \rho_0 \left(1 - \cos \frac{t}{\rho_0}\right) \\ &= \rho_0 \left[1 - c_x(t)\right] \end{aligned} \tag{2.7}$$

The path-length matrix elements are obtained from Eq. 1.35:

$$\begin{gathered} R_{51} = \int_0^t \cos \frac{\tau}{\rho_0} d\frac{\tau}{\rho_0} = \sin \frac{t}{\rho_0} = \frac{s_x(t)}{\rho_0} \\ R_{52} = \int_0^t \sin \frac{\tau}{\rho_0} d\tau = \rho_0 \left(1 - \cos \frac{t}{\rho_0}\right) = \rho_0 \left[1 - c_x(t)\right] \\ R_{56} = \int_0^t \left[1 - c_x(\tau)\right] d\tau = t - s_x(t) \end{gathered} \tag{2.8}$$

So, together with Eq. 2.6, the first-order transfer map for a pure dipole is described by the following matrix,

$$\mathbf{R}^{\text{dipole}} = \begin{pmatrix} c_x & s_x & 0 & 0 & 0 & \rho_0 (1 - c_x) \\ -s_x/\rho_0^2 & c_x & 0 & 0 & 0 & s_x/\rho_0 \\ 0 & 0 & 1 & t & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ s_x/\rho_0 & \rho_0 (1 - c_x) & 0 & 0 & 1 & t - s_x \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.9}$$

2.2.2 Nonlinear Matrix Elements

In order to obtain physically useful results, it is necessary to introduce a rectangular coordinate system. So far, the discussion of matrix elements' computation has concerned a single magnetic element. All derivatives have been with respect to distance along the reference trajectory, assuming constant radius of curvature. We want to be able to calculate the first-, second-, and third-order transfer matrices from those of the individual element. To this end, we must express nonlinear matrix elements in a coordinate system which is consistent in passing from one element to the next. The curvilinear coordinate (x, y, t) is not consistent in this regard, since the curvature h of the central trajectory changes abruptly at the boundary of a dipole magnet.

Therefore, we will employ a rectangular system whose transverse coordinates are the same as those of the curvilinear system at one particular point and whose longitudinal axis points in the instantaneous direction of $\hat{t}$, at that one reference point. The Cartesian systems (x, y, z) at the entrance and the exit of a bending magnet are shown in Fig. 2.1. We define the "TRANSPORT coordinates" as follows,

$$\begin{aligned}
x_1 &= x \\
x_2 &= \frac{dx}{dz} = \frac{x'}{z'} = \frac{x'}{1+hx} \\
x_3 &= y \\
x_4 &= \frac{dy}{dz} = \frac{y'}{z'} = \frac{y'}{1+hx} \\
x_5 &= l \\
x_6 &= \delta
\end{aligned} \tag{2.10}$$

The difference between the coordinate systems has an effect only on nonlinear terms. Expressing the matrix elements in the rectangular system in terms of those in the curvilinear system, which we denote with the superscript c, we have for the second-order terms:

$T_{112} = T^c_{112} + hs_x$

$T_{211} = T^c_{211} - hc_x c'_x$

$T_{212} = T^c_{212} + hs'_x - h(c_x s'_x - c'_x s_x)$

$T_{216} = T^c_{216} - h(c_x d'_x + c'_x d_x)$

$T_{222} = T^c_{222} - hs_x s'_x$

$T_{226} = T^c_{226} - h(s_x d'_x + s'_x d_x)$

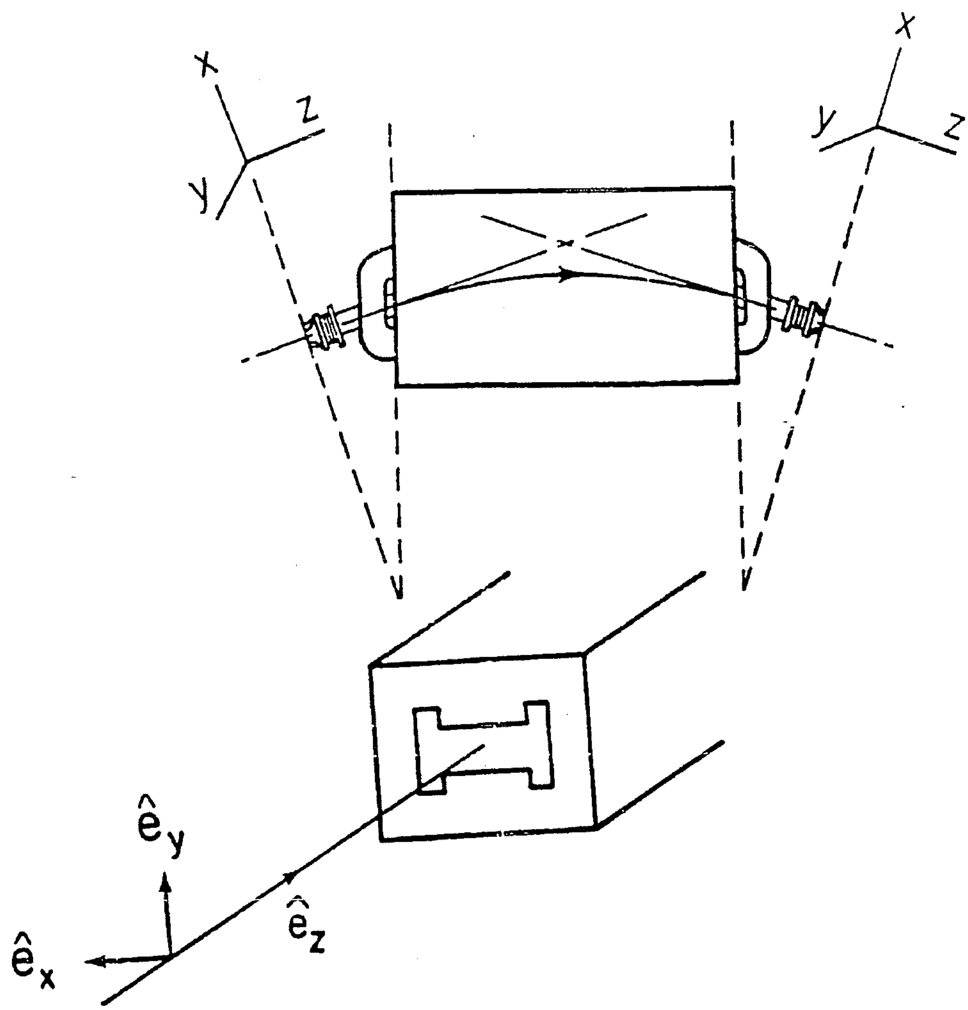

Figure 2.1: Cartesian systems at the entrance and the exit of a dipole magnet. The z-axis points in the instantaneous direction of the reference trajectory.

$$T_{266} = T^c_{266} - hd_x d'_x$$
$$T_{314} = T^c_{314} + hs_y$$
$$T_{413} = T^c_{413} - hc_x c'_y$$
$$T_{414} = T^c_{414} + hs'_y - hc_x s'_y$$
$$T_{423} = T^c_{423} - hs_x c'_y$$
$$T_{424} = T^c_{424} - hs_x c'_y$$
$$T_{436} = T^c_{436} - hc'_y d_x$$
$$T_{446} = T^c_{446} - hs'_y d_x$$

Only the elements different in the two systems are listed. Similarly, for the third-order terms, the connection formulas are:

$$U_{1112} = U^c_{1112} + T^c_{112}h$$
$$U_{1122} = U^c_{1122} + 2T^c_{122}h$$
$$U_{1126} = U^c_{1126} + T^c_{126}h$$
$$U_{1134} = U^c_{1134} + T^c_{134}h$$
$$U_{1144} = U^c_{1144} + 2T^c_{144}h$$
$$U_{2111} = U^c_{2111} + c_x^2 c'_x h^2 - (T^c_{111}c'_x + T^c_{211}c_x)h$$
$$U_{2112} = U^c_{2112} + ((c_x^2 - c_x)h^2 - T^c_{111}h)s'_x + ((2c_x - 1)c'_x h^2 - T^c_{211}h)s_x$$
$$-(T^c_{112}c'_x + T^c_{212}c_x - T^c_{212})h$$
$$U_{2116} = U^c_{2116} + (c_x^2 d'_x + 2c_x c'_x d_x)h^2 - (T^c_{111}d'_x + T^c_{211}d_x + T^c_{116}c'_x + T^c_{216}c_x)h$$
$$U_{2122} = U^c_{2122} + ((2c_x - 2)h^2 s_x - T^c_{112}h)s'_x + c'_x h^2 s_x^2 - T^c_{212}hs_x$$
$$-(T^c_{122}c'_x + T^c_{222}c_x - 2T^c_{222})h$$
$$U_{2126} = U^c_{2126} + ((2c_x - 1)d_x h^2 - T^c_{116}h)s'_x + (((2c_x - 1)d'_x + 2c'_x d_x)h^2 - T^c_{216}h)s_x$$
$$-(T^c_{112}d'_x + T^c_{212}d_x + T^c_{126}c'_x + T^c_{226})h$$
$$U_{2133} = U^c_{2133} - (T^c_{133}c'_x + T^c_{233}c_x)h$$
$$U_{2134} = U^c_{2134} - (T^c_{134}c'_x + T^c_{234}c_x - T^c_{234})h$$
$$U_{2144} = U^c_{2144} - (T^c_{144}c'_x + T^c_{244}c_x - 2T^c_{244})h$$
$$U_{2166} = U^c_{2166} + (2c_x d_x d'_x + c'_x d_x^2)h^2 - (T^c_{116}d'_x + T^c_{216}d_x + T^c_{166}c'_x + T^c_{266}c_x)h$$
$$U_{2222} = U^c_{2222} + (h^2 s_x^2 - T^c_{122}h)s'_x - T^c_{222}hs_x$$

$$U_{2226} = U^c_{2226} + (2d_x h^2 s_x - T^c_{126}h)s'_x + d'_x h^2 s_x^2 - T^c_{226}hs_x - (T^c_{122}d'_x + T^c_{222}d_x)h$$
$$U_{2233} = U^c_{2233} - T^c_{133}hs'_x - T^c_{233}hs_x$$
$$U_{2234} = U^c_{2234} - T^c_{134}hs'_x - T^c_{234}hs_x$$
$$U_{2244} = U^c_{2244} - T^c_{144}hs'_x - T^c_{244}hs_x$$
$$U_{2266} = U^c_{2266} + (d_x^2 h^2 - T^c_{166}h)s'_x + (2d_x d'_x h^2 - T^c_{266}h)s_x - (T^c_{126}d'_x + T^c_{226}d_x)h$$
$$U_{2336} = U^c_{2336} - (T^c_{133}d'_x + T^c_{233}d_x)h$$
$$U_{2346} = U^c_{2346} - (T^c_{134}d'_x + T^c_{234}d_x)h$$
$$U_{2446} = U^c_{2446} - (T^c_{144}d'_x + T^c_{244}d_x)h$$
$$U_{2666} = U^c_{2666} + d_x^2 d'_x h^2 - (T^c_{166}d'_x + T^c_{266}d_x)h$$
$$U_{3114} = U^c_{3114} + T^c_{314}h$$
$$U_{3123} = U^c_{3123} + T^c_{323}h$$
$$U_{3124} = U^c_{3124} + 2T^c_{324}h$$
$$U_{3146} = U^c_{3146} + T^c_{346}h$$
$$U_{4113} = U^c_{4113} + c_x^2 c'_y h^2 - (T^c_{111}c'_y + T^c_{413}c_x)h$$
$$U_{4114} = U^c_{4114} + ((c_x^2 - c_x)h^2 - T^c_{111}h)s'_y + (T^c_{414} - T^c_{414}c_x)h$$
$$U_{4123} = U^c_{4123} + ((2c_x - 1)c'_y h^2 - T^c_{413}h)s_x - (T^c_{112}c'_y + T^c_{423}c_x - T^c_{423})h$$
$$U_{4124} = U^c_{4124} + ((2c_x - 2)h^2 s_x - T^c_{112}h)s'_y - T^c_{414}hs_x + (2T^c_{424} - T^c_{424}c_x)h$$
$$U_{4136} = U^c_{4136} + 2c_x c'_y d_x h^2 - (T^c_{413}d_x + T^c_{116}c'_y + T^c_{436}c_x)h$$
$$U_{4146} = U^c_{4146} + ((2c_x - 1)d_x h^2 - T^c_{116}h)s'_y - (T^c_{414}d_x + T^c_{446}c_x - T^c_{446})h$$
$$U_{4223} = U^c_{4223} + c'_y h^2 s_x^2 - T^c_{423}hs_x - T^c_{122}c'_y h$$
$$U_{4224} = U^c_{4224} + (h^2 s_x^2 - T^c_{122}h)s'_y - T^c_{424}hs_x$$
$$U_{4236} = U^c_{4236} + (2c'_y d_x h^2 - T^c_{436}h)s_x - (T^c_{423}d_x + T^c_{126}c'_y)h$$
$$U_{4246} = U^c_{4246} + (2d_x h^2 s_x - T^c_{126}h)s'_y - T^c_{446}hs_x - T^c_{424}d_x h$$
$$U_{4333} = U^c_{4333} - T^c_{133}c'_y h$$
$$U_{4334} = U^c_{4334} - T^c_{133}hs'_y - T^c_{134}c'_y h$$
$$U_{4344} = U^c_{4344} - T^c_{134}hs'_y - T^c_{144}c'_y h$$
$$U_{4366} = U^c_{4366} + c'_y d_x^2 h^2 - (T^c_{436}d_x + T^c_{166}c'_y)h$$
$$U_{4444} = U^c_{4444} - T^c_{144}hs'_y$$
$$U_{4466} = U^c_{4466} + (d_x^2 h^2 - T^c_{166}h)s'_y - T^c_{446}d_x h$$

Then, using the Green's function integral with

$$G_x(t,\tau) = \rho_0 \sin\frac{(t-\tau)}{\rho_0}$$

$$G_y(t,\tau) = (t-\tau)$$

and the driving terms given in Tables 1.1 and 1.2, we obtain the transverse matrix elements,

$$s_x = \rho_0 \sin(t/\rho_0)$$
$$c_x = \cos(t/\rho_0)$$

$$T_{111} = -(1-c_x^2)/(2\rho_0)$$
$$T_{112} = s_x c_x/\rho_0$$
$$T_{116} = s_x^2$$
$$T_{122} = \rho_0 s_x(1-c_x)$$
$$T_{126} = s_x(1-c_x)$$
$$T_{144} = -\rho_0(1-c_x)/2$$
$$T_{166} = -s_x^2/(2\rho_0)$$
$$T_{314} = ts_x/\rho_0$$
$$T_{324} = \rho_0 t(1-c_x)$$
$$T_{346} = t - s_x$$

$$U_{1111} = c_x^3h^4/8 - c_xh^4/8 + c_x^5h^2/8 - c_x^3h^2/4 + c_xh^2/8$$
$$U_{1112} = c_x^2h^4s_x/8 + c_x^4h^2s_x/8 - c_x^2h^2s_x/8$$
$$\begin{aligned}U_{1116} = {} & h^3s_xt/2 - hs_xt/2 - 11c_xh^5s_x^4/8 - h^5s_x^4/2 + c_xh^3s_x^4 + 5c_xh^5s_x^2/8 \\ & -c_x^3h^3s_x^2/2 + h^3s_x^2 + c_x^3hs_x^2/2 + c_x^5h/4 + c_x^4h/2 - c_x^2h/2 - c_xh/4\end{aligned}$$
$$\begin{aligned}U_{1122} = {} & -h^2s_xt/4 - 3c_x^2h^2s_x/8 - 3c_x^4s_x/8 - c_x^2s_x/8 - c_xs_x + 3s_x/2 \\ & -c_x^3h^2/8 + c_xh^2/8 - c_x^5/8 + c_x^3/2 + c_x^2 - 11c_x/8\end{aligned}$$
$$U_{1126} = -c_xht/2 + c_xt/(2h) + c_x^2h^3s_x/4 + c_x^4hs_x/4 - c_x^4s_x/(4h) - c_x^2s_x/(4h)$$
$$U_{1144} = -h^2s_xt^2/4 + h^2s_xt/2 + c_xt/4 - s_x/4 + c_x^2/2 - 1/2$$
$$U_{1166} = -h^2s_xt/2 + s_xt/2 + c_xh^4s_x^4 + h^4s_x^4 - c_xh^2s_x^4 - c_xh^4s_x^2/2$$

$$+c_x^3h^2s_x^2/2 + c_xh^2s_x^2/2 - 2h^2s_x^2 - c_x^3s_x^2/2 - c_x^4 + c_x^2$$

$$U_{1222} = c_xt/4 + c_x^2h^2s_x/4 + c_x^4s_x/4 - s_x/2 - 3c_x^5/(8h^2) + 5c_x^3/(4h^2) - c_x^2/h^2$$
$$+c_x/(8h^2) - 3c_x^3/8 + 3c_x/8$$

$$U_{1226} = hs_xt/4 + 3c_x^2hs_x/8 + 3c_x^4s_x/(8h) + c_x^2s_x/(8h) - c_xs_x/h + s_x/(2h)$$
$$+c_x^3h/8 - c_xh/8 + c_x^5/(8h) - c_x^3/(2h) - c_x^2/(2h) + 7c_x/(8h)$$

$$U_{1244} = s_xt^2/4 - c_xt/(4h^2) - c_xt/2 + s_x/(4h^2) + c_xs_x/2 + c_x/h^2 - 1/h^2$$

$$U_{1446} = c_x/(2h) - c_x^2/(2h)$$

$$U_{1666} = 1/(2h) - c_x^2/(2h)$$

$$U_{3114} = -h^2t^2/4 + c_xh^2s_xt/4 + h^2s_xt + h^2t/4 - c_xh^2s_x/4 - h^2s_x$$

$$U_{3124} = h^2t^3/12 + t^2/4 - c_xs_xt/4 - 2s_xt - c_x^2t/4 + 5t/4$$
$$+s_x + c_x^2/2 + 2c_x - 5/2$$

$$U_{3146} = ht^2/4 - c_xhs_xt/4 - 3ht/4 + t/(2h) + 3c_xhs_x/4 - c_xs_x/(2h)$$

$$U_{3224} = -t^3/12 + c_x^2t/(4h^2) - t/(4h^2) - t/2 + c_xs_x/2$$
$$+c_x^2/(2h^2) - 2c_x/h^2 + 3/(2h^2)$$

$$U_{3246} = -ht^2/4 - t^2/(4h) + c_xs_xt/(4h) + t/h - s_x/h$$
$$-c_x^2/(4h) - c_x/h + 5/(4h)$$

$$U_{3444} = s_x/2 - t/2$$

Only non-zero "position" matrix elements are listed above. The "angle" matrix elements T_{2jk}, T_{4jk} and U_{2jkl}, U_{4jkl} are found from the connection formulas and differentiation of the "position" matrix elements. The basic trigonometric integrals used in evaluating dipole terms through the Green's function integration are given in Table 2.1.

For the path-length difference terms, we have the following relations between the two coordinate systems:

$$T_{512} = T^c_{512} + hR^c_{52}$$

$$\begin{aligned}
U_{5112} &= U^c_{5112} + hT^c_{512} \\
U_{5122} &= U^c_{5122} + 2hT^c_{522} \\
U_{5126} &= U^c_{5126} + hT^c_{526} \\
U_{5134} &= U^c_{5134} + hT^c_{534} \\
U_{5144} &= U^c_{5144} + 2hT^c_{544}
\end{aligned}$$

Again, only the differing terms are shown. The matrix elements are given by the integrals

Table 2.1: Basic Integrals Used in Evaluating Dipole Matrix Elements

$I(t)$	$\int_0^t I(\tau)d\tau$
c_x	s_x
c_x^2	$(t + c_x s_x)/2$
c_x^3	$s_x - h^2 s_x^3/3$
c_x^4	$(3t - c_x h^2 s_x^3 + c_x^3 s_x + 4c_x s_x)/8$
s_x	$(1 - c_x)/h^2$
$c_x s_x$	$s_x^2/2$
$c_x^2 s_x$	$(1 - c_x^3)/(3h^2)$
$c_x^3 s_x$	$(1 - c_x^4)/(4h^2)$
s_x^2	$(t - c_x s_x)/(2h^2)$
$c_x s_x^2$	$s_x^3/3$
$c_x^3 s_x^2$	$(t + c_x h^2 s_x - c_x^3 s_x)/(8h^2)$
s_x^3	$(c_x^3 - 3c_x + 2)/(3h^4)$
$c_x s_x^3$	$s_x^4/4$
s_x^4	$(3t - c_x h^2 s_x^3 + c_x^3 s_x - 4c_x s_x)/(8h^4)$
$c_x t$	$(h^2 s_x t + c_x - 1)/h^2$
$c_x^2 t$	$(t^2 + 2c_x s_x t - s_x^2)/4$
$s_x t$	$(s_x - c_x t)/h^2$
$c_x s_x t$	$(h^2 s_x^2 t - c_x^2 t + c_x s_x)/(4h^2)$
$s_x^2 t$	$(t^2 - 2c_x s_x t + s_x^2)/(4h^2)$
$c_x^2 t^2$	$t^3/6 + c_x s_x (t^2/2 - 1/(4h^2)) - s_x^2 t/4 + c_x^2 t/(4h^2)$
$c_x s_x t^2$	$(-(c_x^2 - h^2 s_x^2)(2h^2 t^2 - 1) + 4c_x h^2 s_x t - 1)/(8h^4)$
$s_x^2 t^2$	$(t^3/6 - c_x s_x (t^2/2 - 1/(4h^2)) + s_x^2 t/4 - c_x^2 t/(4h^2))/h^2$

in the Tables 1.3, 1.4 for the second and third order respectively.

Thus, we have succeded in obtaining transfer matrices for a dipole magnet. We can express the results with respect to the angle through which the beam is bent, $\phi = t/\rho$. Additional optical effect is produced by the boundaries of the dipole if they are not normal to the reference trajectory. We discuss this point in great detail in the next chapter.

Next, we consider higher multipoles.

2.3 Quadrupole

2.3.1 First-Order Matrix

A pure quadrupole, whose cross-sectional configuration is shown in Fig. 2.2, produces the field described by the following scalar potential ϕ,

$$\phi = \frac{B_0 xy}{a} \tag{2.11}$$

where a is the pole-tip radius and B_0 is the magnitude of the field at the pole-tip radius.

The reference trajectory through a quadrupole is a straight line, known as the optical axis. Since $h = 0$ for a quadrupole, there is no need to introduce a new rectangular coordinate system as we did for the case of a dipole. The magnetic field components are given by the gradient of the scalar potential,

$$\begin{aligned} B_x &= \frac{B_0 y}{a} = gy \\ B_y &= \frac{B_0 x}{a} = gx \end{aligned} \tag{2.12}$$

with g denoting the magnitude of the field gradient. To obtain the equation of motion, we put in Eq. 1.27

$$\begin{aligned} h &= 0 \\ k_1 h^2 &= -\frac{qg}{p_0} \\ k_2 &= 0 \\ k_3 &= 0 \end{aligned} \tag{2.13}$$

Defining

$$k_q = \frac{qg}{p_0}$$

we obtain from Eq. 1.28,

$$\begin{aligned} x'' + k_q^2 x &= 0 \\ y'' - k_q^2 y &= 0 \end{aligned} \tag{2.14}$$

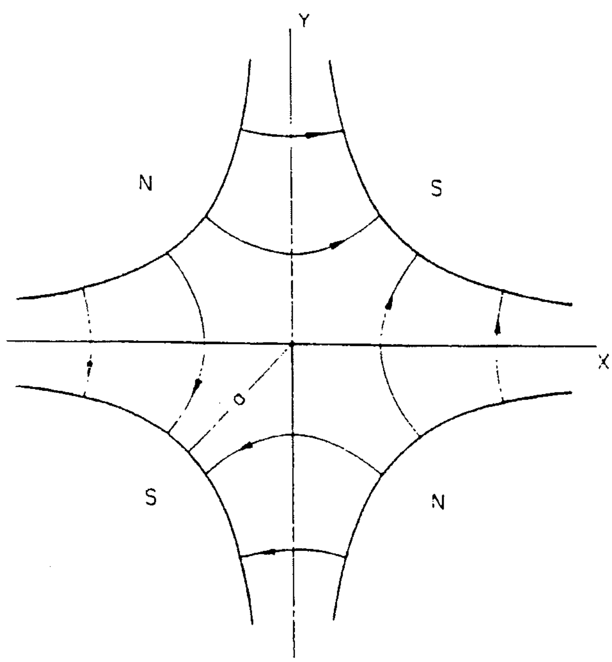

Figure 2.2: The cross section of a quadrupole; a is the distance from the optical axis to the nearest point on the pole tip.

In the usual convention the pole-tip field is positive, so that $k_q^2 > 0$ and the magnetic field tends to restore the trajectory toward the optical axis in the horizontal plane and to deflect it away from the optical axis in the vertical plane. The characteristic rays for the quadrupole are:

$$\begin{aligned}
s_x(t) &= \frac{1}{k_q}\sin k_q t \\
c_x(t) &= \cos k_q t \\
d_x(t) &= 0 \\
s_y(t) &= \frac{1}{k_q}\sinh k_q t \\
c_y(t) &= \cosh k_q t
\end{aligned} \tag{2.15}$$

The first-order transfer matrix is then given by,

$$\mathbf{R}^{\mathrm{quad}} = \begin{pmatrix} c_x & s_x & 0 & 0 & 0 & 0 \\ -k_q^2 s_x & c_x & 0 & 0 & 0 & 0 \\ 0 & 0 & c_y & s_y & 0 & 0 \\ 0 & 0 & k_q^2 s_y & c_y & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.16}$$

A single quadrupole always focuses the beam in one plane and defocuses it in the other.

2.3.2 Nonlinear Matrix Elements

The Green's functions for the quadrupole are:

$$G_x(t,\tau) = \frac{\sin k_q(t-\tau)}{k_q}$$

$$G_y(t,\tau) = \frac{\sinh k_q(t-\tau)}{k_q}$$

The only non-zero second-order matrix elements are the chromatic, δ-dependent, terms,

$$\begin{aligned} T_{116} &= k_q^2 t s_x/2 \\ T_{126} &= (s_x - tc_x)/2 \\ T_{336} &= -k_q^2 t s_y/2 \\ T_{346} &= (s_y - tc_y)/2 \end{aligned}$$

The third-order terms are evaluated with the help of integrals given in Table 2.2:

$$\begin{aligned}
U_{1111} &= (3k_q^6 s_x s_x(4t) + 24k_q^6 c_x s_x^4 - 12k_q^6 t s_x)/64 \\
U_{1112} &= ((6k_q^4 + 3k_q^2)c_x s_x(4t) - 24k_q^2 c_x s_x(2t) - 24k_q^6 s_x^5 + (48k_q^2 - 48k_q^2 c_x^4)s_x \\
&\quad +(36k_q^2 - 24k_q^4)tc_x)/64 \\
U_{1122} &= -((6k_q^4 + 3k_q^2)s_x s_x(4t) + 24k_q^2 s_x s_x(2t) + 48k_q^4 c_x s_x^4 + (36k_q^2 - 24k_q^4)t s_x \\
&\quad +24c_x^5 - 24c_x)/64 \\
U_{1133} &= -((k_q^4 c_x s_x(2t) - k_q^4 s_x c_x(2t) + 2k_q^4 s_x)s_y(2t) + (3k_q^4 s_x c_y(2t) - 2k_q^4 s_x)s_x(2t) \\
&\quad +k_q^2 c_x c_y^2(2t) + (2k_q^2 c_x c_x(2t) - 6k_q^2 c_x)c_y(2t) - 4k_q^4 t s_x + 3k_q^2 c_x)/32 \\
U_{1134} &= -((3k_q^4 s_x s_x(2t) + k_q^2 c_x c_x(2t) - 4k_q^2 c_x)s_y(2t) + k_q^2 c_x c_y(2t)s_x(2t) - k_q^2 s_x c_y^2(2t) \\
&\quad +2k_q^2 s_x c_y(2t) - k_q^2 s_x)/16 \\
U_{1144} &= -((k_q^2 c_x s_x(2t) - k_q^2 s_x c_x(2t) + 2k_q^2 s_x)s_y(2t) + (3k_q^2 s_x c_y(2t) + 2k_q^2 s_x)s_x(2t) \\
&\quad +(3c_x c_x(2t) - 4c_x)c_y(2t) + 2c_x c_x(2t) + 4k_q^2 t s_x - c_x)/32 \\
U_{1166} &= ((5k_q^2 s_x + 2k_q^2 tc_x)s_x(2t) - 2k_q^2 t s_x c_x(2t) + 6k_q^2 c_x s_x^2 - 8k_q^2 t s_x - 2k_q^2 t^2 c_x)/16 \\
U_{1222} &= -(3k_q^2 c_x s_x(4t) + (24 - 24c_x^4)s_x - 12k_q^2 tc_x)/64 \\
U_{1233} &= ((k_q^4 s_x s_x(2t) + k_q^2 c_x c_x(2t) + 2k_q^2 c_x)s_y(2t) + (2k_q^2 c_x - 3k_q^2 c_x c_y(2t))s_x(2t) \\
&\quad +k_q^2 s_x c_y^2(2t) + (2k_q^2 s_x c_x(2t) - 2k_q^2 s_x)c_y(2t) + 4k_q^2 s_x c_x(2t) - 5k_q^2 s_x - 4k_q^2 tc_x)/32 \\
U_{1234} &= -((3k_q^2 c_x s_x(2t) - 3k_q^2 s_x c_x(2t) - 4k_q^2 s_x)s_y(2t) - k_q^2 s_x c_y(2t)s_x(2t) \\
&\quad -c_x c_y^2(2t) - 2c_x c_x(2t) + 3c_x)/16
\end{aligned}$$

$$U_{1244} = ((k_q^2 s_x s_x(2t) + 3c_x c_x(2t) + 2c_x)s_y(2t) + (-3c_x c_y(2t) - 2c_x)s_x(2t)$$
$$+3s_x c_x(2t)c_y(2t) + 6s_x c_x(2t) - 9s_x + 4tc_x)/32$$
$$U_{1266} = -((2k_q^2 ts_x + 3c_x)s_x(2t) + 2tc_x c_x(2t) + 2k_q^2 s_x^3 + 2k_q^2 t^2 s_x - 8tc_x)/16$$
$$U_{3113} = -((k_q^4 c_y s_x(2t) + k_q^4 s_y c_x(2t) - 2k_q^4 s_y)s_y(2t) + (2k_q^4 s_y - k_q^4 s_y c_y(2t))s_x(2t)$$
$$-2k_q^2 c_y c_y^2(2t) + (6k_q^2 c_y - k_q^2 c_y c_x(2t))c_y(2t) - 4k_q^4 ts_y - 3k_q^2 c_y)/32$$
$$U_{3114} = ((k_q^4 s_y s_x(2t) + 3k_q^2 c_y c_x(2t) - 2k_q^2 c_y)s_y(2t) + (-k_q^2 c_y c_y(2t) - 2k_q^2 c_y)s_x(2t)$$
$$+(2k_q^2 s_y - 3k_q^2 s_y c_x(2t))c_y(2t) - 4k_q^2 s_y c_x(2t) + 5k_q^2 s_y + 4k_q^2 tc_y)/32$$
$$U_{3123} = -((3k_q^4 s_y s_x(2t) - k_q^2 c_y c_x(2t))s_y(2t) + (4k_q^2 c_y - 3k_q^2 c_y c_y(2t))s_x(2t)$$
$$+(k_q^2 s_y c_x(2t) - 2k_q^2 s_y)c_y(2t) + k_q^2 s_y)/16$$
$$U_{3124} = ((3k_q^2 c_y s_x(2t) + k_q^2 s_y c_x(2t))s_y(2t) + (-3k_q^2 s_y c_y(2t) - 4k_q^2 s_y)s_x(2t) - c_y c_y^2(2t)$$
$$+2c_y c_x(2t)c_y(2t) + 2c_y c_x(2t) - 3c_y)/16$$
$$U_{3223} = ((k_q^2 c_y s_x(2t) + 3k_q^2 s_y c_x(2t) + 2k_q^2 s_y)s_y(2t) + (2k_q^2 s_y - k_q^2 s_y c_y(2t))s_x(2t)$$
$$-2c_y c_y^2(2t) + (4c_y - c_y c_x(2t))c_y(2t) - 2c_y c_x(2t) + 4k_q^2 ts_y + c_y)/32$$
$$U_{3224} = -((k_q^2 s_y s_x(2t) + 3c_y c_x(2t) + 2c_y)s_y(2t) + (-c_y c_y(2t) - 2c_y)s_x(2t)$$
$$-3s_y c_x(2t)c_y(2t) - 6s_y c_x(2t) + 9s_y - 4tc_y)/32$$
$$U_{3333} = (3k_q^6 s_y s_y(4t) - 24k_q^3 c_y s_y^4 - 12k_q^6 ts_y)/64$$
$$U_{3334} = -((3k_q^6 + 6k_q^4)c_y s_y(4t) - 24k_q^6 c_y s_y(2t) - 24k_q^3 s_y^5 + (-48k_q^4 c_y^2 - 48k_q^4)s_y^3$$
$$+(36k_q^6 - 24k_q^4)tc_y)/64$$
$$U_{3344} = ((6k_q^4 + 3k_q^2)s_y s_y(4t) + 24k_q^2 s_y s_y(2t) - 48k_q c_y s_y^4 + (-24k_q^2 c_y^3 - 24k_q^2 c_y)s_y^2$$
$$+(36k_q^2 - 24k_q^4)ts_y)/64$$
$$U_{3366} = ((3k_q^2 s_y - 2k_q^2 tc_y)s_y(2t) + 2k_q^2 ts_y c_y(2t) - 6k_q^2 c_y s_y^2 + 8k_q^2 ts_y + 2k_q^2 t^2 c_y)/16$$
$$U_{3444} = -(3k_q^2 c_y s_y(4t) + (-24k_q^2 c_y^2 - 24k_q^2)s_y^3 - 12k_q^2 tc_y)/64$$
$$U_{3466} = ((2k_q^2 ts_y - c_y)s_y(2t) - 2tc_y c_y(2t) + 6k_q^2 s_y^3 + 2k_q^2 t^2 s_y + 4tc_y)/16$$

Again, only the non-zero "position" matrix elements are given; corresponding "angle" terms are obtained by differentiation.

Table 2.2: Basic Integrals Used in Evaluating Quadrupole Matrix Elements

$I(t)$	$\int_0^t I(\tau)d\tau$
c_y^4	$\left[12t + s_y(4t) + 8s_y(2t)\right]/32$
$s_y c_y^3$	$s_y^2(t)\left[1 + c_y^2(t)\right]/4$
$s_y^2 c_y^2$	$-\left[4t - s_y(4t)\right]/32$
$c_y^3 s_y$	$s_y^4(t)/\left(4k_q^3\right)$
s_y^4	$\left[12t + s_y(4t) - 8s_y(2t)\right]/32$
$c_x s_y c_y^2$	$\left\{\left[s_y(3t) + 5s_y(t)\right]s_x(t) + \left[3c_y(3t) + 5c_y(t)\right]c_x(t)/k_q^2 - 8\right\}/40$
$c_x s_y^2 c_y$	$\left\{\left[c_y(3t) - 5c_y(t)\right]s_x(t) + \left[3s_y(3t) - 5s_y(t)\right]c_x(t)\right\}/\left(40k_q^2\right)$
$c_x^2 c_y^2$	$\left\{s_x(2t)\left[c_y(2t) + 2\right] + s_y(2t)\left[c_x(2t) + 2\right] + 4t\right\}/16$
$c_x^2 s_y c_y$	$\left\{s_x(2t)s_y(2t) + \left[c_y(2t) + 2\right]c_x(2t)/k_q^2 - 3/k_q^2\right\}/16$
$c_x^2 s_y^2$	$\left\{s_x(2t)\left[c_y(2t) - 2\right] + s_y(2t)\left[c_x(2t) + 2\right] - 4t\right\}/\left(16k_q^2\right)$
c_x^4	$\left[12t + s_x(4t) + 8s_x(2t)\right]/32$
$s_x s_y c_y^2$	$\left\{\left[3c_y(3t) + 5c_y(t)\right]s_x(t) - \left[s_y(3t) + 5s_y(t)\right]c_x(t)\right\}/\left(40k_q^2\right)$
$s_x s_y^2 c_y$	$\left\{\left[3s_y(3t) - 5c_y(t)\right]s_x(t) - \left[c_y(3t) - 5c_y(t)\right]c_x(t)/k_q^2 - 4/k_q^2\right\}/\left(40k_q^2\right)$
$s_x c_x c_y^2$	$\left\{s_x(2t)s_y(2t) - \left[c_y(2t) + 2\right]c_x(2t)/k_q^2 + 3/k_q^2\right\}/16$
$s_x c_x s_y c_y$	$\left[s_x(2t)c_y(2t) - c_x(2t)s_y(2t)\right]/\left(16k_q^2\right)$
$s_x c_x s_y^2$	$\left\{s_x(2t)s_y(2t) - \left[c_y(2t) - 2\right]c_y(2t)/k_q^2 - 1/k_q^2\right\}/\left(16k_q^2\right)$
$s_x c_x^2 c_y$	$\left\{s_x(3t)s_y(t) - 3c_x(3t)c_y(t)/k_q^2 + 5s_x(t)s_y(t) - 5c_x(t)c_y(t)/k_q^2 + 8/k_q^2\right\}/40$
$s_x c_x^2 s_y$	$\left[s_x(3t)c_y(t) - 3c_x(3t)s_y(t) + 5s_x(t)c_y(t) - 5c_x(t)s_y(t)\right]/\left(40k_q^2\right)$
$s_x c_x^3$	$\left[1 - c_x^4(t)\right]/\left(4k_q^2\right)$
$s_x^2 c_y^2$	$-\left\{\left[c_y(2t) + 2\right]s_x(2t) - s_y(2t)\left[c_x(2t) + 2\right] - 4t\right\}/\left(16k_q^2\right)$
$s_x^2 s_y c_y$	$-\left\{s_x(2t)s_y(2t) + \left[c_x(2t) - 2\right]c_y(2t)/k_q^2 + 1/k_q^2\right\}/\left(16k_q^2\right)$
$s_x^2 s_y^2$	$-\left\{\left[c_y(2t) - 2\right]s_x(2t) + s_y(2t)\left[c_x(2t) - 2\right] + 4t\right\}/\left(16k_q^4\right)$
$s_x^2 c_x c_y$	$-\left[3s_x(3t)c_y(t) + c_x(3t)s_y(t) - 5s_x(t)c_y(t) - 5c_x(t)s_y(t)\right]/\left(40k_q^2\right)$
$s_x^2 c_x s_y$	$-\left[3s_x(3t)s_y(t) + c_x(3t)c_y(t)/k_q^2 - 5s_x(t)s_y(t) - 5c_x(t)c_y(t)/k_q^2 + 4/k_q^2\right]/\left(40k_q^2\right)$
$s_x^2 c_x^2$	$\left[4t - s_x(4t)\right]/32$
$s_x^3 c_x$	$s_x^4(t)/4$
s_x^4	$\left[12t + s_x(4t) - 8s_x(2t)\right]/\left(32k_q^4\right)$

2.4 Sextupole

2.4.1 First-Order Matrix

The sextupole is purely non-linear element in a sense that its first-order transfer matrix is identical to that of a drift space. It has six poles, as shown in Fig. 2.3, and its scalar potential ϕ is a solution of the Laplace's equation symmetric under a 120° rotation. It is also odd in the vertical coordinate y because of the midplane symmetry:

$$\phi = \frac{B_0}{3a^2}(3x^2y - y^3) \tag{2.17}$$

where a is the pole-tip radius and B_0 is the magnitude of the field at the pole-tip radius.

The gradient of ϕ gives the components of the field,

$$\begin{aligned} B_x &= 2\frac{B_0}{a^2}xy \\ B_y &= \frac{B_0}{a^2}(x^2 - y^2) \end{aligned} \tag{2.18}$$

Like the quadrupole, the sextupole possesses an optical axis. To obtain the equation of motion, we put in Eq. 1.27

$$\begin{aligned} h &= 0 \\ k_1 &= 0 \\ k_2h^3 &= \frac{qB_0}{a^2p_0} \equiv k_s^2 \\ k_3 &= 0 \end{aligned} \tag{2.19}$$

Since the fields have no linear dependence, a pure sextupole has the first-order properties of a drift space and the characteristic rays are of particularly simple form,

$$\begin{aligned} s_x(t) &= s_y(t) = t \\ c_x(t) &= c_y(t) = 1 \\ d_x(t) &= 0 \end{aligned} \tag{2.20}$$

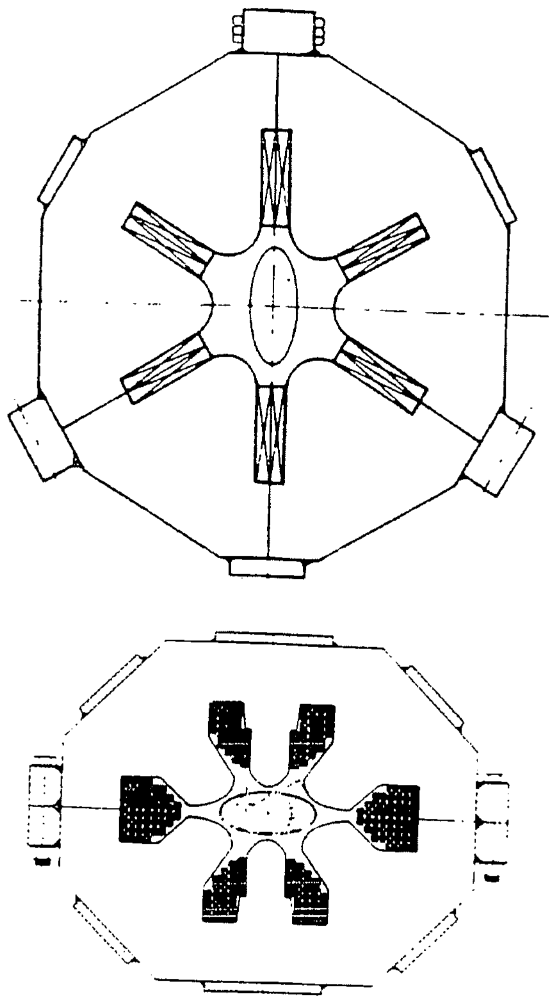

Figure 2.3: The cross section of a sextupole; a is the distance from the optical axis to the nearest point on the pole tip.

2.4.2 Nonlinear Matrix Elements

The sextupole Green's function is quite simple,

$$G_x(t,\tau) = G_y(t,\tau) = t - \tau$$

The non-zero "position" elements are obtained in a straightforward manner from the driving terms of Tables 1.1 and 1.2,

$$T_{111} = -k_s^2 t^2/2$$
$$T_{112} = -k_s^2 t^3/3$$
$$T_{122} = -k_s^2 t^4/12$$
$$T_{133} = k_s^2 t^2/2$$
$$T_{134} = k_s^2 t^3/3$$
$$T_{144} = k_s^2 t^4/12$$
$$T_{313} = k_s^2 t^2$$
$$T_{314} = k_s^2 t^3/3$$
$$T_{323} = k_s^2 t^3/3$$
$$T_{324} = k_s^2 t^4/6$$

$$U_{1111} = k_s^4 t^4/12$$
$$U_{1112} = k_s^4 t^5/12$$
$$U_{1116} = k_s^2 t^2/2$$
$$U_{1122} = k_s^4 t^6/36$$
$$U_{1126} = k_s^2 t^3/3$$
$$U_{1133} = k_s^4 t^4/12$$
$$U_{1134} = k_s^4 t^5/10$$
$$U_{1144} = k_s^4 t^6/60$$
$$U_{1222} = k_s^4 t^7/252$$
$$U_{1226} = k_s^2 t^4/12$$
$$U_{1233} = -k_s^4 t^5/60$$

$U_{1234} = k_s^4t^6/90$
$U_{1244} = k_s^4t^7/252$
$U_{1336} = -k_s^2t^2/2$
$U_{1346} = -k_s^2t^3/3$
$U_{1446} = -k_s^2t^4/12$
$U_{3113} = k_s^4t^4/12$
$U_{3114} = -k_s^4t^5/60$
$U_{3123} = k_s^4t^5/10$
$U_{3124} = k_s^4t^6/90$
$U_{3136} = -k_s^2t^2$
$U_{3146} = -k_s^2t^3/3$
$U_{3223} = k_s^4t^6/60$
$U_{3224} = k_s^4t^7/252$
$U_{3236} = -k_s^2t^3/3$
$U_{3246} = -k_s^2t^4/6$
$U_{3333} = k_s^4t^4/12$
$U_{3334} = k_s^4t^5/12$
$U_{3344} = k_s^4t^6/36$
$U_{3444} = k_s^4t^7/252$

2.5 Octupole

The scalar potential of an octupole, another purely non-linear element, is given by

$$\phi = \frac{B_0}{a^3}(x^3y - xy^3) \tag{2.21}$$

Fig. 2.4 shows the cross section of an octupole along its optical axis. The field components are:

$$B_x = \frac{B_0}{a^3}(3x^2y - y^3)$$
$$B_y = \frac{B_0}{a^3}(x^3 - 3xy^2) \tag{2.22}$$

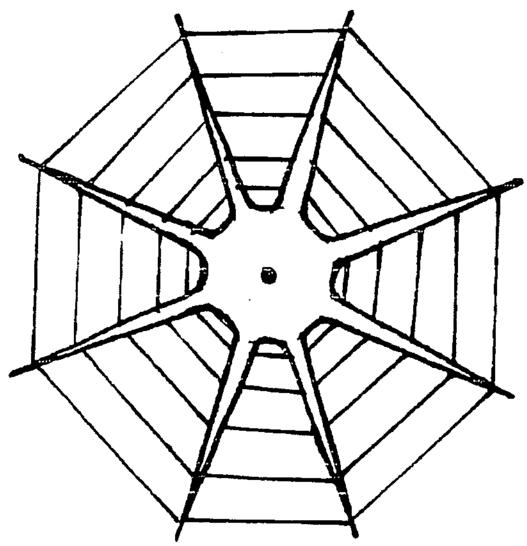

Figure 2.4: The cross section of an octupole; a is the distance from the optical axis to the nearest point on the pole tip.

First-order transfer matrix is that of a drift space with characteristic functions given by Eq. 2.20 and

$$G_x(t,\tau) = G_y(t,\tau) = t - \tau$$

The driving terms are obtained by putting

$$\begin{aligned} h &= 0 \\ k_1 &= 0 \\ k_2 &= 0 \\ k_3 h^4 &= \frac{qB_0}{a^3 p_0} \equiv k_o^2 \end{aligned} \tag{2.23}$$

The non-zero "position" elements are:

$U_{1111} = -k_o^2 t^2/2$
$U_{1112} = -k_o^2 t^3/2$
$U_{1122} = -k_o^2 t^4/4$
$U_{1133} = 3k_o^2 t^2/2$
$U_{1134} = k_o^2 t^3$
$U_{1144} = k_o^2 t^4/4$
$U_{1222} = -k_o^2 t^5/20$
$U_{1233} = k_o^2 t^3/2$
$U_{1234} = k_o^2 t^4/2$
$U_{1244} = 3k_o^2 t^5/20$
$U_{3113} = 3k_o^2 t^2/2$
$U_{3114} = k_o^2 t^3/2$
$U_{3123} = k_o^2 t^3$
$U_{3124} = k_o^2 t^4/2$
$U_{3223} = k_o^2 t^4/4$
$U_{3224} = 3k_o^2 t^5/20$
$U_{3333} = -k_o^2 t^2/2$
$U_{3334} = -k_o^2 t^3/2$

$$U_{3344} = -k_o^2 t^4/4$$
$$U_{3444} = -k_o^2 t^5/20$$

All the second-order matrix elements are zero for the octupole since the field components have no quadratic terms. Thus, the first and second order optics of a system is not changed by an octupole addition.

2.6 System of Elements

The magnetic elements discussed in this chapter are the only multipoles that contribute to the third-order optics. One can combine them in different systems to satisfy specific requirements of an accelerator or a beam line. To this end, we must know how to concatenate the transfer matrices of individual elements to obtain the transfer matrix of a system.

Consider a succession of two elements, the first taking a particle from t_0 to t_1, the second – from t_1 to t_0. We would like to find matrices $R_{ij}^{0\mapsto 2}$, $T_{ijk}^{0\mapsto 2}$, $U_{ijkl}^{0\mapsto 2}$ describing the transformation from t_0 to t_2 given the transfer matrices for the two elements.

The transformation through the first element is described up to the third order by

$$x_i(1) = \sum_{j=1}^{6} R_{ij}^{(0\mapsto 1)} x_j(0) + \sum_{j=1}^{6}\sum_{k=1}^{6} T_{ijk}^{(0\mapsto 1)} x_j(0) x_k(0) + \sum_{j=1}^{6}\sum_{k=1}^{6}\sum_{l=1}^{6} U_{ijkl}^{(0\mapsto 1)} x_j(0) x_k(0) x_l(0) \quad (2.24)$$

Here our definition of the matrices $\mathbf{T}$ and $\mathbf{U}$ differs from the one given by Eq. 1.11 in that the matrices here are rectangular rather than triangular. The use of rectangular matrices makes the nonlinear concatenations easier to derive. The difference between the two sets of matrices is that the off-diagonal elements of the rectangular matrices are exactly half and third of the corresponding elements of the triangular matrices for the second and third order terms respectively.

Similarly, the transformation through the second element is given by

$$x_i(2) = \sum_{j=1}^{6} R_{ij}^{(1\mapsto 2)} x_j(1) + \sum_{j=1}^{6}\sum_{k=1}^{6} T_{ijk}^{(1\mapsto 2)} x_j(1) x_k(1) + \sum_{j=1}^{6}\sum_{k=1}^{6}\sum_{l=1}^{6} U_{ijkl}^{(1\mapsto 2)} x_j(1) x_k(1) x_l(1) \quad (2.25)$$

Substituting Eq. 2.24 into Eq. 2.25, we can obtain the transfer matrices for the combined transformation through both elements,

$$\begin{aligned}
R_{ij}^{(0\mapsto 2)} &= \sum_k R_{ik}^{(1\mapsto 2)} R_{kj}^{(0\mapsto 1)} \\
T_{ijk}^{(0\mapsto 2)} &= \sum_l R_{il}^{(1\mapsto 2)} T_{ljk}^{(0\mapsto 1)} + \sum_{lm} T_{ilm}^{(1\mapsto 2)} R_{lj}^{(0\mapsto 1)} R_{mk}^{(0\mapsto 1)} \\
U_{ijkl}^{(0\mapsto 2)} &= \sum_m R_{im}^{(1\mapsto 2)} U_{mjkl}^{(0\mapsto 1)} + \sum_{mn} T_{imn}^{(1\mapsto 2)} \left[R_{mj}^{(0\mapsto 1)} T_{nkl}^{(0\mapsto 1)} + T_{mkl}^{(0\mapsto 1)} R_{nj}^{(0\mapsto 1)}\right] \\
&\quad + \sum_{mnp} U_{imnp}^{(1\mapsto 2)} R_{mj}^{(0\mapsto 1)} R_{nk}^{(0\mapsto 1)} R_{pl}^{(0\mapsto 1)}
\end{aligned} \quad (2.26)$$

Using Eq. 2.26, we can obtain transfer matrices for any system of optical elements.

Next, we turn our attention to the case neglected in this chapter, namely the t-dependent magnetic fringe regions.

Chapter 3

Fringe Field Optics

3.1 Introduction

In this chapter we consider the effect of the magnetic boundary on the particles' trajectories. In the fringe region, the midplane field acquires longitudinal dependence, i. e. the quantities h, k_1, k_2, k_3 will in general no longer be constant but rather will be functions of t. This fact presents special problems in obtaining the transfer matrices.

The fringing fields of a dipole are especially troublesome because they affect the reference trajectory and violate the implicit assumption of it having a piecewise constant curvature. Put differently, a particle with the reference momentum which starts out following the reference trajectory will no longer do so after it passes through the fringing field region; it will experience a *zeroth* order shift.

One way to avoid the complication is simply to neglect the fringe field. In the sharp cut-off approximation, one assumes that the field abruptly goes to zero at the effective boundary, a fictitious edge placed so that the field line integral is conserved. First and second order transfer matrices in this model are expected to give a useful approximation to the actual optics. The third order elements, however, would contain divergent integrals leading to infinities due to the assumed field discontinuity [3,11].

With the advances of more precise magnet manufacturing and with more stringent requirements put on the accelerator optics, such as a micron range final focusing at the Stanford Linear Collider and effective handling of high intensity medical accelerator beams, one needs to give a more earnest consideration to the fringing field problem. In the language of the transfer matrix theory it means to know how to obtain a solution at least to the third order.

For the sharp cut-off approximation, the problem has been solved to the second order [13]. For a less sudden, more realistic field behavior, a practical solution exists for the first order only [10] and even that relies on considering special cases rather than giving a full treatment.

To have the full third order treatment, one has to deal with the extended fringing fields. Mathematically, the net effect of the boundary can be represented by a "thin lens", a fictitious optical element of zero thickness, located at the plane perpendicular to the assumed reference trajectory. The influence of the field on the reference trajectory itself is described by a zeroth order transfer matrix. The really hard part is to solve the equations of motion through the fringing regions, expressing the solution in terms of physically useful parameters. One such parameter is the gap size between the poles, which measures the spatial extent of the fringe field region; another – the pole face rotation angle, which the boundary of the dipole makes with the reference trajectory. Various line integrals of the field will also enter into the matrix solution.

The computation of the transfer matrices for the fringing field of a dipole involves solving nonlinear differential equations. Although conceptually straight-forward, the calculations are quite involved and algebra (especially for the third-order case) is exceedingly tedious because of the multitude of terms encountered in obtaining a solution. Throughout the calculations, much use is made of a symbolic algebraic manipulation computer package MACSYMA [15].

Higher multipoles – quadrupoles, sextupoles, etc. – do not present the problems of a dipole. Since they possess an optical axis, where the field is zero, the central trajectory passing through it will not be affected in the fringe region. No new mathematical developments are needed since the fringing field can be modelled as a succession of thin multipoles successively weakening in strength, each treated in the framework of the previous chapter. We take up the case of a quadrupole to illustrate this point in the following section.

3.2 Transfer Map for Quadrupole Fringe Region

In the fringe region of length T, the quadrupole strength $k_q(t)$ varies smoothly from its constant value inside the quadrupole to zero in the field-free region. We can consider this region as being made up of tiny quadrupoles of different k_q's stacked up tightly next to each other. Suppose there are N such quadrupoles, each of length Δt, with varying strengths k_q^i, $i = 0, \ldots, N$.

In the first order, a short quadrupole can be treated as a thin lens with the principal plane located at its center. From Eq. 2.16, the transfer matrix for the i^{th} element is given by

$$\mathbf{R}^i = \begin{pmatrix} 1 \mp \frac{1}{2}\Delta t^2 & \Delta t \\ \mp(k_q^i)^2 \Delta t & 1 \mp \frac{1}{2}\Delta t^2 \end{pmatrix} \tag{3.1}$$

where the $\mp$ sign refers to the t-x and t-y planes respectively.

The matrix for the whole region is obtained my multiplying the N individual matrices,

$$\begin{aligned} \mathbf{R}(T) &= \prod_{i=1}^{N} \mathbf{R}^i = \mathbf{R}^N \mathbf{R}^{N-1} \ldots \mathbf{R}^1 \\ &= \begin{pmatrix} 1 \mp \frac{1}{2}\sum_{i=1}^{N}\left[2(N-i)+1\right](k_q^i)^2 \Delta t^2 & N\Delta t \\ \mp \sum_{i=1}^{N}(k_q^i)^2 \Delta t & 1 \mp \frac{1}{2}\sum_{i=1}^{N}(2i-1)(k_q^i)^2\Delta t^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \mp N\Delta t \sum_{i=1}^{N}(k_q^i)^2 \Delta t + O(N\Delta t^2) & N\Delta t \\ \mp \sum_{i=1}^{N}(k_q^i)^2 \Delta t & 1 + O(N\Delta t^2) \end{pmatrix} \end{aligned} \tag{3.2}$$

Taking the limits $\Delta t \to 0$, $N \to \infty$, and $(N\Delta t) \to T$, we get the approximate linear map for the quadrupole fringe region,

$$\mathbf{R}(T) = \begin{pmatrix} 1 \mp T\int_0^T k_q^2(t)dt & T \\ \mp \int_0^T k_q^2(t)dt & 1 \end{pmatrix} \tag{3.3}$$

In the second order, the only non-zero terms are the chromatic ones. For the i^{th} thin lens element, we have

$$\begin{aligned} T_{116}^i = T_{226}^i = -T_{336}^i = -T_{446}^i &= \frac{1}{2}(k_q^i)^2 \Delta t^2 \\ T_{126}^i = T_{346}^i &= 0 \\ T_{216}^i = -T_{436}^i &= (k_q^i)^2 \Delta t \end{aligned} \tag{3.4}$$

We can write the transformation between the elements i and $(i+1)$ as follows,

$$\begin{pmatrix} z_{i+1} \\ z'_{i+1} \end{pmatrix} = \mathbf{R}^i \begin{pmatrix} z_i \\ z'_i \end{pmatrix} + \begin{pmatrix} \pm\frac{1}{2}(k_q^i)^2\Delta t^2 & 0 \\ \pm(k_q^i)^2\Delta t & \pm\frac{1}{2}(k_q^i)^2\Delta t^2 \end{pmatrix} \begin{pmatrix} z_i\delta \\ z'_i\delta \end{pmatrix} \tag{3.5}$$

where z is a generic variable denoting both x and y, $\mathbf{R}^i$ is given by Eq. 3.1, and the $\pm$ sign refers to the t-x and t-y planes respectively.

Transforming Eq. 3.5 through all N elements, we get

$$\begin{aligned} \begin{pmatrix} z_N \\ z'_N \end{pmatrix} &= \mathbf{R}^N\mathbf{R}^{N-1}\ldots\mathbf{R}^1 \begin{pmatrix} z_0 \\ z'_0 \end{pmatrix} \\ &\quad + \begin{pmatrix} \pm\frac{1}{2}\sum_{i=1}^N\left[2(N-i)+1\right](k_q^i)^2\Delta t^2 & 0 \\ \pm\sum_{i=1}^N(k_q^i)^2\Delta t & \pm\frac{1}{2}\sum_{i=1}^N(2i-1)(k_q^i)^2\Delta t^2 \end{pmatrix} \begin{pmatrix} z_0\delta \\ z'_0\delta \end{pmatrix} \\ &= \mathbf{R}(T)\begin{pmatrix} z_0 \\ z'_0 \end{pmatrix} + \begin{pmatrix} \pm N\Delta t\sum_{i=1}^N(k_q^i)^2\Delta t + O(N\Delta t^2) & 0 \\ \pm\sum_{i=1}^N(k_q^i)^2\Delta t & O(N\Delta t^2) \end{pmatrix} \begin{pmatrix} z_0\delta \\ z'_0\delta \end{pmatrix} \end{aligned} \tag{3.6}$$

Taking the limits, we obtain the following non-zero second order terms,

$$\begin{aligned} T_{116}(T) &= -T_{336}(T) = T\int_0^T k_q^2(t)dt \\ T_{216}(T) &= -T_{436}(T) = \int_0^T k_q^2(t)dt \end{aligned} \tag{3.7}$$

The third order elements can be found by concatenating U_{ijkl}'s for the thin lens elements according to the last equation of Eq. 2.26. In that equation, the second summation gives a zero contribution because all the first-order chromatic terms, which get multiplied by the non-zero second-order terms, are identically zero. However, the concatenations are quite tedious and the general results will not be given here.

In the following section, we turn our attention to the dipole fringe fields for the first time.

3.3 Matrix Approach to Dipole Fringe Optics

3.3.1 Mathematical Formulation

A dipole magnet can have its field boundary not perpendicular to the reference trajectory at the entrance or exit pole. The field then provides an additional focusing on a particle. A

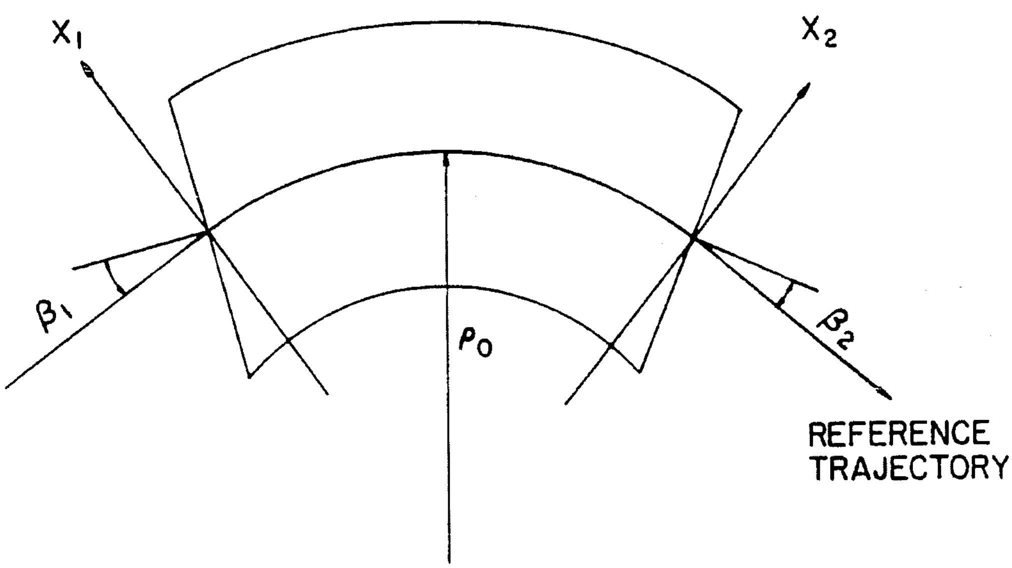

Figure 3.1: A dipole magnet with non-normal entrance and exit field boundaries.

dipole magnet with rotated pole faces is shown in Fig. 3.1. The entrance pole face rotation angle is labelled β_1 and the exit pole face rotation angle is labelled β_2. The angles are both positive. In general, a pole face rotation angle is defined as positive if magnet iron is removed for positive x and added for negative x.

The field in the air gap makes a gradual transition from the longitudinally uniform interior field to the field-free region external to the magnet. We would like to obtain transfer matrices for the fringe region of a rotated boundary.

We consider the entrance of the bending magnet shown in Fig. 3.2. The net effect of the fringe field of an inclined boundary can be mathematically represented by a fictitious optical element of zero thickness, located at the reference plane. The transfer matrix for such a lens is given by a product of three transformations,

$$\mathcal{M}^{0\mapsto f} = \mathcal{M}^{2\mapsto f}\mathcal{M}^{1\mapsto 2}\mathcal{M}^{0\mapsto 1} \tag{3.8}$$

where

1. $\mathcal{M}^{0\mapsto 1}$ is a transformation from the reference plane to the beginning of the fringe region through the pure drift field;

2. $\mathcal{M}^{1\mapsto 2}$ is the transformation through the fringe region;

3. $\mathcal{M}^{2\mapsto f}$ is the transformation from the end of the fringe region back to the reference plane through the pure bend field.

A word about coordinate systems is appropriate here. We have to distinguish between the *beam system* and the *magnet system* (Fig. 3.2). The beam system (z, x, y) is centered around the point where the imagined reference trajectory intersects the field's boundary. The z-axis points in the direction of the reference trajectory and $(x, z = 0)$ is the reference plane. The magnet system (s, u, y) is centered at the same point as the beam system and has its s-axis pointing in the direction perpendicular to the field boundary. Two systems are rotated with respect to each other around the y-axis by an angle β. Our final matrix solution has to be expressed in terms of the *beam coordinates* at $z = 0$.

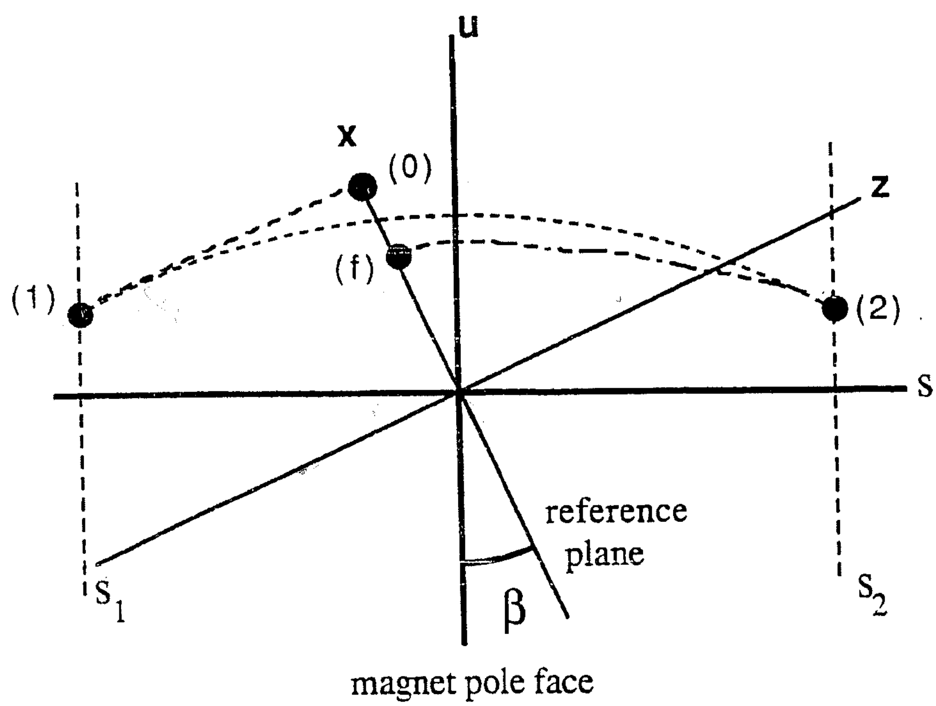

Figure 3.2: Dipole entrance. Reference plane is normal to the design trajectory.

3.3.2 First Order Calculation

We define coordinate s to be normal to the actual pole face and coordinate u – tangent to it. It is convenient to measure the extent of the transition field region in terms of the vertical separation of the magnetic poles d. We assume that d is small compared to the radius of curvature ρ of the reference trajectory in the interior of the magnet, and will derive the effect of the fringe region as an expansion in the ratio $\epsilon = d/\rho$.

We assume the midplane field depends on s only, i. e. we neglect the effects connected to the finite width of the magnet. We define

$$h(s) = \frac{B_y(y=0)}{B_0} \tag{3.9}$$

where B_0 is the constant field well inside the dipole. We assume further that the effective boundary is chosen so that

$$\int_{-\infty}^{s^*} h(s)ds = s^* \tag{3.10}$$

for $s^* \gg d$.

Below, we obtain first-order matrix elements expanded to the second order in the "field extent" parameter ϵ. We follow the procedure suggested by Eq. 3.8.

Transformation $\mathcal{M}^{0\mapsto 1}$: Drift Region

First, we transform from the reference plane $z = 0$ to the beginning of the fringe region $s = s_1$ through a pure drift. (Point s_1 is chosen somewhat arbitrarily since the fringe field never truly goes to zero; we take s_1 to be the point where the field is zero to the accuracy of a measurement.)

The equations of motion are simply

$$\frac{d^2x}{dz^2} = \frac{d^2y}{dz^2} = 0 \tag{3.11}$$

which are solved by

$$\begin{aligned} x(z) &= x_0 + x_0'z \\ y(z) &= y_0 + y_0'z \end{aligned} \tag{3.12}$$

At the plane $s = s_1$, we have

$$z_1 = z(s = s_1) = x_1 \tan\beta + s_1 \sec\beta \tag{3.13}$$

where, from Eq. 3.12,

$$x_1 = x_0 + x_0' z_1 \tag{3.14}$$

Solving Eq. 3.13 and Eq. 3.14 simultaneously for z_1, we get

$$z_1 = \frac{x_0 \tan\beta + s_1 \sec\beta}{1 - x_0' \tan\beta} \tag{3.15}$$

Substituting z_1 into Eq. 3.12 and Taylor expanding to the first order in the initial conditions, we obtain

$$\begin{aligned}
z_1 &= s_1 \sec\beta + x_0 \tan\beta + x_0' s_1 \sec\beta \tan\beta + \cdots \\
x_1 &= x_0 + x_0' s_1 \sec\beta + \cdots \\
x_1' &= x_0' \\
y_1 &= y_0 + y_0' s_1 \sec\beta + \cdots \\
y_1' &= y_0'
\end{aligned} \tag{3.16}$$

Let us introduce the scaled variables:

$$\bar{x} = \frac{x}{\rho}$$

$$\bar{y} = \frac{y}{\rho}$$

$$\bar{u} = \frac{u}{\rho}$$

$$\bar{s} = \frac{s}{d}$$

Then, the linear drift map simply looks as follows,

$$\mathbf{R}^{\bar{x}}_{0\mapsto 1} = \mathbf{R}^{\bar{y}}_{0\mapsto 1} = \begin{pmatrix} 1 & \epsilon \bar{s}_1 \sec\beta \\ 0 & 1 \end{pmatrix} \tag{3.17}$$

We can also write the relationships between the two sets of coordinates,

$$
\begin{aligned}
u &= z\sin\beta + x\cos\beta \\
s &= z\cos\beta - x\sin\beta \\
\\
\frac{du}{ds} &= \frac{du/dz}{ds/dz} = \frac{x' + \tan\beta}{1 - x'\tan\beta} \\
\frac{dy}{ds} &= \frac{dy/dz}{ds/dz} = \frac{y'\sec\beta}{1 - x'\tan\beta}
\end{aligned}
\tag{3.18}
$$

Using Eq. 3.18, we can write the transformation $\mathcal{M}^{0\mapsto1}$ in the $(\bar{s}, \bar{u}, \bar{y})$ system. Expanding to the first order in the initial conditions, we get

$$
\begin{aligned}
\bar{u}_1 &= \epsilon\bar{s}_1\tan\beta + (\bar{x}_0\sec\beta + \epsilon\bar{s}_1\sec^2\beta x_0') \\
\dot{u}_1 &= \tan\beta + \sec^2\beta x_0' \\
\bar{y}_1 &= \bar{y}_0 + \epsilon\bar{s}_1\sec\beta y_0' \\
\dot{y}_1 &= y_0'\sec\beta
\end{aligned}
\tag{3.19}
$$

where the dot denotes d/ds. Anticipating future expansions, we notice that the variable $\dot{u}_1$ may not be assumed "small", in a sense of being proportional to the initial conditions, because of the zero order term $\tan\beta$. We define a new variable $w(s)$ as a deviation from the reference trajectory,

$$
\begin{aligned}
u(s) &= w(s) + \Delta(s) \\
\dot{u}(s) &= \dot{w}(s) + \dot{\Delta}(s)
\end{aligned}
\tag{3.20}
$$

where

$$\Delta(s) = u^{\mathrm{ref}}(s) = u(s)\Big|_{x_0=x_0'=y_0=y_0'=\delta=0}$$

At $s = s_1$, we have

$$
\begin{aligned}
\Delta_1 &= \epsilon\bar{s}_1\tan\beta \\
\dot{\Delta}_1 &= \tan\beta
\end{aligned}
$$

$$
\begin{aligned}
w_1 &= \bar{x}_0 \sec\beta + \epsilon \bar{s}_1 \sec^2\beta x_0' \\
\dot{w}_1 &= \sec^2\beta x_0'
\end{aligned}
\tag{3.21}
$$

Transformation $\mathcal{M}^{1\mapsto 2}$: Fringe Region

Here, we transform from the beginning of the fringe region at $s = s_1$ to its end at $s = s_2$. The normalized field $h(s)$ takes on values 0 and 1 at these respective endpoints and

$$
\int_{s_1}^{s_2} h(s)ds = s_2 \tag{3.22}
$$

The equations of motion can be written as follows in the (s, u, y) coordinates,

$$
\begin{aligned}
\frac{d}{ds}\left(\frac{du/ds}{T}\right) &= \frac{q}{p}\left(\frac{dy}{ds}B_s - B_y\right) \\
\frac{d}{ds}\left(\frac{dy/ds}{T}\right) &= \frac{q}{p}\left(B_u - \frac{du}{ds}B_s\right)
\end{aligned}
\tag{3.23}
$$

where

$$
T = \sqrt{1 + \left(\frac{du}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2}
$$

Using Maxwell's equations and the midplane symmetry we can expand the magnetic field components around $b(s) \equiv B_y(s, u, y = 0)$:

$$
\begin{aligned}
B_s(s,u,y) &= \frac{db(s)}{ds}y - \frac{1}{6}\frac{d^3b(s)}{ds^3}y^3 + \cdots \\
B_u(s,u,y) &= 0 \\
B_y(s,u,y) &= b(s) - \frac{1}{2}\frac{d^2b(s)}{ds^2}y^2 + \cdots
\end{aligned}
\tag{3.24}
$$

Using the above expansions and the fact that

$$
b(s)\frac{q}{p} = \frac{b(s)}{B_0}\frac{1}{\rho(1+\delta)} = h(s)\frac{1}{\rho(1+\delta)}
$$

we can write the equations of motion as follows,

$$
\begin{aligned}
\ddot{u} &= -T\left[f\left(1 + \dot{u}^2\right) + g\dot{u}\dot{y}\right] \\
\ddot{y} &= -T\left[g\left(1 + \dot{y}^2\right) + f\dot{u}\dot{y}\right]
\end{aligned}
\tag{3.25}
$$

where the dot, as before, denotes d/ds and

$$f = \frac{1}{\rho(1+\delta)}\left(h - \dot{h}y\dot{y} - \frac{1}{2}\ddot{h}y^2\right)$$

$$g = \frac{1}{\rho(1+\delta)}\left(\dot{h}\dot{u}y - \frac{\dddot{h}}{6}\dot{u}y^3\right)$$

We can use Eq. 3.20 to expand the equations of motion in the variables representing the deviations from the reference trajectory $\Delta(s)$. We can write the equation for Δ by putting $u(s) = \Delta(s)$, $y = \dot{y} = \delta = 0$ in Eq. 3.25:

$$\ddot{\Delta} = -\frac{h}{\rho}(1+\dot{\Delta}^2)^{\frac{3}{2}} \tag{3.26}$$

with the initial conditions

$$\Delta(s_1) = s_1 \tan\beta \qquad \dot{\Delta}(s_1) = \tan\beta$$

The equations for $w(s)$ and $y(s)$ can then be written to the first order in "small" quantities as follows,

$$\begin{aligned} \ddot{w} &= -(1+\dot{\Delta}^2)^{\frac{3}{2}}\left(\frac{3\dot{\Delta}}{1+\dot{\Delta}^2}\dot{w} - \delta\right)\frac{h}{\rho} \\ \ddot{y} &= -\dot{\Delta}(1+\dot{\Delta}^2)^{\frac{1}{2}}\frac{d}{ds}\left(\frac{h}{\rho}y\right) \end{aligned} \tag{3.27}$$

Eq. 3.26 can be integrated to yield

$$\dot{\Delta}(\bar{s}) = \dot{\Delta}(\bar{s}_1) - \epsilon\int_{\bar{s}_1}^{\bar{s}}\left[1+\dot{\Delta}^2(s)\right]^{\frac{3}{2}} h(s)ds \tag{3.28}$$

The above equation can readily be solved by iteration to $O(\epsilon^2)$:

$$\dot{\Delta}(\bar{s}) = \tan\beta - \epsilon\sec^3\beta\int_{\bar{s}_1}^{\bar{s}} h(s)ds + \frac{3}{2}\epsilon^2\tan\beta\sec^4\beta\left(\int_{\bar{s}_1}^{\bar{s}} h(s)ds\right)^2 \tag{3.29}$$

Integrating, we get $\bar{\Delta}$:

$$\bar{\Delta}(\bar{s}) = \epsilon\bar{s}\tan\beta - \epsilon^2\sec^3\beta\int_{\bar{s}_1}^{\bar{s}}\int_{\bar{s}_1}^{s} h(s')ds'ds \tag{3.30}$$

Eq. 3.27 can be rewritten in the dimensionless form as follows,

$$\begin{aligned}
\frac{d}{d\bar{s}}\bar{w} &= \epsilon\dot{w} \\
\frac{d}{d\bar{s}}\dot{w} &= -\epsilon h(\bar{s})(1+\dot{\Delta}^2)^{\frac{3}{2}}\left(\frac{3\dot{\Delta}}{1+\dot{\Delta}^2}\dot{w}-\delta\right) \\
\frac{d}{d\bar{s}}\bar{y} &= \epsilon\dot{y} \\
\frac{d}{d\bar{s}}\dot{y} &= -\dot{\Delta}(1+\dot{\Delta}^2)^{\frac{1}{2}}\frac{d}{d\bar{s}}(h\bar{y})
\end{aligned} \tag{3.31}$$

where the initial conditions are

$$\bar{w}(\bar{s}_1)=\bar{w}_1 \quad \dot{w}(\bar{s}_1)=\dot{w}_1$$

$$\bar{y}(\bar{s}_1)=\bar{y}_1 \quad \dot{y}(\bar{s}_1)=\dot{y}_1$$

Let us solve Eq. 3.31 to $O(\epsilon^2)$ using order-by-order iteration procedure:

$$\begin{aligned}
\bar{w}^{(n)}(\bar{s}) &= \bar{w}_1+\epsilon\int_{\bar{s}_1}^{\bar{s}}\dot{w}^{(n-1)}(s)ds \\
\dot{w}^{(n)}(\bar{s}) &= \dot{w}_1-3\epsilon\int_{\bar{s}_1}^{\bar{s}}p_\epsilon(s)h(s)\dot{w}^{(n-1)}(s)ds+\delta\int_{\bar{s}_1}^{\bar{s}}q_\epsilon(s)h(s)ds \\
\bar{y}^{(n)}(\bar{s}) &= \bar{y}_1+\epsilon\int_{\bar{s}_1}^{\bar{s}}\dot{y}^{(n-1)}(s)ds \\
\dot{y}^{(n)}(\bar{s}) &= \dot{y}_1-p_\epsilon(\bar{s})h(\bar{s})\bar{y}^{(n-1)}(\bar{s})+\int_{\bar{s}_1}^{\bar{s}}\frac{dp_\epsilon}{ds}h(s)\bar{y}^{(n-1)}(s)ds
\end{aligned} \tag{3.32}$$

where

$$p_\epsilon(\bar{s})=\dot{\Delta}(1+\dot{\Delta}^2)^{\frac{1}{2}}$$

$$q_\epsilon(\bar{s})=\epsilon(1+\dot{\Delta}^2)^{\frac{3}{2}}$$

We can use Eq. 3.29 to expand p_ϵ and q_ϵ to $O(\epsilon^2)$:

$$p_\epsilon(\bar{s})=\frac{\sin\beta}{\cos^2\beta}-\epsilon\frac{1+\sin^2\beta}{\cos^4\beta}\int_{\bar{s}_1}^{\bar{s}}h(s)ds+\epsilon^2\frac{\sin\beta(3+\sin^2\beta)}{\cos^6\beta}\left(\int_{\bar{s}_1}^{\bar{s}}h(s)ds\right)^2+\cdots$$

$$q_\epsilon(\bar{s})=\epsilon\sec^3\beta-3\epsilon^2\frac{\sin\beta}{\cos^5\beta}\int_{\bar{s}_1}^{\bar{s}}h(s)ds+\cdots$$

and then

$$\frac{dp_\epsilon}{d\bar{s}} = -\epsilon\frac{1+\sin^2\beta}{\cos^4\beta}h + 2\epsilon^2\frac{\sin\beta(3+\sin^2\beta)}{\cos^6\beta}h\int_{\bar{s}_1}^{\bar{s}} h(s)ds + \cdots$$

Starting with the initial conditions as the zeroth order iterates, the successive iterations of Eq. 3.32 produce the transfer map at the endpoint $\bar{s} = \bar{s}_2$:

$$\begin{aligned}
\bar{w}_2 &= \bar{w}_1 + \dot{w}_1\left[\epsilon(\bar{s}_2 - \bar{s}_1) - 3\epsilon^2\sin\beta\sec^2\beta\int_{\bar{s}_1}^{\bar{s}_2}\int_{\bar{s}_1}^{s} h(s')ds'ds\right] \\
&\quad +\delta\epsilon^2\sec^3\beta\int_{\bar{s}_1}^{\bar{s}_2}\int_{\bar{s}_1}^{s} h(s')ds'ds \\
\dot{w}_2 &= \dot{w}_1\left[1 - 3\epsilon\bar{s}_2\sin\beta\sec^2\beta + \frac{3}{2}\epsilon^2\bar{s}_2^2(1+4\sin^2\beta)\sec^4\beta\right] \\
&\quad +\delta\left[\epsilon\bar{s}_2\sec^3\beta - 3\epsilon^2\bar{s}_2^2\sin\beta\sec^5\beta\right] \\
\bar{y}_2 &= \bar{y}_1\Big[1 - \epsilon\bar{s}_2\sin\beta\sec^2\beta \\
&\quad +\epsilon^2\sec^4\beta\left(\frac{1}{2}\bar{s}_2^2(1+2\sin^2\beta) - (1+\sin^2\beta)\int_{\bar{s}_1}^{\bar{s}_2}\int_{\bar{s}_1}^{s} h^2(s')ds'ds\right)\Big] \\
&\quad +\dot{y}_1\left[\epsilon(\bar{s}_2 - \bar{s}_1) - \epsilon^2\sin\beta\sec^2\beta\left(\bar{s}_2(\bar{s}_2 - \bar{s}_1) - \int_{\bar{s}_1}^{\bar{s}_2}\int_{\bar{s}_1}^{s} h(s')ds'ds\right)\right] \qquad (3.33) \\
\dot{y}_2 &= -\bar{y}_1\Big[\sin\beta\sec^2\beta - \epsilon\sec^4\beta\left((1+2\sin^2\beta)\bar{s}_2 - (1+\sin^2\beta)\int_{\bar{s}_1}^{\bar{s}_2} h^2(s)ds\right) \\
&\quad +\epsilon^2\sin\beta\sec^6\beta\Big(\frac{3}{2}(3+2\sin^2\beta)\bar{s}_2^2 - (1+\sin^2\beta)\int_{\bar{s}_1}^{\bar{s}_2}\int_{\bar{s}_1}^{s} h^2(s')ds'ds \\
&\quad -(7+3\sin^2\beta)\int_{\bar{s}_1}^{\bar{s}_2}\int_{\bar{s}_1}^{s} h^2(s)h(s')ds'ds\Big)\Big] + \dot{y}_1\Big[1 - \epsilon\sin\beta\sec^2\beta(\bar{s}_2 - \bar{s}_1) \\
&\quad +\epsilon^2\sec^4\beta\Big(\left((1+2\sin^2\beta)\bar{s}_2 - (1+\sin^2\beta)\int_{\bar{s}_1}^{\bar{s}_2} h^2(s)ds\right)(\bar{s}_2 - \bar{s}_1) \\
&\quad +(1+\sin^2\beta)\int_{\bar{s}_1}^{\bar{s}_2}\int_{\bar{s}_1}^{s} h^2(s')ds'ds - \sin^2\beta\int_{\bar{s}_1}^{\bar{s}_2}\int_{\bar{s}_1}^{s} h(s')ds'ds\Big)\Big]
\end{aligned}$$

where we have used Eq. 3.22 to evaluate the field integral at $\bar{s}_2$.

Transformation $\mathcal{M}^{2\mapsto 3}$: Pure Bend Region

The final transformation is through a pure bend from the end of the fringe region at $s = s_2$ back to the initial plane $z = 0$.

The equations of motion are :

$$
\begin{aligned}
\frac{d}{dz}\left(\frac{x'}{\left(1+x'^2+y'^2\right)^{\frac{1}{2}}}\right) &= -\frac{q}{p}B_0 \\
\frac{d}{dz}\left(\frac{y'}{\left(1+x'^2+y'^2\right)^{\frac{1}{2}}}\right) &= 0
\end{aligned}
\tag{3.34}
$$

We can rewrite the above equations as follows,

$$
\begin{aligned}
x'' &= -\frac{1}{\rho(1+\delta)}\left(1+x'^2\right)\left(1+x'^2+y'^2\right)^{\frac{1}{2}} \\
y'' &= -\frac{1}{\rho(1+\delta)}x'\,y'\left(1+x'^2+y'^2\right)^{\frac{1}{2}}
\end{aligned}
\tag{3.35}
$$

The equations can be expanded

$$
\begin{aligned}
x'' &= -\frac{1}{\rho(1+\delta)}\left(1+\frac{3\,x'^2+y'^2}{2}+\cdots\right) \\
y'' &= -\frac{1}{\rho(1+\delta)}x'y'+\cdots
\end{aligned}
\tag{3.36}
$$

and solved as a power series in z; we have the following dimensionless solution at $z=0$,

$$
\begin{aligned}
\bar{x}_3 &= \bar{x}_2 - x_2'\bar{z}_2 - \frac{1}{2}\bar{z}_2^2(1-\delta) \\
x_3' &= x_2' + \bar{z}_2(1-\delta) + \frac{3}{2}x_2'\bar{z}_2\left(x_2'+\bar{z}_2\right) + \frac{1}{2}\bar{z}_2^3 \\
\bar{y}_3 &= \bar{y}_2 - y_2'\bar{z}_2 \\
y_3' &= y_2' + x_2'y_2'\bar{z}_2 + \frac{1}{2}y_2'\bar{z}_2^2
\end{aligned}
\tag{3.37}
$$

Next, we would like to connect $(\bar{z}_2, \bar{x}_2, x_2', \bar{y}_2, y_2')$ to $(\bar{s}_2, \bar{w}_2, \dot{w}_2, \bar{y}_2, \dot{y}_2)$. Using Eq. 3.18 and Eq. 3.20, we get

$$
\begin{aligned}
\bar{z}_2 &= \epsilon\sec\beta\bar{s}_2 - \epsilon^2\sin\beta\sec^3\beta\int_{\bar{s}_1}^{\bar{s}_2}\int_{\bar{s}_1}^{s}h(s')ds'ds + w_2\sin\beta \\
\bar{x}_2 &= -\epsilon^2\sec^2\beta\int_{\bar{s}_1}^{\bar{s}_2}\int_{\bar{s}_1}^{s}h(s')ds'ds + \bar{w}_2\cos\beta \\
x_2' &= -\epsilon\bar{s}_2\sec\beta + \frac{\epsilon^2}{2}\sin\beta\sec^3\beta\bar{s}_2^2 + \dot{w}_2\cos^2\beta + 2\epsilon\dot{w}_2\bar{s}_2\sin\beta \\
y_2' &= \dot{y}_2\cos\beta + \epsilon\dot{y}_2\bar{s}_2\tan\beta - \frac{\epsilon^2}{2}\dot{y}_2\tan^2\beta\bar{s}_2^2\sec\beta
\end{aligned}
\tag{3.38}
$$

Putting Eq. 3.38 into Eq. 3.37 and expanding to the appropriate orders, we obtain the transfer map for the uniform bend region:

$$
\begin{aligned}
\bar{x}_3 &= \epsilon^2 \sec^2\beta \left(\frac{\bar{s}_2^2}{2} - \int_{\bar{s}_1}^{\bar{s}_2}\int_{\bar{s}_1}^{s} h(s')ds'ds\right) + (\bar{w}_2 - \epsilon\dot{w}_2\bar{s}_2)\cos\beta \\
&\quad -\epsilon^2 \left[\bar{w}_2 \sin^2\beta \sec^3\beta \left(\frac{\bar{s}_2^2}{2} - \int_{\bar{s}_1}^{\bar{s}_2}\int_{\bar{s}_1}^{s} h(s')ds'ds\right)\right. \\
&\quad \left. +\dot{w}_2 \tan\beta \left(2\bar{s}_2^2 - \int_{\bar{s}_1}^{\bar{s}_2}\int_{\bar{s}_1}^{s} h(s')ds'ds\right) + \delta\frac{\bar{s}_2^2}{2}\sec^2\beta\right] \\
x_3' &= \epsilon^2 \sin\beta\sec^3\beta \left(\frac{\bar{s}_2^2}{2} - \int_{\bar{s}_1}^{\bar{s}_2}\int_{\bar{s}_1}^{s} h(s')ds'ds\right) + \bar{w}_2\sin\beta + \dot{w}_2\cos^2\beta \qquad (3.39) \\
&\quad +\epsilon\bar{s}_2(2\dot{w}_2\sin\beta - \delta\sec\beta) - \epsilon^2\left(\frac{3}{2}\dot{w}_2\bar{s}_2^2 - \delta\sin\beta\sec^3\beta\int_{\bar{s}_1}^{\bar{s}_2}\int_{\bar{s}_1}^{s} h(s')ds'ds\right) \\
\bar{y}_3 &= \bar{y}_2 - \epsilon\dot{y}_2\bar{s}_2 - \epsilon^2\dot{y}_2\sin\beta\sec^2\beta\left(\bar{s}_2^2 - \int_{\bar{s}_1}^{\bar{s}_2}\int_{\bar{s}_1}^{s} h(s')ds'ds\right) \\
y_3' &= \dot{y}_2\cos\beta + \epsilon\dot{y}_2\bar{s}_2\tan\beta - \frac{\epsilon^2}{2}\dot{y}_2\bar{s}_2^2\sec^3\beta
\end{aligned}
$$

The coordinate independent, or zeroth order, terms can be seen present in the expansions for the horizontal variables. These terms are second order in ϵ.

Now, we are ready for the final step of putting all the transfer maps together.

Combined Transformation $\mathcal{M}^{0\mapsto f}$

The net transfer map for the fringing field region is given by a matrix product of the three individual transformations,

$$\mathcal{M}^{0\mapsto f} = \mathcal{M}^{2\mapsto 3}\mathcal{M}^{1\mapsto 2}\mathcal{M}^{0\mapsto 1} \qquad (3.40)$$

In the derivations above, we have assumed that at each stage the expansions are carried out to the order that assures the correct final result to the first order in the initial conditions (x_0, x_0', y_0, y_0') and to the second order in the "field extent" parameter ϵ. The need to keep track of the orders arises from the fact that we solve the equations of motion by looking for a power series solution, in effect, in two independent "small" variables: ϵ and the phase space coordinates' deviation. In other words, we need to decide a priori to what order in either parameter we want the final map to be, and then carry out the expansions at each stage accordingly.

We perform the following transformations,

$$\begin{pmatrix} \bar{x}_0 \\ x'_0 \\ \bar{y}_0 \\ y'_0 \end{pmatrix} \mapsto \begin{pmatrix} \bar{w}_1 \\ \dot{w}_1 \\ \bar{y}_1 \\ \dot{y}_1 \end{pmatrix} \mapsto \begin{pmatrix} \bar{w}_2 \\ \dot{w}_2 \\ \bar{y}_2 \\ \dot{y}_2 \end{pmatrix} \mapsto \begin{pmatrix} \bar{x}_3 \\ x'_3 \\ \bar{y}_3 \\ y'_3 \end{pmatrix}$$

The matrix elements depend on some trigonometric functions of the angle β and line integrals of the field. There are four such independent integrals which are given below together with $I_5 = I_1 + I_4$:

$$\begin{aligned}
I_1 &= \int_{\bar{s}_1}^{\bar{s}_2} ds \int_{\bar{s}_1}^{\bar{s}} ds'[h_0(s') - h(s')] \\
I_2 &= \int_{\bar{s}_1}^{\bar{s}_2} ds[1 - h(s)]h(s) \\
I_3 &= \int_{\bar{s}_1}^{\bar{s}_2} ds[1 - h(s)] \int_{\bar{s}_1}^{\bar{s}} ds' h^2(s') \\
I_4 &= \int_{\bar{s}_1}^{\bar{s}_2} ds[1 - h^2(s)] \int_{\bar{s}_1}^{\bar{s}} ds' h(s') \\
I_5 &= \int_{\bar{s}_1}^{\bar{s}_2} ds \left(\int_{\bar{s}_1}^{\bar{s}} ds' h_0(s') - h^2(s) \int_{\bar{s}_1}^{\bar{s}} ds' h(s') \right)
\end{aligned} \tag{3.41}$$

where h_0 is the sharp-cutoff fringing field function given by a simple step function,

$$h_0(\bar{s}) = \Theta(\bar{s})$$

and we have

$$\frac{\bar{s}_2^2}{2} = \int_{\bar{s}_1}^{\bar{s}_2} ds \int_{\bar{s}_1}^{\bar{s}} ds' h_0(s')$$

The five dimensionless integrals defined above have one property in common: their integrands go to zero at both the lower and the upper limits. This fact removes the ambiguity in determining s_1 and s_2. In fact, the limits of integration can be taken respectively to $-\infty$ and $+\infty$. In practice, one just needs to measure the line integrals between a point where the field is sufficiently close to zero and a point where it is equal to its deep-inside-the-magnet value.

The complete transfer map for the final coordinates is given below,

$$
\begin{aligned}
\bar{x}_f &= \epsilon^2 \frac{I_1}{\cos^2\beta} + \bar{x}_0 \left[1 - \epsilon^2 I_1 \frac{\sin^2\beta}{\cos^4\beta}\right] + x_0' \epsilon^2 I_1 \frac{2\sin\beta}{\cos^3\beta} - \delta\epsilon^2 \frac{I_1}{\cos^2\beta} \\
x_f' &= \epsilon^2 I_1 \frac{\sin\beta}{\cos^3\beta} + \bar{x}_0 \tan\beta + x_0' \left[1 + \epsilon^2 I_1 \frac{3\sin^2\beta}{\cos^4\beta}\right] - \delta\epsilon^2 I_1 \frac{2\sin\beta}{\cos^3\beta} \\
\bar{y}_f &= \bar{y}_0 \left[1 - \frac{\epsilon^2}{\cos^4\beta}\left(I_1 - (I_3 + I_4)(1 + \sin^2\beta)\right)\right] - y_0' \epsilon^2 I_1 \frac{2\sin\beta}{\cos^3\beta} \\
y_f' &= -\bar{y}_0 \left[\tan\beta - \epsilon I_2 \frac{(1 + \sin^2\beta)}{\cos^3\beta} + \epsilon^2 \frac{\sin\beta}{\cos^5\beta}\left(2I_5(3 + \sin^2\beta) - I_3(1 + \sin^2\beta)\right)\right] \\
&\quad + y_0' \left[1 + \frac{\epsilon^2}{\cos^4\beta}\left(I_1 \sin^2\beta + (I_3 + I_5)(1 + \sin^2\beta)\right)\right]
\end{aligned}
\tag{3.42}
$$

If we put $\epsilon = 0$ in Eq. 3.42, we will obtain the well-known matrix elements for the linear sharp-cutoff approximation [3,10].

3.3.3 Nonlinear Transfer Map for the Fringe Region

Let us look for the transfer map up to the third order in the phase-space coordinates and, for simplicity, up to the first order in ϵ. The three individual transformations $\mathcal{M}^{0\mapsto 1}$, $\mathcal{M}^{1\mapsto 2}$, and $\mathcal{M}^{2\mapsto 3}$ will now have to be expanded to third order in the "small" variables.

Here, we look at the fringe region map; the complete transformation $\mathcal{M}^{0\mapsto f}$ will be given in the next section. We go back to Eq. 3.25 and expand it to the third order in w, $\dot{w}$, y, $\dot{y}$, and δ. The result can be written in the form of the dimensionless equations as follows,

$$
\begin{aligned}
\frac{d\bar{w}}{d\bar{s}} &= \epsilon\dot{w} \\
\frac{d\dot{w}}{d\bar{s}} &= -\epsilon h\left(3\dot{w}\Gamma_1^2\Gamma_2 - \delta\Gamma_1^3\right) \\
&\quad + \left[\frac{1}{2\epsilon}\frac{d^2h}{d\bar{s}^2}\bar{y}^2\Gamma_1^3 + \frac{dh}{d\bar{s}}\bar{y}\dot{y}\Gamma_1 - \epsilon h\left(\frac{3}{2}\dot{w}^2\Gamma_3 - 3\dot{w}\delta\Gamma_1^2\Gamma_2 + \frac{1}{2}\dot{y}^2\Gamma_1 + \delta^2\Gamma_1^3\right)\right] \\
&\quad + \left[\frac{1}{2\epsilon}\frac{d^2h}{d\bar{s}^2}\left(3\dot{w}\bar{y}^2\Gamma_1^2\Gamma_2 - \bar{y}^2\delta\Gamma_1^3\right) + \frac{dh}{d\bar{s}}\left(\dot{w}\bar{y}\dot{y}\Gamma_2 - \bar{y}\dot{y}\delta\Gamma_1\right)\right. \\
&\quad \left. - \epsilon h\left(\frac{1}{2}\dot{w}^3\Gamma_4 - \frac{3}{2}\dot{w}^2\delta\Gamma_3 + \frac{1}{2}\dot{w}\dot{y}^2\Gamma_2 + 3\dot{w}\delta^2\Gamma_1^2\Gamma_2 - \frac{1}{2}\dot{y}^2\delta\Gamma_1 - \delta^3\Gamma_1^3\right)\right] \qquad (3.43) \\
\frac{d\bar{y}}{d\bar{s}} &= \epsilon\dot{y}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\dot{y}}{d\bar{s}} &= -\frac{d}{d\bar{s}}(h\bar{y})\left[\Gamma_1^2\Gamma_2 + \left(\dot{w}\Gamma_3 - \delta\Gamma_1^2\Gamma_2\right) + \left(\frac{1}{2}\dot{w}^2\Gamma_4 - \dot{w}\delta\Gamma_3 + \frac{1}{2}\dot{y}^2\Gamma_2 + \delta^2\Gamma_1^2\Gamma_2\right)\right] \\
&+\frac{1}{6\epsilon^2}\frac{d}{d\bar{s}}\left(\frac{d^2h}{d\bar{s}^2}\bar{y}^3\right)\Gamma_1^2\Gamma_2
\end{aligned}
$$

where

$$\Gamma_1 = \left(1+\dot{\Delta}^2\right)^{\frac{1}{2}} = \sec\beta - \epsilon\sin\beta\sec^3\beta\int_{\bar{s}_1}^{\bar{s}} h(s)ds + \cdots$$

$$\Gamma_2 = \frac{\dot{\Delta}}{\left(1+\dot{\Delta}^2\right)^{\frac{1}{2}}} = \sin\beta - \epsilon\int_{\bar{s}_1}^{\bar{s}} h(s)ds + \cdots$$

$$\Gamma_3 = \frac{\left(1+2\dot{\Delta}^2\right)}{\left(1+\dot{\Delta}^2\right)^{\frac{1}{2}}} = (1+\sin^2\beta)\sec\beta - \epsilon\sin\beta\sec^3\beta\left(2+\cos^2\beta\int_{\bar{s}_1}^{\bar{s}} h(s)ds\right) + \cdots$$

$$\Gamma_4 = \frac{\dot{\Delta}\left(3+2\dot{\Delta}^2\right)}{\left(1+\dot{\Delta}^2\right)^{\frac{3}{2}}} = \sin\beta(2+\cos^2\beta) - 3\epsilon\cos^2\beta\int_{\bar{s}_1}^{\bar{s}} h(s)ds + \cdots$$

and the initial conditions are

$$\bar{w}(\bar{s}_1) = \bar{w}_1 \quad \dot{w}(\bar{s}_1) = \dot{w}_1$$

$$\bar{y}(\bar{s}_1) = \bar{y}_1 \quad \dot{y}(\bar{s}_1) = \dot{y}_1$$

We will try to solve Eq. 3.43 using the order-by-order method similar to the Green's function integration employed for the case of an ideal magnet.

Second Order Coefficients

We can write the most general second-order solution to Eq. 3.43 as follows,

$$
\begin{aligned}
\bar{w} &= R_{11}\bar{w}_1 + R_{12}\dot{w}_1 + R_{16}\delta \\
&+T_{122}\dot{w}_1^2 + T_{126}\dot{w}_1\delta + T_{133}\bar{y}_1^2 + T_{134}\bar{y}_1\dot{y}_1 + T_{144}\dot{y}_1^2 + T_{166}\delta^2 \\
\dot{w} &= R_{22}\dot{w}_1 + R_{26}\delta \\
&+T_{222}\dot{w}_1^2 + T_{226}\dot{w}_1\delta + T_{233}\bar{y}_1^2 + T_{234}\bar{y}_1\dot{y}_1 + T_{244}\dot{y}_1^2 + T_{266}\delta^2 \qquad (3.44) \\
\bar{y} &= R_{33}\bar{y}_1 + R_{34}\dot{y}_1
\end{aligned}
$$

$$
\begin{aligned}
&\quad +T_{323}\dot{w}_1\bar{y}_1 + T_{324}\dot{w}_1\dot{y}_1 + T_{336}\bar{y}_1\delta + T_{346}\dot{y}_1\delta \\
\dot{y} &= R_{43}\bar{y}_1 + R_{44}\dot{y}_1 \\
&\quad +T_{423}\dot{w}_1\bar{y}_1 + T_{424}\dot{w}_1\dot{y}_1 + T_{436}\bar{y}_1\delta + T_{446}\dot{y}_1\delta
\end{aligned}
$$

where matrices R and T are both functions of $\bar{s}$.

Putting Eq. 3.44 into Eq. 3.43 and equating second-order terms multiplying the same products of the initial conditions, we get the following equations for T_{ijk}'s,

$$
\begin{aligned}
\frac{d}{d\bar{s}}T_{1ij} &= \epsilon T_{2ij} \\
\frac{d}{d\bar{s}}T_{2ij} &= -3\epsilon\Gamma_1^2\Gamma_2 h T_{2ij} + f_{2ij} \\
\frac{d}{d\bar{s}}T_{3ij} &= \epsilon T_{4ij} \\
\frac{d}{d\bar{s}}T_{4ij} &= -\Gamma_1^2\Gamma_2\frac{d}{d\bar{s}}(hT_{3ij}) + f_{4ij}
\end{aligned}
\tag{3.45}
$$

The initial conditions are simply

$$T_{ijk}(\bar{s} = \bar{s}_1) = 0$$

The functions f_{2ij} and f_{4ij} are given in Table 3.1, where the prime is used to denote $d/d\bar{s}$.

First order matrix elements R_{ij} were obtained in the previous section. We give them here again for completeness:

$$
\begin{aligned}
R_{11} &= 1 \\
R_{12} &= \epsilon(\bar{s} - \bar{s}_1) - 3\epsilon^2 \sin\beta \sec^2\beta \int_{\bar{s}_1}^{\bar{s}}\int_{\bar{s}_1}^{s} h(s')ds'ds \\
R_{16} &= \epsilon^2 \sec^3\beta \int_{\bar{s}_1}^{\bar{s}}\int_{\bar{s}_1}^{s} h(s')ds'ds \\
R_{22} &= 1 - 3\epsilon\sin\beta\sec^2\beta\int_{\bar{s}_1}^{\bar{s}} h(s)ds + \frac{3}{2}\epsilon^2(1+4\sin^2\beta)\sec^4\beta\left(\int_{\bar{s}_1}^{\bar{s}} h(s)ds\right)^2 \\
R_{26} &= \epsilon\sec^3\beta\int_{\bar{s}_1}^{\bar{s}} h(s)ds - 3\epsilon^2\sin\beta\sec^5\beta\left(\int_{\bar{s}_1}^{\bar{s}} h(s)ds\right)^2 \\
R_{33} &= 1 - \epsilon\sin\beta\sec^2\beta\int_{\bar{s}_1}^{\bar{s}} h(s)ds \\
&\quad +\epsilon^2\sec^4\beta\left[\frac{1}{2}(1+2\sin^2\beta)\left(\int_{\bar{s}_1}^{\bar{s}} h(s)ds\right)^2 - (1+\sin^2\beta)\int_{\bar{s}_1}^{\bar{s}}\int_{\bar{s}_1}^{s} h^2(s')ds'ds\right]
\end{aligned}
$$

$$
\begin{aligned}
R_{34} &= \epsilon(\bar{s}-\bar{s}_1) - \epsilon^2 \sin\beta \sec^2\beta \left[(\bar{s}-\bar{s}_1)\int_{\bar{s}_1}^{\bar{s}} h(s)ds - \int_{\bar{s}_1}^{\bar{s}}\int_{\bar{s}_1}^{s} h(s')ds'ds\right] \\
R_{43} &= -\sin\beta\sec^2\beta h + \epsilon\sec^4\beta\left[(1+2\sin^2\beta)h\int_{\bar{s}_1}^{\bar{s}} h(s)ds - (1+\sin^2\beta)\int_{\bar{s}_1}^{\bar{s}} h^2(s)ds\right] \\
&\quad -\epsilon^2\sin\beta\sec^6\beta\left[\frac{3}{2}(3+2\sin^2\beta)\left(\int_{\bar{s}_1}^{\bar{s}} h(s)ds\right)^2 - (1+\sin^2\beta)\int_{\bar{s}_1}^{\bar{s}}\int_{\bar{s}_1}^{s} h^2(s')ds'ds\right. \\
&\quad \left. -(7+3\sin^2\beta)\int_{\bar{s}_1}^{\bar{s}}\int_{\bar{s}_1}^{s} h^2(s)h(s')ds'ds\right] \\
R_{44} &= 1 - \epsilon\sin\beta\sec^2\beta(\bar{s}-\bar{s}_1)h \\
&\quad +\epsilon^2\sec^4\beta\left[(\bar{s}-\bar{s}_1)\left((1+2\sin^2\beta)\int_{\bar{s}_1}^{\bar{s}} h(s)ds - (1+\sin^2\beta)\int_{\bar{s}_1}^{\bar{s}} h^2(s)ds\right)\right. \\
&\quad \left. +(1+\sin^2\beta)\int_{\bar{s}_1}^{\bar{s}}\int_{\bar{s}_1}^{s} h^2(s')ds'ds - \sin^2\beta\int_{\bar{s}_1}^{\bar{s}}\int_{\bar{s}_1}^{s} h(s')ds'ds\right] \\
R_{66} &= 1
\end{aligned}
$$

Eq. 3.45 can be solved by iteration to a desired order in ϵ. We get the following expansion series:

$$
\begin{aligned}
T_{1ij}(\bar{s}) &= \epsilon\int_{\bar{s}_1}^{\bar{s}} T_{2ij}(s)ds \\
T_{2ij}(\bar{s}) &= \int_{\bar{s}_1}^{\bar{s}} f_{2ij}(s)ds - 3\epsilon\int_{\bar{s}_1}^{\bar{s}}\Gamma_1^2(s)\Gamma_2(s)h(s)\int_{\bar{s}_1}^{s} f_{2ij}(s')ds'ds \\
&\quad +9\epsilon^2\int_{\bar{s}_1}^{\bar{s}}\Gamma_1^2(s)\Gamma_2(s)h(s)\int_{\bar{s}_1}^{s}\Gamma_1^2(s')\Gamma_2(s')h(s')\int_{\bar{s}_1}^{s'} f_{2ij}(s'')ds''ds'ds + \cdots \qquad (3.46) \\
T_{3ij}(\bar{s}) &= \epsilon\int_{\bar{s}_1}^{\bar{s}}\int_{\bar{s}_1}^{s} f_{4ij}(s')ds'ds - \epsilon^2\int_{\bar{s}_1}^{\bar{s}}\Gamma_1^2(s)\Gamma_2(s)h(s)\int_{\bar{s}_1}^{s}\int_{\bar{s}_1}^{s'} f_{4ij}(s'')ds''ds'ds + \cdots \\
T_{4ij}(\bar{s}) &= \int_{\bar{s}_1}^{\bar{s}} f_{4ij}(s)ds - \Gamma_1^2(\bar{s})\Gamma_2(\bar{s})h(\bar{s})T_{3ij}(\bar{s}) + \int_{\bar{s}_1}^{\bar{s}}\frac{d}{ds}\left[\Gamma_1^2(s)\Gamma_2(s)\right]h(s)T_{3ij}(s)ds
\end{aligned}
$$

non-zero second order coefficients calculated to $O(\epsilon)$ are given below,

$$
\begin{aligned}
T_{133} &= \frac{h}{2\cos^3\beta} - \epsilon\frac{\sin\beta}{2\cos^5\beta}\left[5h\int_{\bar{s}_1}^{\bar{s}} h(s)ds - (5+\sin^2\beta)\int_{\bar{s}_1}^{\bar{s}} h^2(s)ds\right] \\
T_{134} &= \frac{\epsilon}{\cos^3\beta}\left[(\bar{s}-\bar{s}_1)h - (1+\sin^2\beta)\int_{\bar{s}_1}^{\bar{s}} h(s)ds\right] \\
T_{222} &= -\frac{3}{2}\epsilon\frac{(1+\sin^2\beta)}{\cos\beta}\int_{\bar{s}_1}^{\bar{s}} h(s)ds \\
T_{226} &= 3\epsilon\frac{\sin\beta}{\cos^2\beta}\int_{\bar{s}_1}^{\bar{s}} h(s)ds
\end{aligned}
$$

$$
\begin{aligned}
T_{233} &= \frac{1}{2\epsilon\cos^3\beta}\frac{dh}{d\bar{s}} - \frac{\sin\beta}{2\cos^5\beta}\left[5\frac{dh}{d\bar{s}}\int_{\bar{s}_1}^{\bar{s}} h(s)ds - \sin^2\beta h^2\right] \\
&+\frac{\epsilon}{\cos^7\beta}\left[\sin^2\beta(2+\sin^2\beta\cos^2\beta)h\int_{\bar{s}_1}^{\bar{s}} h^2(s)ds - \frac{\sin^2\beta\cos^2\beta}{2}\int_{\bar{s}_1}^{\bar{s}} h^3(s)ds\right. \\
&-\frac{\sin^2\beta}{2}(4+5\sin^2\beta-2\sin^4\beta)h^2\int_{\bar{s}_1}^{\bar{s}} h(s)ds \\
&\left.+\frac{5}{4}(1+6\sin^2\beta)\frac{dh}{d\bar{s}}\left(\int_{\bar{s}_1}^{\bar{s}} h(s)ds\right)^2 - (1+\sin^2\beta)\frac{dh}{d\bar{s}}\int_{\bar{s}_1}^{\bar{s}}\int_{\bar{s}_1}^{s} h^2(s')ds'ds\right] \\
T_{234} &= \frac{1}{\cos^3\beta}\left[(\bar{s}-\bar{s}_1)\frac{dh}{d\bar{s}} - \sin^2\beta h\right] \\
&-\epsilon\frac{\sin\beta}{\cos^5\beta}\left[(4+\cos^2\beta)(\bar{s}-\bar{s}_1)h^2 - 2(1+\sin^2\beta)h\int_{\bar{s}_1}^{\bar{s}} h(s)ds\right. \\
&\left.-(8+\cos^2\beta)\int_{\bar{s}_1}^{\bar{s}} h^2(s)ds + 5(\bar{s}-\bar{s}_1)\frac{dh}{d\bar{s}}\int_{\bar{s}_1}^{\bar{s}} h(s)ds - \frac{dh}{d\bar{s}}\int_{\bar{s}_1}^{\bar{s}}\int_{\bar{s}_1}^{s} h(s')ds'ds\right] \\
T_{244} &= \frac{\epsilon}{2\cos^3\beta}\left[(\bar{s}-\bar{s}_1)(\frac{dh}{d\bar{s}}(\bar{s}-\bar{s}_1-2\sin^2\beta h) + (2-3\cos^2\beta)\int_{\bar{s}_1}^{\bar{s}} h(s)ds\right] \\
T_{266} &= -\frac{\epsilon}{\cos^3\beta}\int_{\bar{s}_1}^{\bar{s}} h(s)ds \\
T_{323} &= -\epsilon\frac{(1+\sin^2\beta)}{\cos\beta}\int_{\bar{s}_1}^{\bar{s}} h(s)ds \\
T_{336} &= \epsilon\frac{\sin\beta}{\cos^2\beta}\int_{\bar{s}_1}^{\bar{s}} h(s)ds \\
T_{423} &= -\frac{(1+\sin^2\beta)}{\cos\beta}h + 2\epsilon\frac{\sin\beta}{\cos^3\beta}\left[(4+2\sin^2\beta)h\int_{\bar{s}_1}^{\bar{s}} h(s)ds - (3+\sin^2\beta)\int_{\bar{s}_1}^{\bar{s}} h^2(s)ds\right] \\
T_{424} &= -\epsilon\frac{1+\sin^2\beta}{\cos\beta}(\bar{s}-\bar{s}_1)h \\
T_{436} &= \frac{\sin\beta}{\cos^2\beta}h - \frac{2\epsilon}{\cos^4\beta}\left[(1+2\sin^2\beta)h\int_{\bar{s}_1}^{\bar{s}} h(s)ds - (1+\sin^2\beta)\int_{\bar{s}_1}^{\bar{s}} h^2(s)ds\right] \\
T_{446} &= \epsilon\frac{\sin\beta}{\cos^2\beta}(\bar{s}-\bar{s}_1)h
\end{aligned}
$$

The above coefficients can be evaluated at the final point $\bar{s} = \bar{s}_2$ by using Eq. 3.22 and the boundary values $h(\bar{s}_2) = 1$, $[dh/d\bar{s}](\bar{s}_2) = 0$.

Third Order Coefficients

The third order terms are obtained in much the same fashion as the second order terms, although their calculation is much more cumbersome. We add the third order terms with coefficients U_{ijkl} to the expansions in Eq. 3.44. Then, we put Eq. 3.44 into Eq. 3.43 and equate third-order terms multiplying the same products of the initial conditions. We obtain the following equations for U_{ijkl}'s,

$$\begin{aligned}
\frac{d}{d\bar{s}}U_{1ijk} &= \epsilon U_{2ijk} \\
\frac{d}{d\bar{s}}U_{2ijk} &= -3\epsilon\Gamma_1^2\Gamma_2 h U_{2ijk} + g_{2ijk} \\
\frac{d}{d\bar{s}}U_{3ijk} &= \epsilon U_{4ijk} \\
\frac{d}{d\bar{s}}U_{4ijk} &= -\Gamma_1^2\Gamma_2\frac{d}{d\bar{s}}(hU_{3ijk}) + g_{4ijk}
\end{aligned} \tag{3.47}$$

with initial conditions

$$U_{ijkl}(\bar{s} = \bar{s}_1) = 0$$

The functions g_{2ijk} and g_{4ijk} depend on Γ_n's, R_{ij}'s and T_{ijk}'s, and are given in Table 3.2 (again, the prime is used to denote $d/d\bar{s}$).

Eq. 3.47 can also be solved by iteration; matrix elements U_{ijkl} satisfy the same integral expansion as do the elements T_{ijk} in Eq. 3.46. The iterative integration is straightforward, and we do not explicitly list here the 20 non-zero matrix elements. It is worth mentioning, however, that after we evaluate all the integrals at $\bar{s} = \bar{s}_2$, there remains one coefficient which contains a divergent (in the sharp cut-off approximation) ϵ^{-1} term:

$$U_{4333} = -\frac{2(1+\sin^2\beta)}{3\epsilon\cos^4\beta}\int_{\bar{s}_1}^{\bar{s}_2}\left(\frac{dh(s')}{ds'}\right)^2 ds' + \cdots$$

The complete list of the field line integrals appearing in the final fringe region map will be given in the following section.

Table 3.1: Driving Terms For Second-Order Matrix Elements

T_{ijk}	f_{ijk}
T_{222}	$-3\epsilon R_{22}^2\Gamma_3 h/2$
T_{226}	$3\epsilon(R_{22}\Gamma_1^2\Gamma_2 - R_{22}R_{26}\Gamma_3)h$
T_{233}	$(R_{33}^2\Gamma_1^3 h'' + 2\epsilon R_{33}R_{43}\Gamma_1 h' - \epsilon^2 R_{43}^2\Gamma_1 h)/(2\epsilon)$
T_{234}	$(R_{33}R_{34}\Gamma_1^3 h'' + \epsilon(R_{33}R_{44} + R_{34}R_{43})\Gamma_1 h' - \epsilon^2 R_{43}R_{44}\Gamma_1 h)/\epsilon$
T_{244}	$(R_{34}^2\Gamma_1^3 h'' + 2\epsilon R_{34}R_{44}\Gamma_1 h' - \epsilon^2 R_{44}^2\Gamma_1 h)/(2\epsilon)$
T_{266}	$-\epsilon(3R_{26}^2\Gamma_3 - 6R_{26}\Gamma_1^2\Gamma_2 + 2\Gamma_1^3)h/2$
T_{423}	$-R_{22}R_{33}\Gamma_3 h' - \epsilon R_{22}R_{43}\Gamma_3 h$
T_{424}	$-R_{22}R_{34}\Gamma_3 h' - \epsilon R_{22}R_{44}\Gamma_3 h$
T_{436}	$(R_{33}\Gamma_1^2\Gamma_2 - R_{26}R_{33}\Gamma_3)h' + \epsilon(R_{43}\Gamma_1^2\Gamma_2 - R_{26}R_{43}\Gamma_3)h$
T_{446}	$(R_{34}\Gamma_1^2\Gamma_2 - R_{26}R_{34}\Gamma_3)h' + \epsilon(R_{44}\Gamma_1^2\Gamma_2 - R_{26}R_{44}\Gamma_3)h$

Table 3.2: Driving Terms For Third-Order Matrix Elements

U_{ijkl}	g_{ijkl}
U_{2222}	$-(R_{22}^3\Gamma_4 + 6R_{22}T_{222}\Gamma_3)h\epsilon/2$
U_{2226}	$-(3R_{22}^2R_{26}\Gamma_4 + (6T_{222}R_{26} + 6R_{22}T_{226} - 3R_{22}^2)\Gamma_3 - 6T_{222}\Gamma_1^2\Gamma_2)h\epsilon/2$
U_{2233}	$(3R_{22}R_{33}^2\Gamma_1^2\Gamma_2 + 2T_{323}R_{33}\Gamma_1^3)h''/(2\epsilon) + (R_{22}R_{33}R_{43}\Gamma_2$ $+(T_{323}R_{43} + R_{33}T_{423})\Gamma_1)h' - (6R_{22}T_{233}\Gamma_3 + R_{22}R_{43}^2\Gamma_2 + 2T_{423}R_{43}\Gamma_1)h\epsilon/2$
U_{2234}	$(3R_{22}R_{33}R_{34}\Gamma_1^2\Gamma_2 + (T_{323}R_{34} + T_{324}R_{33})\Gamma_1^3)h''/\epsilon$ $+((R_{22}R_{33}R_{44} + R_{22}R_{34}R_{43})\Gamma_2 + (T_{323}R_{44} + T_{324}R_{43} + R_{33}T_{424} + R_{34}T_{423})\Gamma_1)h'$ $+(-3R_{22}T_{234}\Gamma_3 - R_{22}R_{43}R_{44}\Gamma_2 + (-T_{423}R_{44} - T_{424}R_{43})\Gamma_1)h\epsilon$
U_{2244}	$(3R_{22}R_{34}^2\Gamma_1^2\Gamma_2 + 2T_{324}R_{34}\Gamma_1^3)h''/(2\epsilon) + (R_{22}R_{34}R_{44}\Gamma_2 + (T_{324}R_{44} + R_{34}T_{424})\Gamma_1)h'$ $-(6R_{22}T_{244}\Gamma_3 + R_{22}R_{44}^2\Gamma_2 + 2T_{424}R_{44}\Gamma_1)h\epsilon/2$
U_{2266}	$-(3R_{22}R_{26}^2\Gamma_4 + (6R_{22}T_{266} - 6R_{22}R_{26} + 6T_{226}R_{26})\Gamma_3 + (-6T_{226} + 6R_{22})\Gamma_1^2\Gamma_2)h\epsilon/2$
U_{2336}	$(3R_{26}R_{33}^2\Gamma_1^2\Gamma_2 + (2R_{33}T_{336} - R_{33}^2)\Gamma_1^3)h''/(2\epsilon)$ $+(R_{26}R_{33}R_{43}\Gamma_2 + (R_{33}T_{436} - R_{33}R_{43} + T_{336}R_{43})\Gamma_1)h'$ $-(6T_{233}R_{26}\Gamma_3 + (-6T_{233}\Gamma_1^2 + R_{26}R_{43}^2)\Gamma_2 + (2R_{43}T_{436} - R_{43}^2)\Gamma_1)h\epsilon/2$
U_{2346}	$(3R_{26}R_{33}R_{34}\Gamma_1^2\Gamma_2 + (R_{33}T_{346} - R_{33}R_{34} + T_{336}R_{34})\Gamma_1^3)h''/\epsilon + ((R_{26}R_{33}R_{44}$ $+R_{26}R_{34}R_{43})\Gamma_2 + (R_{33}T_{446} - R_{33}R_{44} + T_{336}R_{44} + R_{34}T_{436} - R_{34}R_{43} + T_{346}R_{43})\Gamma_1)h'$ $+(-3T_{234}R_{26}\Gamma_3 + (3T_{234}\Gamma_1^2 - R_{26}R_{43}R_{44})\Gamma_2 + (-R_{43}T_{446} + R_{43}R_{44} - T_{436}R_{44})\Gamma_1)h\epsilon$
U_{2446}	$(3R_{26}R_{34}^2\Gamma_1^2\Gamma_2 + (2R_{34}T_{346} - R_{34}^2)\Gamma_1^3)h''/(2\epsilon)$ $+(R_{26}R_{34}R_{44}\Gamma_2 + (R_{34}T_{446} - R_{34}R_{44} + T_{346}R_{44})\Gamma_1)h'$ $-(6T_{244}R_{26}\Gamma_3 + (-6T_{244}\Gamma_1^2 + R_{26}R_{44}^2)\Gamma_2 + (2R_{44}T_{446} - R_{44}^2)\Gamma_1)h\epsilon/2$
U_{2666}	$-(R_{26}^3\Gamma_4 + (6R_{26}T_{266} - 3R_{26}^2)\Gamma_3 + (-6T_{266} + 6R_{26})\Gamma_1^2\Gamma_2 - 2\Gamma_1^3)h\epsilon/2$
U_{4223}	$-(R_{22}^2R_{33}\Gamma_4 + (2T_{222}R_{33} + 2R_{22}T_{323})\Gamma_3)h'/2 - (R_{22}^2R_{43}\Gamma_4 + (2T_{222}R_{43} + 2R_{22}T_{423})\Gamma_3)h\epsilon/2$
U_{4224}	$-(R_{22}^2R_{34}\Gamma_4 + (2T_{222}R_{34} + 2R_{22}T_{324})\Gamma_3)h'/2 - (R_{22}^2R_{44}\Gamma_4 + (2T_{222}R_{44} + 2R_{22}T_{424})\Gamma_3)h\epsilon/2$
U_{4236}	$(-R_{22}R_{26}R_{33}\Gamma_4 + (-R_{22}T_{336} + (-T_{226} + R_{22})R_{33} - R_{26}T_{323})\Gamma_3 + T_{323}\Gamma_1^2\Gamma_2)h'$ $+(-R_{22}R_{26}R_{43}\Gamma_4 + (-R_{22}T_{436} + (-T_{226} + R_{22})R_{43} - R_{26}T_{423})\Gamma_3 + T_{423}\Gamma_1^2\Gamma_2)h\epsilon$
U_{4246}	$(-R_{22}R_{26}R_{34}\Gamma_4 + (-R_{22}T_{346} + (-T_{226} + R_{22})R_{34} - R_{26}T_{324})\Gamma_3 + T_{324}\Gamma_1^2\Gamma_2)h'$ $+(-R_{22}R_{26}R_{44}\Gamma_4 + (-R_{22}T_{446} + (-T_{226} + R_{22})R_{44} - R_{26}T_{424})\Gamma_3 + T_{424}\Gamma_1^2\Gamma_2)h\epsilon$
U_{4333}	$R_{33}^3\Gamma_1^2\Gamma_2h'''/(6\epsilon^2) + R_{33}^2R_{43}\Gamma_1^2\Gamma_2h''/(2\epsilon) - (2T_{233}R_{33}\Gamma_3 + R_{33}R_{43}^2\Gamma_2)h'/2$ $-(2T_{233}R_{43}\Gamma_3 + R_{43}^3\Gamma_2)h\epsilon/2$
U_{4334}	$R_{33}^2R_{34}\Gamma_1^2\Gamma_2h'''/(2\epsilon^2) + (R_{33}^2R_{44} + 2R_{33}R_{34}R_{43})\Gamma_1^2\Gamma_2h''/(2\epsilon) - ((2T_{233}R_{34} + 2T_{234}R_{33})\Gamma_3$ $+(2R_{33}R_{43}R_{44} + R_{34}R_{43}^2)\Gamma_2)h'/2 - ((2T_{233}R_{44} + 2T_{234}R_{43})\Gamma_3 + 3R_{43}^2R_{44}\Gamma_2)h\epsilon/2$
U_{4344}	$R_{33}R_{34}^2\Gamma_1^2\Gamma_2h'''/(2\epsilon^2) + (2R_{33}R_{34}R_{44} + R_{34}^2R_{43})\Gamma_1^2\Gamma_2h''/(2\epsilon) - ((2T_{234}R_{34} + 2T_{244}R_{33})\Gamma_3$ $+(R_{33}R_{44}^2 + 2R_{34}R_{43}R_{44})\Gamma_2)h'/2 - ((2T_{234}R_{44} + 2T_{244}R_{43})\Gamma_3 + 3R_{43}R_{44}^2\Gamma_2)h\epsilon/2$
U_{4366}	$-(R_{26}^2R_{33}\Gamma_4 + (2R_{26}T_{336} + (2T_{266} - 2R_{26})R_{33})\Gamma_3 + (-2T_{336} + 2R_{33})\Gamma_1^2\Gamma_2)h'/2$ $-(R_{26}^2R_{43}\Gamma_4 + (2R_{26}T_{436} + (2T_{266} - 2R_{26})R_{43})\Gamma_3 + (-2T_{436} + 2R_{43})\Gamma_1^2\Gamma_2)h\epsilon/2$
U_{4444}	$R_{34}^3\Gamma_1^2\Gamma_2h'''/(6\epsilon^2) + R_{34}^2R_{44}\Gamma_1^2\Gamma_2h''/(2\epsilon) - (2T_{244}R_{34}\Gamma_3 + R_{34}R_{44}^2\Gamma_2)h'/2$ $-(2T_{244}R_{44}\Gamma_3 + R_{44}^3\Gamma_2)h\epsilon/2$
U_{4466}	$-(R_{26}^2R_{34}\Gamma_4 + (2R_{26}T_{346} + (2T_{266} - 2R_{26})R_{34})\Gamma_3 + (-2T_{346} + 2R_{34})\Gamma_1^2\Gamma_2)h'/2$ $-(R_{26}^2R_{44}\Gamma_4 + (2R_{26}T_{446} + (2T_{266} - 2R_{26})R_{44})\Gamma_3 + (-2T_{446} + 2R_{44})\Gamma_1^2\Gamma_2)h\epsilon/2$

3.4 Complete Second and Third Order Map for Dipole Fringe Field

3.4.1 Drift Region Transformation

Expanding Eq. 3.12 and Eq. Eq. 3.18 to the third order in the initial conditions, we obtain the drift region map $(\bar{x}_0, x_0', \bar{y}_0, y_0') \mapsto (\bar{w}_1, \dot{w}_1, \bar{y}_1, \dot{y}_1)$,

$$\begin{aligned}
\bar{w}_1 &= (\bar{x}_0 + \epsilon\bar{s}_1 x_0' \sec\beta)\sec\beta + \left(\bar{x}_0 x_0' + \epsilon\bar{s}_1\, {x_0'}^2 \sec\beta\right)\sec\beta\tan\beta \\
&\quad + \left(\bar{x}_0\, {x_0'}^2 + \epsilon\bar{s}_1\, {x_0'}^3 \sec\beta\right)\sec\beta\tan^2\beta \\
\dot{w}_1 &= x_0'\sec^2\beta + {x_0'}^2\sec^2\beta\tan\beta + {x_0'}^3\sec^2\beta\tan^2\beta \\
\bar{y}_1 &= (\bar{y}_0 + \epsilon\bar{s}_1 y_0'\sec\beta) + (\bar{x}_0 y_0' + \epsilon\bar{s}_1 x_0' y_0'\sec\beta)\tan\beta \\
&\quad + \left(\bar{x}_0 x_0' y_0' + \epsilon\bar{s}_1\, {x_0'}^2\, y_0'\sec\beta\right)\tan^2\beta \\
\dot{y}_1 &= y_0'\sec\beta + x_0' y_0'\sec\beta\tan\beta + {x_0'}^2\, y_0'\sec\beta\tan^2\beta
\end{aligned} \tag{3.48}$$

3.4.2 Transformation through Pure Bend Region

For the uniform field region, we need to solve Eq. 3.35 at the endpoint $z = 0$ to the third order in the final coordinates of the transformation through the fringe region, $(\bar{w}_2, \dot{w}_2, \bar{y}_2, \dot{y}_2)$. The equations can be solved exactly for x and y as follows.

Dividing one equation by the other, we obtain

$$\frac{y''}{x''} = \frac{dy'}{dx'} = \frac{x'y'}{1 + {x'}^2} \tag{3.49}$$

Next, we separate the variables and integrate both parts from the initial point $z = z_2$,

$$\int_{z_2}^{z} \frac{x'dx'}{1 + {x'}^2} = \int_{z_2}^{z} \frac{dy'}{y'} \tag{3.50}$$

$$\frac{1}{2}\ln\left(\frac{1 + {x'}^2}{1 + {x_2'}^2}\right) = \ln\frac{y'}{y_2'} \tag{3.51}$$

Solving for y'

$$y' = \frac{y_2'}{\sqrt{1 + {x_2'}^2}}\sqrt{1 + {x'}^2} \tag{3.52}$$

and putting the above expression into Eq. 3.35, we obtain the decoupled equation for x',

$$\frac{dx'}{dz} = -\frac{1}{\rho(1+\delta)}\sqrt{1+\frac{{y_2'}^2}{1+{x_2'}^2}}\left(1+{x'}^2\right)^{\frac{3}{2}} \tag{3.53}$$

Separating variables and integrating,

$$\int_{z_2}^{z}\frac{dx'}{\left(1+{x'}^2\right)^{\frac{3}{2}}} = -\frac{1}{\rho(1+\delta)}\sqrt{1+\frac{{y_2'}^2}{1+{x_2'}^2}}\int_{z_2}^{z}dz \tag{3.54}$$

we get

$$\frac{x'}{\left(1+{x'}^2\right)^{\frac{1}{2}}} = \frac{x_2'}{\left(1+{x_2'}^2\right)^{\frac{1}{2}}} - \frac{1}{\rho(1+\delta)}\sqrt{1+\frac{{y_2'}^2}{1+{x_2'}^2}}(z-z_2) \tag{3.55}$$

which we can solve for x' explicitly,

$$x' = \frac{x_2'(1+\delta) - \sqrt{1+{x_2'}^2+{y_2'}^2}(\bar{z}-\bar{z}_2)}{\left[(1+\delta)^2 + 2x_2'(1+\delta)\sqrt{1+{x_2'}^2+{y_2'}^2}(\bar{z}-\bar{z}_2) - (1+{x_2'}^2+{y_2'}^2)(\bar{z}-\bar{z}_2)^2\right]^{\frac{1}{2}}} \tag{3.56}$$

Putting Eq. 3.56 into Eq. 3.52, we get the solution for y',

$$y' = \frac{y_2'}{\left[(1+\delta)^2 + 2x_2'(1+\delta)\sqrt{1+{x_2'}^2+{y_2'}^2}(\bar{z}-\bar{z}_2) - (1+{x_2'}^2+{y_2'}^2)(\bar{z}-\bar{z}_2)^2\right]^{\frac{1}{2}}} \tag{3.57}$$

The final point of integration is $\bar{z}_3 = 0$, so

$$\begin{aligned} x_3' &= x'(\bar{z}=0) \\ y_3' &= y'(\bar{z}=0) \end{aligned} \tag{3.58}$$

We can integrate Eq. 3.56 and Eq. 3.57 from $\bar{z}_2$ to 0 to obtain $\bar{x}$ and $\bar{y}$ respectively:

$$\begin{aligned} \bar{x}_3 &= \bar{x}_2 \\ &\quad -\frac{(1+\delta)^2 - \left[(1+\delta)^2 - 2x_2'(1+\delta)\bar{z}_2\sqrt{1+{x_2'}^2+{y_2'}^2} - \bar{z}_2^2(1+{x_2'}^2+{y_2'}^2)\right]^{\frac{1}{2}}}{\sqrt{1+{x_2'}^2+{y_2'}^2}} \end{aligned} \tag{3.59}$$

$$\begin{aligned} \bar{y}_3 &= \bar{y}_2 \\ &\quad +\frac{(1+\delta)}{\sqrt{1+{x_2'}^2+{y_2'}^2}}\left\{\sin^{-1}\left[\frac{x_2'}{\sqrt{1+{x_2'}^2}}\right] - \sin^{-1}\left[\frac{x_2'(1+\delta)+\bar{z}_2\sqrt{1+{x_2'}^2+{y_2'}^2}}{(1+\delta)\sqrt{1+{x_2'}^2}}\right]\right\} \end{aligned}$$

Next, we need to solve for $(\bar{z}_2, \bar{x}_2, x_2', \bar{y}_2, y_2')$ to the third order in $(\bar{w}_2, \dot{w}_2, \bar{y}_2, \dot{y}_2)$ and the first order in ϵ. Using Eq. 3.18 and Eq. 3.20, we get

$$
\begin{aligned}
\bar{z}_2 &= (\bar{w}_2 + \bar{\Delta}_2)\sin\beta + \epsilon\bar{s}_2\cos\beta = \epsilon\bar{s}_2\sec\beta + \bar{w}_2\sin\beta + \cdots \\
\bar{x}_2 &= (\bar{w}_2 + \bar{\Delta}_2)\cos\beta - \epsilon\bar{s}_2\sin\beta = \bar{w}_2\cos\beta + \cdots \\
x_2' &= \frac{(\dot{w}_2 + \dot{\Delta}_2) - \tan\beta}{1 + (\dot{w}_2 + \dot{\Delta}_2)\tan\beta} = -\epsilon\bar{s}_2\sec\beta + \dot{w}_2\left(\cos^2\beta + 2\epsilon\bar{s}_2\sin\beta\right) \\
&\quad -\dot{w}_2^2\sin\beta\cos\beta\left(\cos^2\beta + 3\epsilon\bar{s}_2\sin\beta\right) + \dot{w}_2^3\sin^2\beta\cos^2\beta\left(\cos^2\beta + 4\epsilon\bar{s}_2\sin\beta\right) + \cdots \\
\bar{y}_2 &= \bar{y}_2 \\
y_2' &= \frac{\dot{y}_2\sec\beta}{1 + (\dot{w}_2 + \dot{\Delta}_2)\tan\beta} = \dot{y}_2(\cos\beta + \epsilon\bar{s}_2\tan\beta) \\
&\quad -\dot{w}_2\dot{y}_2\sin\beta\cos\beta(\cos\beta + 2\epsilon\bar{s}_2\tan\beta) + \dot{w}_2^2\dot{y}_2\sin^2\beta\cos^2\beta(\cos\beta + 3\epsilon\bar{s}_2\tan\beta) + \cdots
\end{aligned}
\tag{3.60}
$$

Putting Eq. 3.60 into Eq. 3.56, Eq. 3.57, and Eq. 3.59 and expanding to the appropriate orders, we get the full transformation through the pure bend field $(\bar{w}_2, \dot{w}_2, \bar{y}_2, \dot{y}_2) \mapsto (\bar{x}_3, x_3', \bar{y}_3, y_3')$,

$$
\begin{aligned}
\bar{x}_3 &= [\bar{w}_2 - \epsilon\bar{s}_2\dot{w}_2]\cos\beta \\
&\quad -\left[\bar{w}_2^2\frac{\sin\beta}{2} + \bar{w}_2\dot{w}_2(\cos^2\beta + 2\epsilon\bar{s}_2\sin\beta) - \bar{w}_2\delta\epsilon\bar{s}_2\sec\beta - \dot{w}_2^2\epsilon\bar{s}_2\cos^2\beta\right]\sin\beta \\
&\quad +\left[\bar{w}_2^2\delta\frac{\sin\beta}{2} + \bar{w}_2\dot{w}_2^2\left(\sin\beta\cos^2\beta - \frac{3}{2}\epsilon(1 - 3\sin^2\beta)\right)\cos\beta\right. \\
&\quad \left. -\bar{w}_2\dot{y}_2^2\epsilon\bar{s}_2\frac{\cos\beta}{2} - \bar{w}_2\delta^2\epsilon\bar{s}_2\sec\beta - \dot{w}_2^3\epsilon\bar{s}_2\sin\beta\cos^3\beta\right]\sin\beta + \cdots \\
x_3' &= \left[\bar{w}_2\sin\beta + \dot{w}_2(\cos^2\beta + 2\epsilon\bar{s}_2\sin\beta) - \delta\epsilon\bar{s}_2\sec\beta\right] \\
&\quad -\left[\bar{w}_2\delta\sin\beta + \dot{w}_2^2\left(\sin\beta\cos^2\beta - \frac{3}{2}\epsilon\bar{s}_2(1 - 3\sin^2\beta)\right)\cos\beta\right. \\
&\quad \left. -\dot{y}_2^2\epsilon\bar{s}_2\frac{\cos\beta}{2} - \delta^2\epsilon\bar{s}_2\sec\beta\right] + \left[\bar{w}_2^3\frac{\sin^3\beta}{2} + \bar{w}_2^2\dot{w}_2(\frac{3}{2}\cos^2\beta + 3\epsilon\bar{s}_2\sin\beta)\sin^2\beta\right. \\
&\quad -\frac{3}{2}\bar{w}_2^2\delta\epsilon\bar{s}_2\sin\beta\tan\beta + \bar{w}_2\dot{w}_2^2(\frac{3}{2}\cos^2\beta + 6\epsilon\bar{s}_2\sin\beta)\sin\beta\cos^2\beta \\
&\quad -3\bar{w}_2\dot{w}_2\delta\epsilon\bar{s}_2\sin\beta\cos\beta + \bar{w}_2\dot{y}_2^2(\frac{\cos^2\beta}{2} + \epsilon\bar{s}_2\sin\beta)\sin\beta + \bar{w}_2\delta^2\sin\beta \\
&\quad +\dot{w}_2^3\left(\cos^2\beta\sin\beta - \epsilon\bar{s}_2(3 - 7\sin^2\beta)\right)\sin\beta\cos^2\beta - \frac{3}{2}\dot{w}_2^2\delta\epsilon\bar{s}_2\cos^3\beta
\end{aligned}
\tag{3.61}
$$

$$
\begin{aligned}
& -\dot{w}_2\dot{y}_2^2\epsilon\bar{s}_2\sin\beta\cos^2\beta - \dot{y}_2^2\delta\epsilon\bar{s}_2\frac{\cos\beta}{2} - \delta^3\epsilon\bar{s}_2\sec\beta\Big] + \cdots \\
\bar{y}_3 &= [\bar{y}_2 - \dot{y}_2\epsilon\bar{s}_2] - [\bar{w}_2\dot{y}_2(\cos\beta + \epsilon\bar{s}_2\tan\beta) - \dot{w}_2\dot{y}_2\epsilon\bar{s}_2\cos\beta]\sin\beta \\
& + \left[\bar{w}_2\dot{w}_2\dot{y}_2\left(\sin\beta\cos^2\beta - \epsilon\bar{s}_2(1-3\sin^2\beta)\right) - \dot{w}_2^2\bar{y}_2\epsilon\bar{s}_2\sin\beta\cos^2\beta\right]\sin\beta + \cdots \\
y_3' &= \dot{y}_2(\cos\beta + \epsilon\bar{s}_2\tan\beta) - \dot{w}_2\bar{y}_2\left(\sin\beta\cos^2\beta - \epsilon\bar{s}_2(1-3\sin^2\beta)\right) \\
& + \Big[\bar{w}_2^2\dot{y}_2(\cos\beta + \epsilon\bar{s}_2\tan\beta)\frac{\sin^2\beta}{2} + \bar{w}_2\dot{w}_2\bar{y}_2(\cos^2\beta + 3\epsilon\bar{s}_2\sin\beta)\sin\beta\cos\beta \\
& -\bar{w}_2\dot{y}_2\delta\epsilon\bar{s}_2\sin\beta + \dot{w}_2^2\bar{y}_2\left(\sin\beta\cos^2\beta - \epsilon\bar{s}_2(2-5\sin^2\beta)\right)\sin\beta\cos\beta \\
& -\dot{w}_2\dot{y}_2\delta\epsilon\bar{s}_2\cos^2\beta\Big] + \cdots
\end{aligned}
$$

3.4.3 Matrix Elements for Combined Nonlinear Map

Combining Eq. 3.48 and Eq. 3.61 with the results of the previous section on the fringe region transformation, we can obtain the whole map $(\bar{x}_0, x_0', \bar{y}_0, y_0') \mapsto (\bar{x}^f, x'^f, \bar{y}^f, y'^f)$ as the sought power series,

$$
x_i^f = \sum_{j=1}^{6} R_{ij}x_j + \sum_{j=1}^{6}\sum_{k=j}^{6} T_{ijk}x_jx_k + \sum_{j=1}^{6}\sum_{k=j}^{6}\sum_{l=k}^{6} U_{ijkl}x_jx_kx_l \tag{3.62}
$$

where we again use the TRANSPORT notation of Eq. 2.10. The matrix elements R_{ij}, T_{ijk}, U_{ijkl} contain dimensionless field line integrals, some of which were given in Eq. 3.41. Their integrands go rapidly to zero at both integration limits. As noted above, this fact allows to remove the uncertainty in the extent of the fringe region and gives a practical definition for field measurements.

The nonlinear terms for the combined map are obtained from the three individual transformations using Eq. 2.26 expanded to $O(\epsilon)$.

Second Order Matrix Elements

The second-order solution contains two integral form factors,

$$
\begin{aligned}
I_2 &= \int_{-\infty}^{+\infty} d\bar{s}[1-h(\bar{s})]h(\bar{s}) \\
I_6 &= \int_{-\infty}^{+\infty} d\bar{s}[1-h(\bar{s})]h^2(\bar{s})
\end{aligned} \tag{3.63}
$$

The 10 non-zero terms are given below:

$$
\begin{aligned}
T_{111} &= -\frac{\tan^2\beta}{2}+\cdots \\
T_{133} &= \frac{\sec^2\beta}{2}-\epsilon I_2\frac{\sin\beta(5+\sin^2\beta)}{2\cos^4\beta}+\cdots \\
T_{212} &= \tan^2\beta+\cdots \\
T_{216} &= -\tan\beta+\cdots \\
T_{233} &= \frac{\sin\beta(1+\sin^2\beta)}{2\cos^3\beta}-\epsilon\frac{\sin^2\beta}{\cos^5\beta}\left[I_2(5-\cos^4\beta)-I_6\frac{\cos^2\beta}{2}\right]+\cdots \\
T_{234} &= -\tan^2\beta-\epsilon I_2\frac{\sin\beta(8+\cos^2\beta)}{\cos^4\beta}+\cdots \\
T_{313} &= \tan^2\beta-\epsilon I_2\frac{\sin\beta(1+\sin^2\beta)}{\cos^4\beta}+\cdots \\
T_{414} &= -\tan^2\beta+\epsilon I_2\frac{\sin\beta(1+\sin^2\beta)}{\cos^4\beta}+\cdots \\
T_{423} &= -\sec^2\beta+\epsilon I_2\frac{\sin\beta(5+\sin^2\beta)}{\cos^4\beta}+\cdots \\
T_{436} &= \tan\beta-\epsilon I_2\frac{2(1+\sin^2\beta)}{\cos^3\beta}+\cdots
\end{aligned}
\tag{3.64}
$$

The above matrix elements are in perfect agreement with the known sharp-cutoff approximation results [13] , $\epsilon = 0$.

Third Order Matrix Elements

The third-order solution contains nine integral form factors: I_1, I_2, and I_6 encountered in the first and second order solution plus the six more given below, all of which include the square of the field derivative,

$$
\begin{aligned}
J_1 &= \int_{-\infty}^{+\infty} d\bar{s}\left(\frac{dh(\bar{s})}{d\bar{s}}\right)^2 \\
J_2 &= \int_{-\infty}^{+\infty} d\bar{s}\left(\frac{dh(\bar{s})}{d\bar{s}}\right)^2 \bar{s} \\
J_3 &= \int_{-\infty}^{+\infty} d\bar{s}\left(\frac{dh(\bar{s})}{d\bar{s}}\right)^2 \bar{s}^2
\end{aligned}
\tag{3.65}
$$

$$
\begin{aligned}
J_4 &= \int_{-\infty}^{+\infty} d\bar{s} \left(\frac{dh(\bar{s})}{d\bar{s}}\right)^2 \int_{-\infty}^{\bar{s}} ds' h(s') \\
J_5 &= \int_{-\infty}^{+\infty} d\bar{s} \left(\frac{dh(\bar{s})}{d\bar{s}}\right)^2 \bar{s} \int_{-\infty}^{\bar{s}} ds' h(s') \\
J_6 &= \int_{-\infty}^{+\infty} d\bar{s} \left(\frac{dh(\bar{s})}{d\bar{s}}\right)^2 \int_{-\infty}^{\bar{s}} ds' \int_{-\infty}^{s'} ds'' h(s'')
\end{aligned}
$$

The 34 non-zero terms are given below,

$$
\begin{aligned}
U_{1112} &= -\tan^3\beta + \cdots \\
U_{1116} &= \frac{\tan^2\beta}{2} + \cdots \\
U_{1133} &= -\frac{\sin^2\beta(1+\sin^2\beta)}{2\cos^4\beta} + \epsilon\frac{\sin^3\beta}{\cos^6\beta}\left[I_2(5-\cos^4\beta) - I_6\frac{\cos^2\beta}{2}\right] + \cdots \\
U_{1134} &= \frac{\sin\beta(1+\sin^2\beta)}{\cos^3\beta} + \epsilon I_2\frac{2\sin^2\beta(1+\cos^2\beta)}{\cos^5\beta}\cdots \\
U_{1233} &= \tan\beta\sec^2\beta - \epsilon I_2\frac{(5+18\sin^2\beta+\sin^4\beta)}{2\cos^5\beta} + \cdots \\
U_{1336} &= -\frac{\sec^2\beta}{2} + \epsilon I_2\frac{\sin\beta(5+\sin^2\beta)}{\cos^4\beta} + \cdots \\
U_{2111} &= \frac{\tan^3\beta}{2} + \cdots \\
U_{2112} &= \frac{3}{2}\tan^2\beta + \cdots \\
U_{2122} &= \frac{\sin\beta(2+\cos^2\beta)}{2\cos^3\beta} + \cdots \\
U_{2126} &= -\tan^2\beta + \cdots \\
U_{2133} &= \frac{\tan^3\beta}{2} - \epsilon I_2\frac{\sin^2\beta(1+\sin^2\beta)}{\cos^5\beta} + \cdots \\
U_{2134} &= 2\tan^4\beta + \epsilon\frac{\sin\beta}{\cos^6\beta}\left[I_2(1-8\sin^2\beta-5\sin^4\beta+2\sin^6\beta) + I_6\sin^2\beta\cos^2\beta\right] + \cdots \\
U_{2144} &= \frac{\sin\beta(1-3\sin\beta)}{2\cos^3\beta} - \epsilon I_2\frac{\sin^2(8+\cos^2\beta)}{\cos^5\beta} + \cdots \\
U_{2166} &= \tan\beta + \cdots \\
U_{2233} &= 3\tan^2\beta\sec^2\beta \\
&\quad -\epsilon\frac{\sin\beta}{2\cos^6\beta}\left[I_2(9+35\sin^2\beta+2\sin^4\beta+4\sin^6\beta) + 2I_6\sin^2\beta\cos^2\beta\right] + \cdots
\end{aligned}
$$

$$
\begin{aligned}
U_{2234} &= -2\tan\beta\sec^2\beta - \epsilon\frac{I_2}{\cos^5\beta}(9+24\sin^2\beta-\sin^4\beta)+\cdots \\
U_{2336} &= -\frac{\sin\beta(1+\sin^2\beta)}{\cos^3\beta}+\cdots \\
U_{2346} &= \tan^2\beta - \epsilon I_2\frac{2\sin\beta(1+\sin^2\beta)}{\cos^4\beta}+\cdots \\
U_{3114} &= \tan^3\beta - \epsilon I_2\frac{\sin^2\beta(1+\sin^2\beta)}{\cos^5\beta}+\cdots \\
U_{3123} &= \frac{\sin\beta(1+\sin^2\beta)}{\cos^3\beta} - \epsilon I_2\frac{2\sin^2\beta(3+\sin^2\beta)}{\cos^5\beta}+\cdots \\
U_{3136} &= -\tan^2\beta + \epsilon I_2\frac{2\sin\beta(1+\sin^2\beta)}{\cos^4\beta}+\cdots \\
U_{3333} &= \left[\frac{2}{3}\frac{(1+\sin^2\beta)}{4\cos^4\beta}J_2 + \frac{(1+5\sin^2\beta)}{6\cos^4\beta}\right] \\
&\quad -\epsilon\frac{\sin\beta}{6\cos^6\beta}\left[I_2\sin^2\beta\left(7+3\sin^2\beta\right) - 8I_6\sin^2\beta - 6I_1J_1 + 21J_5 - 3J_6\right]+\cdots \\
U_{3334} &= \frac{2}{3}\frac{\sin\beta}{\cos^3\beta} - \frac{\epsilon}{2\cos^5\beta}\left[I_2(1+3\sin^2\beta-\sin^4\beta) - 3J_3\right]+\cdots \\
U_{4113} &= -\frac{\tan^3\beta}{2} + \epsilon I_2\frac{\sin^2\beta(1+\sin^2\beta)}{2\cos^5\beta}+\cdots \\
U_{4114} &= \frac{\tan^2\beta}{2}+\cdots \\
U_{4123} &= -\tan^3\beta + \epsilon I_2\frac{\sin^2\beta(1+\sin^2\beta)}{2\cos^5\beta}+\cdots \\
U_{4124} &= -2\tan^3\beta + \epsilon I_2\frac{2\sin^2\beta(3+\sin^2\beta)}{\cos^5\beta}+\cdots \\
U_{4146} &= \tan^2\beta - \epsilon I_2\frac{2\sin\beta(1+\sin^2\beta)}{\cos^4\beta}+\cdots \\
U_{4223} &= -\frac{\sin\beta(2+\cos^2\beta)}{2\cos^3\beta} + \epsilon I_2\frac{3(1+3\sin^2\beta)}{\cos^5\beta}+\cdots \\
U_{4236} &= \sec^2\beta - \epsilon I_2\frac{2\sin\beta(5+\sin^2\beta)}{\cos^4\beta}+\cdots \\
U_{4333} &= -\frac{J_1}{\epsilon}\frac{2(1+\sin^2\beta)}{3\cos^4\beta} - \frac{\sin\beta}{12\cos^5\beta}\left[\sin^2\beta\left(1+6\cos^2\beta\right)+6J_2-42J_4\right]+\cdots \\
U_{4334} &= \frac{\left[\sin^2\beta(1+6\cos^2\beta)-6J_2\right]}{4\cos^4\beta}+\cdots
\end{aligned}
$$

$$U_{4344} = -\frac{3}{2}\tan\beta + \frac{\epsilon}{2\cos^5\beta}\left[I_2(4+23\sin^2\beta - 4\sin^4\beta) - 3J_3\right] + \cdots$$

$$U_{4366} = -\tan\beta + \epsilon I_2 \frac{3(1+\sin^2\beta)}{\cos^3\beta} + \cdots$$

As a simple exercise, we can model the field function $h(\bar{s})$ by a polynomial whose first and second derivatives vanish at the end-points and whose integral over the fringe region is equal to 1. Taking $\bar{s}_2 = -\bar{s}_1 = 1$, we take the lowest order polynomial satisfying the boundary conditions:

$$h(\bar{s}) = \begin{cases} 0 & |\bar{s}| > 1 \\ \frac{1}{2} + \frac{5}{16}\bar{s} - \frac{5}{8}\bar{s}^3 + \frac{3}{16}\bar{s}^5 & |\bar{s}| > 1 \end{cases} \tag{3.66}$$

The form-factor integrals can then be evaluated and we get the following result,

$$\begin{array}{cc}
I_1 = -0.07 \approx -I_2/3 & J_1 = 0.71 \approx 10I_2/3 \\
I_2 = 0.22 & J_2 = 0.00 \\
I_3 = 0.03 \approx I_2/8 & J_3 = 0.07 \approx I_2/3 \\
I_4 = 0.11 \approx I_2/2 & J_4 = 0.14 \approx 2I_2/3 \\
I_5 = 0.04 \approx I_2/5 & J_5 = 0.03 \approx I_2/6 \\
I_6 = 0.11 \approx I_2/2 & J_6 = 0.04 \approx I_2/5
\end{array}$$

Chapter 4

Hamiltonian Methods

4.1 Introduction

In this chapter we turn our attention to a completely different approach to the charged particle beam optics. Despite successes of the transfer matrix method in giving practical solutions, it has some deficiencies. Basically, the matrix theory is a straightforward attack on nonlinear differential equations which does not take into account the physical origin of the equations. There is no attempt to take advantage of the underlying symmetries of the beam transport arising from its being a Hamiltonian motion. For example, it is clear that not all of the matrix elements are independent and that there are interdependencies between the different optical orders, but it is not clear how to get at them. Also, in the case of the dipole fringe field, we remarked that equations describing various matrix terms were to be solved by iteration as a power series in the "field extent" parameter ϵ. Much care was needed in keeping track of the orders of many different terms to ensure the proper expansion at the end (this was due to the fact that some elements had $O(\epsilon^{-1})$ terms). Is there a way to obtain a solution not based on the ϵ expansion? In short, we would like to have some means to obtain physical rather than purely mathematical insight into the problem of charged particle beam transport.

Hamiltonian methods which will be described in this chapter afford us such means. They use the symplectic, or area preserving, property of the transfer map and some Poisson bracket operators to obtain certain homogeneous polynomials f_n of degree n in the canonical phase-space variables, which uniquely describe the optics to the order $(n-1)$.

We will describe the Hamiltonian machinery required to solve transport problems up to the third order and will apply it to the case of the dipole fringe field. We will also show how to transform the solution into the usual noncanonical TRANSPORT variables and how to obtain the desired transfer matrices from the polynomials f_n.

The methods, largely developed in [7], which use symplecticity of Hamiltonian flow maps are called *Lie Algebraic*, because the Poisson bracket operators introduced to solve the problem form a Lie Algebra.

4.2 Lie Algebraic Approach

4.2.1 Mathematical Tools

Consider motion with three degrees of freedom governed by a Hamiltonian $H(\mathbf{z},t)$, where $\mathbf{z} = (q_1, p_1, q_2, p_2, q_3, p_3)$ is a generalized coordinate-momentum 6-vector and t is an independent variable. Suppose we are interested in trajectories near some given trajectory, $\mathbf{z}^g$. Define

$$\mathbf{Z} = \mathbf{z} - \mathbf{z}^g \tag{4.1}$$

The evolution of $\mathbf{Z}$ is governed by some new Hamiltonian $\mathcal{H}(\mathbf{Z},t)$. In fact, if we expand $\mathcal{H}$ in a power series of $\mathbf{Z}$, we get,

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4 + \cdots \tag{4.2}$$

where each $\mathcal{H}_n$ is a homogeneous polynomial of degree n in $\mathbf{Z}$. Our task is to determine the final state, $\mathbf{Z}^f = \mathbf{Z}(t^f)$, given the initial state, $\mathbf{Z}^i = \mathbf{Z}(t^i)$. A solution to the problem is constituted by the knowledge of the *transfer map*, $\mathcal{M}$,

$$\mathbf{Z}^f = \mathcal{M}\mathbf{Z}^i \tag{4.3}$$

In addition to the matrix representation, there also exists the Lie Algebraic representation of the transfer map $\mathcal{M}$. We start with a few definitions.

Let f be a given function of the phase space and let g be any function. Associated with each f, we define a *Lie operator* acting on general functions g and denoted by $: f :$ as a Poisson bracket operation,

$$: f : g \equiv [f, g] = \sum_{i=1}^{3} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \tag{4.4}$$

Lie operators possess an important property which will be useful to us later. It can be shown from the Jacobi identity for Poisson brackets that the commutator of two Lie operators, denoted by curly brackets, is again a Lie operator, which can be calculated in terms of a Poisson bracket,

$$\{: f :, : g :\} =: [f, g] : \tag{4.5}$$

Next, we define a *Lie transformation* associated with f, $\exp(: f :)$, which is given by the exponential series,

$$\exp(: f :) = \sum_{n=0}^{\infty} \frac{: f :^n}{n!} \tag{4.6}$$

More explicitly, the action of $\exp(: f :)$ on an arbitrary function g is as follows,

$$\exp(: f :) g = g + [f, g] + \frac{1}{2} [f, [f, g]] + \cdots \tag{4.7}$$

It can be shown [9] that Lie transformations, which form a *Lie Algebra* under commutation, are symplectic (Hamiltonian generated) and to every symplectic map there corresponds a unique function f. Therefore, the map $\mathcal{M}$ of Eq. 4.3 can be uniquely represented by a Lie transformation,

$$\mathcal{M} = \exp(: f :) \tag{4.8}$$

Furthermore, there exist a theorem which gives sense to the idea of classification by orders [7,11]:

Theorem 1 (Factorization Theorem) *Let $\mathcal{M}$ be an analytic symplectic map which maps the origin of phase space into itself,*

$$\mathbf{Z}^f = \mathcal{M}\mathbf{Z}$$

Then $\mathcal{M}$ has a unique Lie Algebraic factorization,

$$\mathcal{M} = \ldots \exp(:f_4:)\exp(:f_3:)\exp(:f_2:) \tag{4.9}$$

where each $f_n(\mathbf{Z})$ is a homogeneous polynomial of degree n in $\mathbf{Z}$.

The expansion in Eq. 4.9 up to the f_n term completely specifies the optics up to the order $(n-1)$ in a sense that if we were to write Eq. 4.9 as a power series in $\mathbf{Z}$ using Eq. 4.7 we would obtain matrix expansion correct to the order $(n-1)$. Also, written in the form of Eq. 4.9, the representation of $\mathcal{M}$ remains symplectic truncated at any point, since a product of two symplectic transformations is symplectic.

Our aim is evident now. To obtain the full third order optics, we must derive the polynomials f_2, f_3, and f_4 characterizing $\mathcal{M}$ from the polynomials $\mathcal{H}_2$, $\mathcal{H}_3$, $\mathcal{H}_4$ characterizing $\mathcal{H}$.

4.2.2 Equation of Motion for Map $\mathcal{M}$

We will denote the differentiation with respect to the independent variable t with a dot. Let us start by taking a derivative of an arbitrary function g,

$$\dot{g}\left(\mathbf{Z}(t)\right) = \dot{g}\left(\mathcal{M}(t)\mathbf{Z}^i\right) = \dot{\mathcal{M}}(t) g\left(\mathbf{Z}^i\right) \tag{4.10}$$

Also, the equation of motion for g is given by its Poisson bracket with the Hamiltonian,

$$\dot{g}\left(\mathbf{Z}(t)\right) = \left[g\left(\mathbf{Z}(t)\right), \mathcal{H}\left(\mathbf{Z}(t), t\right)\right] \tag{4.11}$$

Using general properties of symplectic maps [11], we can manipulate the right hand side of Eq. 4.11 in the following manner,

$$\begin{aligned} \left[g\left(\mathbf{Z}\right), \mathcal{H}\left(\mathbf{Z}, t\right)\right] &= \left[g\left(\mathcal{M}\mathbf{Z}^i\right), \mathcal{H}\left(\mathcal{M}\mathbf{Z}^i, t\right)\right] \\ = \left[\mathcal{M}g\left(\mathbf{Z}^i\right), \mathcal{M}\mathcal{H}\left(\mathbf{Z}^i, t\right)\right] &= \mathcal{M}\left[g\left(\mathbf{Z}^i\right), \mathcal{H}\left(\mathbf{Z}^i, t\right)\right] \\ = \mathcal{M}\left[-\mathcal{H}\left(\mathbf{Z}^i, t\right), g\left(\mathbf{Z}^i\right)\right] &= \mathcal{M} : -\mathcal{H}\left(\mathbf{Z}^i, t\right) : g\left(\mathbf{Z}^i\right) \end{aligned} \tag{4.12}$$

Comparing Eq. 4.10 and Eq. 4.12, we find the result,

$$\dot{\mathcal{M}}g\left(\mathbf{Z}^i\right) = \mathcal{M} : -\mathcal{H}\left(\mathbf{Z}^i, t\right) : g\left(\mathbf{Z}^i\right) \tag{4.13}$$

Since the function g is arbitrary, we see that $\mathcal{M}$ satisfies the following equation of motion,

$$\dot{\mathcal{M}} = \mathcal{M} : -\mathcal{H}\left(\mathbf{Z}^i, t\right) : \tag{4.14}$$

For the Hamiltonians with the commuting property

$$\left[\mathcal{H}\left(\mathbf{Z}^i, t'\right), \mathcal{H}\left(\mathbf{Z}^i, t''\right)\right] = 0 \tag{4.15}$$

we can integrate Eq. 4.14 to obtain the exact result

$$\mathcal{M} = \exp\left(- : \int_{t^i}^{t} \mathcal{H}\left(\mathbf{Z}^i, t'\right) dt' :\right) \tag{4.16}$$

Since $\mathcal{H}$ has the form of Eq. 4.2, its integral can be written as follows,

$$-\int_{t^i}^{t} \mathcal{H}\left(\mathbf{Z}^i, t'\right) dt' = \sum_{n=2}^{\infty} h_n\left(\mathbf{Z}^i, t\right) \tag{4.17}$$

where h_n are homogeneous polynomials of degree n in the variables $\mathbf{Z}^i$. Hence, $\mathcal{M}$ can be written in the form

$$\mathcal{M} = \exp\left(: \sum_{n=2}^{\infty} h_n :\right) \tag{4.18}$$

We now turn our attention to expressing Eq. 4.18 in the form of Eq. 4.9. The approach will be similar to the Dyson's expansion employed in quantum field theory, which uses the interaction picture.

4.2.3 Interaction Picture

We can write the map $\mathcal{M}$ in the factored product form

$$\mathcal{M} = \ldots \mathcal{M}_4 \mathcal{M}_3 \mathcal{M}_2 = \mathcal{M}_R \mathcal{M}_2 \tag{4.19}$$

Also, we can decompose $\mathcal{H}$ into a sum of homogeneous polynomials,

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4 + \cdots = \mathcal{H}_2 + \mathcal{H}_R \tag{4.20}$$

The subscript R denotes "remaining" terms.

Differentiation of Eq. 4.19 yields the result

$$\dot{\mathcal{M}} = \dot{\mathcal{M}}_R \mathcal{M}_2 + \mathcal{M}_R \dot{\mathcal{M}}_2 \tag{4.21}$$

Combining Eq. 4.14 with Eq. 4.19 and Eq. 4.20, we obtain

$$\dot{\mathcal{M}}_R \mathcal{M}_2 + \mathcal{M}_R \dot{\mathcal{M}}_2 = \mathcal{M}_R \mathcal{M}_2 : -\mathcal{H}_2 - \mathcal{H}_R := \mathcal{M}_R \mathcal{M}_2 : -\mathcal{H}_2 : + \mathcal{M}_R \mathcal{M}_2 : -\mathcal{H}_R : \tag{4.22}$$

It will be shown later that $\mathcal{M}_2$ is required to satisfy the equation

$$\dot{\mathcal{M}}_2 = \mathcal{M}_2 : -\mathcal{H}_2 : \tag{4.23}$$

Then, it follows from Eq. 4.22 and Eq. 4.23 that

$$\dot{\mathcal{M}}_R \mathcal{M}_2 = \mathcal{M}_R \mathcal{M}_2 : -\mathcal{H}_R : \tag{4.24}$$

or equivalently

$$\dot{\mathcal{M}}_R = \mathcal{M}_R \mathcal{M}_2 : -\mathcal{H}_R : \mathcal{M}_2^{-1} \tag{4.25}$$

The quantity $\mathcal{M}_2 : -\mathcal{H}_R : \mathcal{M}_2^{-1}$ can be simplified using the properties of Lie transformations as follows,

$$\mathcal{M}_2 : -\mathcal{H}_R : \mathcal{M}_2^{-1} =: -\mathcal{H}_R^{\text{int}} : \tag{4.26}$$

where the "interaction" Hamiltonian $\mathcal{H}_R^{\text{int}}$ is given by the expression

$$\mathcal{H}_R^{\text{int}} \left(\mathbf{Z}^i, t\right) = \mathcal{M}_2 \mathcal{H}_R \left(\mathbf{Z}^i, t\right) = \mathcal{H}_R \left(\mathcal{M}_2 \mathbf{Z}^i, t\right) \tag{4.27}$$

With this simplification the equation for $\mathcal{M}_R$ takes the final form

$$\dot{\mathcal{M}}_R = \mathcal{M}_R : -\mathcal{H}_R^{\text{int}} : \tag{4.28}$$

By construction, $\mathcal{M}_R$ and $\mathcal{H}_R$ involve polynomials only of the third degree and higher; the same is true for $\mathcal{H}_R^{\text{int}}$, as will be shown later. So, Eq. 4.28 for $\mathcal{M}_R$ involves polynomials only of the third degree and higher, which correspond to the nonlinear part of the map. Next, we turn to the calculation of the linear part $\mathcal{M}_2$.

4.2.4 Linear Map $\mathcal{M}_2$

Let us denote the action of $\mathcal{M}_2$ on $\mathbf{Z}^i$ with $\bar{\mathbf{Z}}(t)$,

$$\bar{\mathbf{Z}}(t) = \mathcal{M}_2 \mathbf{Z}^i = \exp(: f_2 :)\mathbf{Z}^i \tag{4.29}$$

or, in expanded form,

$$\bar{\mathbf{Z}}(t) = \mathbf{Z}^i + : f_2 : \mathbf{Z}^i + \frac{1}{2} : f_2 :^2 \mathbf{Z}^i + \cdots \tag{4.30}$$

Before proceeding further, let us look at the degrees of various Lie operators and polynomials. For a homogeneous polynomial f_n, we let d denote the degree,

$$d(f_n) = n \tag{4.31}$$

For the operation of taking a Poisson bracket of two polynomials f_k and f_l, which involves multiplication and two differentiations, we have

$$d([f_k, f_l]) = k + l - 2 = d(f_k) + d(f_l) - 2 \tag{4.32}$$

which can be written in terms of a Lie operator as follows

$$d(: f_k : f_l) = d(f_k) + d(f_l) - 2 \tag{4.33}$$

For a set of m homogeneous polynomials $f^1, f^2, \ldots f^m$ (the superscript labels the polynomial, not its degree) and an arbitrary homogeneous polynomial g_n of degree n, Eq. 4.33 can be generalized to give the relation

$$d\left(: f^1 :: f^2 : \ldots : f^m : g_n\right) = n - 2m + \sum_{j=1}^{m} d(f^j) \tag{4.34}$$

Now, if we apply Eq. 4.34 to the right hand side of Eq. 4.30, we will observe that all the terms in the expansion are of degree 1. Hence, $\mathcal{M}_2$ is a *linear* transformation which can be represented by a 6×6 matrix M. In the component form, we can write

$$\bar{Z}_a(t) = \mathcal{M}_2 Z_a^i = \sum_b M_{ab}(t) Z_b^i \tag{4.35}$$

So, the computation of $\mathcal{M}_2$ is equivalent to finding the matrix M.

The time evolution of $\mathcal{M}_2$, and hence of M, is governed by quadratic part of the Hamiltonian $\mathcal{H}_2$, which can be written in the following form,

$$\mathcal{H}_2 = \frac{1}{2}\sum_{ab} S_{ab}(t) Z^i_a Z^i_b \tag{4.36}$$

where S is a symmetric matrix. Next, we check the consistency of Eq. 4.23. Applying $: -\mathcal{H}_2 :$ to $\mathbf{Z}^i$, we get

$$: -\mathcal{H}_2 : Z^i_c = \sum_{ab} J_{ca} S_{ab} Z^i_b \tag{4.37}$$

where the matrix J is defined in terms of the fundamental Poisson bracket,

$$J_{ab} \equiv \left[Z^i_a, Z^i_b \right] \tag{4.38}$$

or, in the matrix form for the 6-dimensional phase space,

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{4.39}$$

Now, we can compute the right hand side of Eq. 4.23,

$$\begin{aligned} \mathcal{M}_2 : -\mathcal{H}_2 : Z^i_d &= \mathcal{M}_2 \sum_{ab} J_{da} S_{ab} Z^i_b \\ = \sum_{ab} J_{da} S_{ab} \mathcal{M}_2 Z^i_b &= \sum_{ab} J_{da} S_{ab} \bar{Z}^i{}_b \\ = \sum_{ab} J_{da} S_{ab} M_{bc} Z^i_c &= \left(JSMZ^i \right)_d \end{aligned} \tag{4.40}$$

or, in the matrix form,

$$\mathcal{M}_2 : -\mathcal{H}_2 : \mathbf{Z}^i = JSM\mathbf{Z}^i \tag{4.41}$$

The left hand side of Eq. 4.23 gives

$$\dot{\mathcal{M}}_2 \mathbf{Z}^i = \dot{\bar{\mathbf{Z}}} = \dot{M}\mathbf{Z}^i \tag{4.42}$$

Comparing right hand sides of Eq. 4.41 and Eq. 4.42, we see that they are both of degree 1 in $\mathbf{Z}^i$, and so Eq. 4.23 is consistent. The matrix M must satisfy the following differential equation,

$$\dot{M} = JSM \tag{4.43}$$

with the initial condition

$$M(\mathbf{Z}^i) = I$$

required by the stipulation that $\mathcal{M}$, and hence both $\mathcal{M}_2$ and $\mathcal{M}_R$, be the identity map $\mathcal{I}$ when $t = t^i$.

Eq. 4.43 can be integrated to yield a unique solution

$$M = \exp\left(\int_{t^i}^{t} JS(t')dt'\right) \tag{4.44}$$

from which we see that

$$: f_2 := \int_{t^i}^{t} JS(t')dt' \tag{4.45}$$

The explicit form of f_2 is not required for the subsequent calculations and so will not be worked out here. Instead, we turn our attention to the nonlinear part of the map.

4.2.5 Computation of Higher Order Polynomials

Integrating both parts of Eq. 4.28, we get the following result,

$$\mathcal{M}_R(t) = \mathcal{I} + \int_{t^i}^{t} dt' \mathcal{M}_R(t') : -\mathcal{H}_R^{\text{int}}(t') : \tag{4.46}$$

We can solve Eq. 4.46 be repeated iteration, obtaining the following time ordered series,

$$\mathcal{M}_R(t) = \mathcal{I} + \int_{t^i}^{t} dt' : -\mathcal{H}_R^{\text{int}}(t') : + \int_{t^i}^{t} dt' \int_{t^i}^{t'} dt'' : -\mathcal{H}_R^{\text{int}}(t'') :: -\mathcal{H}_R^{\text{int}}(t') : + \cdots \tag{4.47}$$

We now use Eq. 4.47 to obtain explicit formulas for the polynomials f_3 and f_4, which we need to specify the optics of a system up to the third order. By definition, $\mathcal{M}_R$ is given by the expansion

$$\mathcal{M}_R = \ldots \exp(: f_4 :) \exp(: f_3 :) \tag{4.48}$$

which can be applied to some homogeneous polynomial g_n to yield with the help of Eq. 4.7

$$\begin{aligned}\mathcal{M}_R g_n &= \ldots(1+:f_4:+\cdots)\left(1+:f_4:+\frac{1}{2}:f_4:^2+\cdots\right)g_n\\ &= g_n+(:f_3:g_n)+\left(\frac{1}{2}:f_3:^2 g_n+:f_4:g_n\right)+\cdots\end{aligned} \tag{4.49}$$

We have grouped the terms according to the degree using Eq. 4.34; the degrees displayed are n, $(n+1)$, and $(n+2)$ respectively.

Next, we apply Eq. 4.47 to the same polynomial g_n using the obvious decomposition

$$\mathcal{H}_R^{\rm int} = \mathcal{H}_2^{\rm int} + \mathcal{H}_3^{\rm int} + \cdots \tag{4.50}$$

We obtain the following result,

$$\begin{aligned}\mathcal{M}_R g_n &= g_n + \left(\int_{t^i}^t dt' : -\mathcal{H}_3^{\rm int}(t') : g_n\right) \qquad (4.51)\\ &+\left(\int_{t^i}^t dt' : -\mathcal{H}_4^{\rm int}(t') : g_n + \int_{t^i}^t dt' \int_{t^i}^{t'} dt'' : -\mathcal{H}_3^{\rm int}(t'') :: -\mathcal{H}_3^{\rm int}(t') : g_n\right)+\cdots\end{aligned}$$

where again the terms have been grouped according to the degree.

Comparing terms of the same degree in Eq. 4.49 and Eq. 4.51, we find the following relations,

$$:f_3: = \int_{t^i}^t dt' : -\mathcal{H}_3^{\rm int}(t') : \tag{4.52}$$

$$\frac{1}{2}:f_3:^2+:f_4: = \int_{t^i}^t dt' : -\mathcal{H}_4^{\rm int}(t') : + \int_{t^i}^t dt' \int_{t^i}^{t'} dt'' : -\mathcal{H}_3^{\rm int}(t'') :: -\mathcal{H}_3^{\rm int}(t') : \tag{4.53}$$

Both sides of Eq. 4.52 are Lie operators. We can remove the colons and write the explicit formula for f_3 as follows,

$$f_3 = -\int_{t^i}^t dt' \mathcal{H}_3^{\rm int}(t') \tag{4.54}$$

Both sides of the above equation are of the same degree, satisfying the requirement for consistency.

Eq. 4.53 can be solved for $:f_4:$ to yield

$$\begin{aligned}:f_4: &= \int_{t^i}^t dt' : -\mathcal{H}_4^{\rm int}(t') : \qquad (4.55)\\ &+\int_{t^i}^t dt' \int_{t^i}^{t'} dt'' : -\mathcal{H}_3^{\rm int}(t'') :: -\mathcal{H}_3^{\rm int}(t') : -\frac{1}{2}\int_{t^i}^t dt' \int_{t^i}^t dt'' : -\mathcal{H}_3^{\rm int}(t'') :: -\mathcal{H}_3^{\rm int}(t') :\end{aligned}$$

where the use have been made of Eq. 4.52. The right hand side of Eq. 4.55 needs to be manipulated to bring the term in the form of a Lie operator. The last term in Eq. 4.55, which is equal to $:f_2:^2/2$, can be rewritten as a time ordered integral:

$$\frac{1}{2}:f_3:^2=\frac{1}{2}\int_{t^i}^{t}dt'\int_{t^i}^{t'}dt''\left(:-\mathcal{H}_3^{\rm int}(t''):: -\mathcal{H}_3^{\rm int}(t'):+:-\mathcal{H}_3^{\rm int}(t')::-\mathcal{H}_3^{\rm int}(t''):\right) \tag{4.56}$$

Substituting Eq. 4.56 into Eq. 4.55, we obtain the result

$$:f_4:=\int_{t^i}^{t}dt':-\mathcal{H}_4^{\rm int}(t'):+\frac{1}{2}\int_{t^i}^{t}dt'\int_{t^i}^{t'}dt''\left\{:-\mathcal{H}_3^{\rm int}(t''):,:-\mathcal{H}_3^{\rm int}(t'):\right\} \tag{4.57}$$

Now we can use Eq. 4.5 to express the commutator in Eq. 4.57 as a Lie operator:

$$\left\{:-\mathcal{H}_3^{\rm int}(t''):,:-\mathcal{H}_3^{\rm int}(t'):\right\}=:\left[-\mathcal{H}_3^{\rm int}(t''),-\mathcal{H}_3^{\rm int}(t')\right]: \tag{4.58}$$

Thanks to Eq. 4.58, we have succeeded in rendering the right hand side of Eq. 4.55 to be in the form of a Lie operator. Removing the colons, we obtain the explicit formula for f_4:

$$f_4=-\int_{t^i}^{t}dt'\mathcal{H}_4^{\rm int}(t')+\frac{1}{2}\int_{t^i}^{t}dt'\int_{t^i}^{t'}dt''\left[-\mathcal{H}_3^{\rm int}(t''),-\mathcal{H}_3^{\rm int}(t')\right] \tag{4.59}$$

4.2.6 Summary of Results

We can summarize the results of this section in the following theorem.

Theorem 2 *Consider a system governed by a Hamiltonian $\mathcal{H}=\mathcal{H}_2+\mathcal{H}_3+\mathcal{H}_4+\cdots$. Then, the polynomials f_2, f_3, f_4 used in the factorization of Eq. 4.9 satisfy the following differential equations,*

$$\begin{aligned}\dot{M} &= JSM\\ \dot{f}_3 &= -\mathcal{H}_3^{\rm int}\\ \dot{f}_4 &= -\mathcal{H}_4^{\rm int}+\frac{1}{2}:f_3:(-\mathcal{H}_3^{\rm int})\end{aligned} \tag{4.60}$$

with the initial conditions

$$M(\mathbf{Z}^i)=I$$

$$f_3(\mathbf{Z}^i)=f_4(\mathbf{Z}^i)=0$$

M is the 6×6 matrix representation of $\exp(: f_2 :)$,

$$\exp(: f_2 :)\mathbf{Z} = M\mathbf{Z}$$

the 6×6 matrix S is obtained from $\mathcal{H}_2$ using

$$\mathcal{H}_2(\mathbf{Z}, t) = \frac{1}{2} \sum_{ab} S_{ab}(t) Z_a Z_b \tag{4.61}$$

and J is the basic Poisson bracket matrix,

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

Finally, $\mathcal{H}_3^{\text{int}}$ and $\mathcal{H}_4^{\text{int}}$ are defined by

$$\mathcal{H}_n^{\text{int}}(\mathbf{Z}) = \mathcal{H}_n(M\mathbf{Z}) \tag{4.62}$$

We are now ready to apply the Hamiltonian methods to the problem of calculating the optical properties of the dipole fringe field. We will compute the matrix M together with the polynomials f_3 and f_4 and then, from them, obtain the coefficients in the Taylor expansion,

$$X_a^{final} = \sum_b R_{ab} X_b + \sum_b \sum_c T_{abc} X_b X_c + \sum_b \sum_c \sum_d U_{abcd} X_b X_c X_d + \cdots \tag{4.63}$$

where $\mathbf{X}$ is the usual TRANSPORT 6-vector, $\mathbf{X} = (x, x', y, y', l, \delta)$.

4.3 Dipole Fringe Region Map

4.3.1 Problem Formulation

The geometry of the problem is shown in Fig. 4.1. We consider the entrance pole face of a dipole. Here, the coordinate s is perpendicular to the magnetic boundary and x and y are the other two coordinates in the orthogonal set (s, x, y). The design trajectory makes an angle β with the s-axis as it enters the fringe region. We would like to relate the coordinates at $s = s_2$ with those at $s = s_1$. We assume that the magnetic field $B(s) = B_y(s, 0, 0)$ goes smoothly from zero at s_1 to the constant value B_0 at s_2.

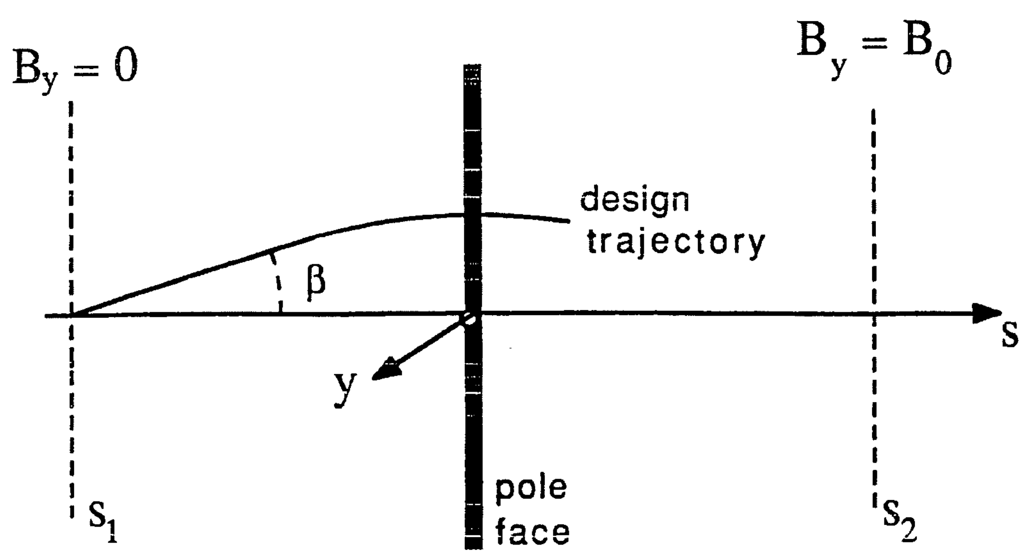

Figure 4.1: Fringe region at the entrance of a dipole.

4.3.2 Fringe Region Hamiltonian

Hamiltonian for the motion of a relativistic charged particle has the form

$$H = e\phi + c[m^2c^2 + (\mathbf{P} - e\mathbf{A}/c)^2]^{\frac{1}{2}} \tag{4.64}$$

We can change the independent variable to s, the distance along the line perpendicular to the pole face, by defining another Hamiltonian,

$$\mathcal{H} \equiv -P_s(x, P_x, y, P_y, t, -H) \tag{4.65}$$

We get ($\phi = 0$),

$$\mathcal{H} = -eA_s/c - [H^2/c^2 - m^2c^2 - (P_x - eA_x/c)^2 - (P_y - eA_y/c)^2]^{\frac{1}{2}} \tag{4.66}$$

For the case of the uniform field inside the magnet, the vector potential has only one component:

$$A_s = A_y = 0$$
$$A_x = B_{-1} - \frac{B_1}{2}y^2 - \frac{B_3}{4}y^4 + \cdots \tag{4.67}$$

where

$$B_{-1} = \int_{-\infty}^{s} B(s')ds'$$
$$B_1 = \frac{dB}{ds}$$
$$B_3 = -\frac{1}{6}\frac{d^3B}{ds^3}$$

and $B(s) = B_y(s, 0, 0)$.

Note that $(t, -H)$ play the role of the third coordinate and conjugate momentum in Eq. 4.66. We can make a canonical transformation to a more convenient pair, (τ, δ), where $\delta = (P - P_0)/P_0$ is the usual fractional deviation from the design momentum and τ is given by

$$\tau = -(P_0 c)\beta t \tag{4.68}$$

where βc is the speed of the off-momentum particle,

$$\beta = \frac{(1+\delta)}{\sqrt{(1+\delta)^2 + (1-\beta_0^2)/\beta_0^2}} \tag{4.69}$$

Rewriting Eq. 4.66, we get,

$$\mathcal{H} = -\left[P_0^2(1+\delta)^2 - (P_x - eA_x/c)^2 - P_y^2\right]^{\frac{1}{2}} \tag{4.70}$$

Next, we rescale the variables in Eq. 4.70,

$$\begin{aligned} p_{x,y} &= \frac{P_{x,y}}{P_0} \\ a_x &= \frac{eA_x}{P_0 c} \\ \mathcal{K} &= \frac{\mathcal{H}}{P_0} \end{aligned}$$

We obtain,

$$\mathcal{K} = -\left[(1+\delta)^2 - (p_x - a_x)^2 - p_y^2\right]^{\frac{1}{2}} \tag{4.71}$$

The scaled field b can be written as follows,

$$b = \frac{e}{p_0 c}B = \frac{1}{\rho}\frac{B}{B_0} \tag{4.72}$$

where B_0 is the constant field well inside the magnet.

Now, we rewrite the Hamiltonian in terms of deviations from the design trajectory. We define vector $\mathbf{x}$,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} x - x^d \\ p_x - p_x^d \\ y \\ p_y \\ \tau - \tau^d \\ \delta \end{pmatrix} \tag{4.73}$$

where the superscript d refers to the design coordinates. Since the Hamiltonian does not explicitly depend on x, $p_x^d = const = \sin\beta$. We can write the full Hamiltonian to the 4^{th} order,

$$\mathcal{K} = -[1 + 2x_6 + x_6^2 - (x_2 + \sin\beta - b_{-1} + \frac{b_1}{2}x_3^2 + \frac{b_3}{4}x_3^4)^2 - x_4^2]^{\frac{1}{2}} \tag{4.74}$$

Defining

$$
\begin{aligned}
g &= \sin\beta - b_{-1} \\
n &= \sqrt{1-g^2}
\end{aligned}
\tag{4.75}
$$

we can write,

$$
\mathcal{K} = -n[1 - V_1 - V_2 - V_3 - V_4]^{\frac{1}{2}} \tag{4.76}
$$

where

$$
\begin{aligned}
V_1 &= \frac{2}{n^2}(gx_2 - x_6) \\
V_2 &= \frac{1}{n^2}(x_2^2 + gb_1x_3^2 + x_4^2 - x_6^2) \\
V_3 &= \frac{1}{n^2}b_1x_2x_3^2 \\
V_4 &= \frac{1}{2n^2}(gb_3 + \frac{b_1^2}{2})x_3^4
\end{aligned}
\tag{4.77}
$$

Expanding Eq. 4.76, we get the Hamiltonian $K(\mathbf{x}, s)$ for the evolution of $\mathbf{x}$,

$$
K = K_2 + K_3 + K_4 + \cdots \tag{4.78}
$$

where

$$
\begin{aligned}
K_2 &= n(\frac{V_2}{2} + \frac{V_1^2}{8}) \\
K_3 &= n(\frac{V_3}{2} + \frac{V_1V_2}{4} + \frac{V_1^3}{16}) \\
K_4 &= n(\frac{V_4}{2} + \frac{V_1V_3}{4} + \frac{V_2^2}{8} + \frac{3V_1^2V_2}{16} + \frac{5V_1^4}{128})
\end{aligned}
\tag{4.79}
$$

Note that our notation here is different from that of the previous chapter. In terms of the normalized field variable $h(\bar{s})$, we have simply

$$
b(\bar{s}) = \frac{h(\bar{s})}{\rho}
$$

and

$$
b_{-1}(\bar{s}) = \epsilon \int_{\bar{s}_1}^{\bar{s}} h(s)ds
$$

4.3.3 Calculation of M

The matrix elements of M are found from the first equation in Eq. 4.60 and Eq. 4.61. Quadratic part of the Hamiltonian K is,

$$K_2 = \frac{1}{2n}(\frac{1}{n^2}x_2^2 + gb_1x_3^2 - \frac{2g}{n^2}x_2x_6 + x_4^2 + \frac{g}{n^2}x_6^2). \tag{4.80}$$

From Eq. 4.61 we obtain S,

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/n^3 & 0 & 0 & 0 & -g/n^3 \\ 0 & 0 & gb_1/n & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -g/n^3 & 0 & 0 & 0 & g/n^3 \end{pmatrix} \tag{4.81}$$

From the first equation in Eq. 4.60 we obtain the following equations,

$$\begin{aligned} M'_{1,i} &= \frac{1}{n^3}M_{2,i} - \frac{g}{n^3}M_{6,i} \\ M'_{2,i} &= M'_{6,i} = 0 \\ M'_{3,i} &= \frac{1}{n}M_{4,i} \\ M'_{4,i} &= -\frac{gb_1}{n}M_{3,i} \\ M'_{5,i} &= -\frac{g}{n^3}M_{2,i} + \frac{g}{n^3}M_{6,i} \end{aligned} \tag{4.82}$$

Here, we have used the prime to denote differentiation with respect to s.

Taking initial conditions into account, we can write the solution matrix,

$$M = \begin{pmatrix} 1 & M_{1,2} & 0 & 0 & 0 & M_{1,6} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_{3,3} & M_{3,4} & 0 & 0 \\ 0 & 0 & M_{4,3} & M_{4,4} & 0 & 0 \\ 0 & M_{5,2} & 0 & 0 & 1 & M_{5,6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.83}$$

Integrating the equations for $M_{i,j}$'s we have,

$$\begin{aligned} M_{1,2} &= \int_{-\infty}^{s} \frac{1}{n^3(s')}ds' \\ M_{1,6} = M_{5,2} = -M_{5,6} &= -\int_{-\infty}^{s} \frac{g(s')}{n^3(s')}ds' \end{aligned} \tag{4.84}$$

The remaining matrix elements, $M_{3,3}$, $M_{3,4}$, $M_{4,3}$, $M_{4,4}$, cannot be obtained by quadratures. From Eq. 4.82, we get the following result,

$$\begin{aligned} M_{3,4} &= \int_{-\infty}^{s} \frac{1}{n(s')} M_{4,4}(s')ds' \\ M_{4,3} &= -\int_{-\infty}^{s} \frac{g(s')b_1(s')}{n(s')} M_{3,3}(s')ds' \end{aligned} \tag{4.85}$$

We can write the following equations for $M_{3,3}$ and $M_{4,4}$,

$$\begin{aligned} M'_{3,3}(s) &= -\frac{1}{n(s)} \int_{-\infty}^{s} \frac{g(s')b_1(s')}{n(s')} M_{3,3}(s')ds' \\ M'_{4,4}(s) &= -\frac{g(s)b_1(s)}{n(s)} \int_{-\infty}^{s} \frac{1}{n(s')} M_{4,4}(s')ds' \end{aligned} \tag{4.86}$$

We can solve Eq. 4.86 by iteration:

$$\begin{aligned} M_{3,3}(s) &= 1 - \int_{-\infty}^{s} \int_{-\infty}^{s_1} \frac{1}{n(s_1)} \frac{g(s_2)b_1(s_2)}{n(s_2)} ds_2 ds_1 \\ &+ \int_{-\infty}^{s} \int_{-\infty}^{s_1} \int_{-\infty}^{s_2} \int_{-\infty}^{s_3} \frac{1}{n(s_1)} \frac{g(s_2)b_1(s_2)}{n(s_2)} \frac{1}{n(s_3)} \frac{g(s_4)b_1(s_4)}{n(s_4)} ds_4 ds_3 ds_2 ds_1 + \cdots \\ M_{4,4}(s) &= 1 - \int_{-\infty}^{s} \int_{-\infty}^{s_1} \frac{g(s_1)b_1(s_1)}{n(s_1)} \frac{1}{n(s_2)} ds_2 ds_1 \\ &+ \int_{-\infty}^{s} \int_{-\infty}^{s_1} \int_{-\infty}^{s_2} \int_{-\infty}^{s_3} \frac{g(s_1)b_1(s_1)}{n(s_1)} \frac{1}{n(s_2)} \frac{g(s_3)b_1(s_3)}{n(s_3)} \frac{1}{n(s_4)} ds_4 ds_3 ds_2 ds_1 + \cdots \end{aligned} \tag{4.87}$$

$M_{3,4}$ and $M_{4,3}$ are then given by Eq. 4.85.

It should be remarked that the iterated solutions given in Eq. 4.87 are just the power series expansions in the "field extent" parameter ϵ. Specifically, the displayed terms are $O(\epsilon^0)$, $O(\epsilon^1)$, and $O(\epsilon^2)$ respectively.

The matrix elements represent the linear map for the canonical deviation variables.

4.3.4 Calculation of Lie Polynomials f_3 and f_4

We must integrate the last two equations in Eq. 4.60 to obtain f_3 and f_4. Expanding K_3 in Eq. 4.79 and using Eq. 4.83, we can write

$$K_3^{\text{int}}(\mathbf{x}, s) = K_3(M\mathbf{x}, s) = \sum_{i=1}^{10} T_i(s)\Upsilon_i \tag{4.88}$$

where Υ_i is a homogeneous polynomial of the 3^{rd} degree in $\mathbf{x}$:

$$\vec{\Upsilon} = \begin{pmatrix} x_2^3 \\ x_2^2 x_6 \\ x_2 x_3^2 \\ x_2 x_3 x_4 \\ x_2 x_4^2 \\ x_2 x_6^2 \\ x_3^2 x_6 \\ x_3 x_4 x_6 \\ x_4^2 x_6 \\ x_6^3 \end{pmatrix} \tag{4.89}$$

Coefficients T_i are tabulated in Table 4.1. Integrating the equation for f_3 in Eq. 4.60, we obtain the following result from Eq. 4.88,

$$f_3 = -\int_{-\infty}^{s} \left(\sum_{i=1}^{10} T_i(s') \Upsilon_i \right) ds' \tag{4.90}$$

Let

$$\tau_i(s) = -\int_{-\infty}^{s} T_i(s') ds' \tag{4.91}$$

Then,

$$f_3 = \sum_{i=1}^{10} \tau_i \Upsilon_i \tag{4.92}$$

The differential equation for f_4 can be written as follows,

$$f_4' = -(K_4^{\text{int}} + \bar{K}_4) \tag{4.93}$$

where

$$\bar{K}_4 = \frac{1}{2} : f_3 : (K_3^{\text{int}}) \tag{4.94}$$

We can write,

$$K_4^{\text{int}}(\mathbf{x}, s) = \sum_{i=1}^{19} U_i(s) \Xi_i \tag{4.95}$$

and

$$\bar{K}_4(\mathbf{x}, s) = \sum_{i=1}^{19} V_i(s) \Xi_i \tag{4.96}$$

where Ξ_i is a homogeneous polynomial of the 4^{th} degree in x, which can be represented by a 19-vector,

$$\Xi = \begin{pmatrix} x_2^4 \\ x_2^3 x_6 \\ x_2^2 x_3^2 \\ x_2^2 x_3 x_4 \\ x_2^2 x_4^2 \\ x_2^2 x_6^2 \\ x_2 x_3^2 x_6 \\ x_2 x_3 x_4 x_6 \\ x_2 x_4^2 x_6 \\ x_2 x_6^3 \\ x_3^4 \\ x_3^3 x_4 \\ x_3^2 x_4^2 \\ x_3^2 x_6^2 \\ x_3 x_4^3 \\ x_3 x_4 x_6^2 \\ x_4^4 \\ x_4^2 x_6^2 \\ x_6^4 \end{pmatrix} \tag{4.97}$$

The coefficients U_i's, V_i's are tabulated in Table 4.2 and Table 4.3 respectively.

Let

$$\omega_i(s) = -\int_{-\infty}^{s} [U_i(s') + V_i(s')]\, ds' \tag{4.98}$$

Then,

$$f_4 = \sum_{i=1}^{19} \omega_i \Xi_i \tag{4.99}$$

We have succeeded in obtaining explicit formulas for the polynomials f_3 and f_4. They depend on some line integrals and the matrix elements of M. If one has measured the field $b(s)$, it is possible to obtain τ_i's and ω_i's numerically. Alternatively, we can go back to the ϵ expansion, expanding the integrals to a desired order. There is no need to decide a priori what that order should be.

Lie algebraic expansion of Eq. 4.9 can be used to describe the optics of a system without reducing it to a Taylor expansion [8]. However, we would like to obtain the TRANSPORT matrices up to the third order to compare with the results of the previous section. Our procedure would be as follows. First, we obtain the canonical matrix representation. Then, we transform to the TRANSPORT variables. Finally, the TRANSPORT matrices are obtained from Taylor expanding in the TRANSPORT variables.

Table 4.1: Coefficients in the Expansion of K_3^{int}

i	T_i
1	$g/(2n^5)$
2	$-(1+2g^2)/(2n^5)$
3	$(M_{3,3}^2 b_1 + M_{4,3}^2 g)/(2n^3)$
4	$(M_{3,3}M_{3,4}b_1 + M_{4,3}M_{4,4}g)/n^3$
5	$(M_{3,4}^2 b_1 + M_{4,4}^2 g)/(2n^3)$
6	$(g(3-n^2))/(2n^5)$
7	$-(M_{3,3}^2 b_1 g + M_{4,3}^2)/(2n^3)$
8	$-(M_{3,3}M_{3,4}b_1 g + M_{4,3}M_{4,4})/(n^3)$
9	$-(M_{3,4}^2 b_1 g + M_{4,4}^2)/(2n^3)$
10	$-g^2/(2n^5)$

Table 4.2: Coefficients in the Expansion of $K_{\underline{4}}^{\text{int}}$

i	U_i
1	$(n^2+5g^2)\,/\,(8n^7)$
2	$-(3gn^2+5g^3)\,/\,(2n^7)$
3	$\left((3M_{3,3}^2b_1g+M_{4,3}^2)n^2+3M_{3,3}^2b_1g^3+3M_{4,3}^2g^2\right)\,/\,(4n^5)$
4	$((3M_{3,3}M_{3,4}b_1g+M_{4,3}M_{4,4})n^2+3M_{3,3}M_{3,4}b_1g^3+3M_{4,3}M_{4,4}g^2)\,/\,(2n^5)$
5	$\left((3M_{3,4}^2b_1g+M_{4,4}^2)n^2+3M_{3,4}^2b_1g^3+3M_{4,4}^2g^2\right)\,/\,(4n^5)$
6	$(2n^4+15g^2)\,/\,(4n^7)$
7	$-\left(M_{3,3}^2b_1n^2+3M_{3,3}^2b_1g^2+3M_{4,3}^2g\right)\,/\,(2n^5)$
8	$-(M_{3,3}M_{3,4}b_1n^2+3M_{3,3}M_{3,4}b_1g^2+3M_{4,3}M_{4,4}g)\,/\,(n^5)$
9	$-\left(M_{3,4}^2b_1n^2+3M_{3,4}^2b_1g^2+3M_{4,4}^2g\right)\,/\,(2n^5)$
10	$(3gn^2-5g)\,/\,(2n^7)$
11	$\left((2M_{3,3}^4b_3g+2b_1^2M_{3,3}^4)n^2+M_{3,3}^4b_1^2g^2+2M_{3,3}^2M_{4,3}^2b_1g+M_{4,3}^4\right)\,/\,(8n^3)$
12	$[(2M_{3,3}^3M_{3,4}b_3g+2b_1^2M_{3,3}^3M_{3,4})n^2+M_{3,3}^3M_{3,4}b_1^2g^2$ $+(M_{3,3}^2M_{4,3}M_{4,4}+M_{3,3}M_{3,4}M_{4,3}^2)b_1g+M_{4,3}^3M_{4,4}]/(2n^3)$
13	$[(6M_{3,3}^2M_{3,4}^2b_3g+6b_1^2M_{3,3}^2M_{3,4}^2)n^2+3M_{3,3}^2M_{3,4}^2b_1^2g^2$ $+(M_{3,3}^2M_{4,4}^2+4M_{3,3}M_{3,4}M_{4,3}M_{4,4}+M_{3,4}^2M_{4,3}^2)b_1g$ $+3M_{4,3}^2M_{4,4}^2]/(4n^3)$
14	$\left(-(M_{3,3}^2b_1g+M_{4,3}^2)n^2+3M_{3,3}^2b_1g+3M_{4,3}^2\right)\,/\,(4n^5)$
15	$[(2M_{3,3}M_{3,4}^3b_3g+2b_1^2M_{3,3}M_{3,4}^3)n^2+M_{3,3}M_{3,4}^3b_1^2g^2$ $+(M_{3,3}M_{3,4}M_{4,4}^2+M_{3,4}^2M_{4,3}M_{4,4})b_1g+M_{4,3}M_{4,4}^3]/(2n^3)$
16	$(-(M_{3,3}M_{3,4}b_1g+M_{4,3}M_{4,4})n^2+3M_{3,3}M_{3,4}b_1g+3M_{4,3}M_{4,4})\,/\,(2n^5)$
17	$\left((2M_{3,4}^4b_3g+2b_1^2M_{3,4}^4)n^2+M_{3,4}^4b_1^2g^2+2M_{3,4}^2M_{4,4}^2b_1g+M_{4,4}^4\right)\,/\,(8n^3)$
18	$\left(-(M_{3,4}^2b_1g+M_{4,4}^2)n^2+3M_{3,4}^2b_1g+3M_{4,4}^2\right)\,/\,(4n^5)$
19	$(g(5-n^2))\,/\,(8n^7)$

Table 4.3: Coefficients in the Expansion of $\tilde{K}_4$

i	V_i
1	0
2	0
3	$\tau_3 T_4 - T_3 \tau_4$
4	$2\tau_3 T_5 - 2T_3 \tau_5$
5	$\tau_4 T_5 - T_4 \tau_5$
6	0
7	$\tau_3 T_8 - T_3 \tau_8 - \tau_4 T_7 + T_4 \tau_7$
8	$2\tau_3 T_9 - 2T_3 \tau_9 - 2\tau_5 T_7 + 2T_5 \tau_7$
9	$\tau_4 T_9 - T_4 \tau_9 - \tau_5 T_8 + T_5 \tau_8$
10	0
11	0
12	0
13	0
14	$\tau_7 T_8 - T_7 \tau_8$
15	0
16	$2\tau_7 T_9 - 2T_7 \tau_9$
17	0
18	$\tau_8 T_9 - T_8 \tau_9$
19	0

4.3.5 Canonical Matrix Representation

We can expand the exponentials in Eq. 4.9 to obtain the following power series,

$$\begin{aligned} x_a^f = \mathcal{M}x_a &= \ldots \exp(:f_4:)\exp(:f_3:)M_{ab}x_b \\ &= \ldots(1+:f_4:+\cdots)(1+:f_3:+\frac{1}{2}:f_3:^2+\cdots)M_{ab}x_b \\ &= M_{ab}x_b+:f_3:M_{ab}x_b+(:f_4:+\frac{1}{2}:f_3:^2)M_{ab}x_b+\cdots \end{aligned} \tag{4.100}$$

where the terms appear according to their order.

Also, we can write a formal power series expansion in the canonical variables as follows,

$$x_a^f = \mathcal{M}x_a = M_{ab}x_b + Q_{abc}x_bx_c + W_{abcd}x_bx_cx_d + \cdots \tag{4.101}$$

where we sum over the repeated indices and take $b \le c \le d$ to avoid the occurrence of the same terms in the sum; $a = 1, \ldots, 6$

We can identify the terms in Eq. 4.100 with the matrices of Eq. 4.101 according to the order:

$$\begin{aligned} :f_3:M_{ab}x_b &\longleftrightarrow Q_{abc}x_bx_c \\ (:f_4:+\frac{1}{2}:f_3:^2)M_{ab}x_b &\longleftrightarrow W_{abcd}x_bx_cx_d \end{aligned} \tag{4.102}$$

There are 72 non-zero matrix elements out of total 498 (6×83): 12 M_{ab}'s, 20 Q_{abc}'s, and 40 W_{abcd}'s. They depend on τ_i's and ω_i's. The elements of Q and W are tabulated in Table 4.4 and Table 4.5 respectively.

Table 4.4: Second Order Canonical Matrix Elements Q_{abc}

$Q_{1,2,2}$	$-3\tau_1$
$Q_{1,3,3}$	$-\tau_3$
$Q_{1,3,4}$	$-\tau_4$
$Q_{1,4,4}$	$-\tau_5$
$Q_{1,2,6}$	$-2\tau_2$
$Q_{1,6,6}$	$-\tau_6$
$Q_{3,2,3}$	$2\tau_3 M_{3,4} - M_{3,3}\tau_4$
$Q_{3,2,4}$	$M_{3,4}\tau_4 - 2M_{3,3}\tau_5$
$Q_{3,3,6}$	$2M_{3,4}\tau_7 - M_{3,3}\tau_8$
$Q_{3,4,6}$	$M_{3,4}\tau_8 - 2M_{3,3}\tau_9$
$Q_{4,2,3}$	$2\tau_3 M_{4,4} - \tau_4 M_{4,3}$
$Q_{4,2,4}$	$\tau_4 M_{4,4} - 2M_{4,3}\tau_5$
$Q_{4,3,6}$	$2M_{4,4}\tau_7 - M_{4,3}\tau_8$
$Q_{4,4,6}$	$M_{4,4}\tau_8 - 2M_{4,3}\tau_9$
$Q_{5,2,2}$	$-\tau_2$
$Q_{5,3,3}$	$-\tau_7$
$Q_{5,3,4}$	$-\tau_8$
$Q_{5,4,4}$	$-\tau_9$
$Q_{5,2,6}$	$-2\tau_6$
$Q_{5,6,6}$	$-3\tau_{10}$

Table 4.5: Third Order Canonical Matrix Elements W_{abcd}

$W_{1,2,2,2}$	$-4\omega_1$
$W_{1,2,3,3}$	$-2\omega_3$
$W_{1,2,3,4}$	$-2\omega_4$
$W_{1,2,4,4}$	$-2\omega_5$
$W_{1,2,2,6}$	$-3\omega_2$
$W_{1,3,3,6}$	$\tau_3\tau_8 - \omega_7 - \tau_4\tau_7$
$W_{1,3,4,6}$	$2\tau_3\tau_9 - \omega_8 - 2\tau_5\tau_7$
$W_{1,4,4,6}$	$-\omega_9 + \tau_4\tau_9 - \tau_5\tau_8$
$W_{1,2,6,6}$	$-2\omega_6$
$W_{1,6,6,6}$	$-\omega_{10}$
$W_{3,2,2,3}$	$-2\tau_3 M_{3,3}\tau_5 - M_{3,3}\omega_4 + M_{3,3}\tau_4^2/2 + 2\omega_3 M_{3,4}$
$W_{3,3,3,3}$	$4M_{3,4}\omega_{11} - M_{3,3}\omega_{12}$
$W_{3,2,2,4}$	$-2M_{3,3}\omega_5 - 2\tau_3 M_{3,4}\tau_5 + M_{3,4}\omega_4 + M_{3,4}\tau_4^2/2$
$W_{3,3,3,4}$	$3M_{3,4}\omega_{12} - 2M_{3,3}\omega_{13}$
$W_{3,3,4,4}$	$2M_{3,4}\omega_{13} - 3M_{3,3}\omega_{15}$
$W_{3,4,4,4}$	$M_{3,4}\omega_{15} - 4M_{3,3}\omega_{17}$
$W_{3,2,3,6}$	$-2\tau_3 M_{3,3}\tau_9 - M_{3,3}\omega_8 + M_{3,3}\tau_4\tau_8 + 2M_{3,4}\omega_7 - 2M_{3,3}\tau_5\tau_7$
$W_{3,2,4,6}$	$-2M_{3,3}\omega_9 - 2\tau_3 M_{3,4}\tau_9 + M_{3,4}\omega_8 + M_{3,4}\tau_4\tau_8 - 2M_{3,4}\tau_5\tau_7$
$W_{3,3,6,6}$	$-M_{3,3}\omega_{16} + 2M_{3,4}\omega_{14} - 2M_{3,3}\tau_7\tau_9 + M_{3,3}\tau_8^2/2$
$W_{3,4,6,6}$	$-2M_{3,3}\omega_{18} + M_{3,4}\omega_{16} - 2M_{3,4}\tau_7\tau_9 + M_{3,4}\tau_8^2/2$
$W_{4,2,2,3}$	$-2\tau_3 M_{4,3}\tau_5 + 2\omega_3 M_{4,4} - (\omega_4 - \tau_4^2)M_{4,3}/2$
$W_{4,3,3,3}$	$4M_{4,4}\omega_{11} - M_{4,3}\omega_{12}$
$W_{4,2,2,4}$	$-2M_{4,3}\omega_5 - 2\tau_3 M_{4,4}\tau_5 + (\omega_4 + \tau_4^2)M_{4,4}/2$
$W_{4,3,3,4}$	$3M_{4,4}\omega_{12} - 2M_{4,3}\omega_{13}$
$W_{4,3,4,4}$	$2M_{4,4}\omega_{13} - 3M_{4,3}\omega_{15}$
$W_{4,4,4,4}$	$M_{4,4}\omega_{15} - 4M_{4,3}\omega_{17}$
$W_{4,2,3,6}$	$-2\tau_3 M_{4,3}\tau_9 - M_{4,3}\omega_8 + \tau_4 M_{4,3}\tau_8 + 2M_{4,4}\omega_7 - 2M_{4,3}\tau_5\tau_7$
$W_{4,2,4,6}$	$-2M_{4,3}\omega_9 - 2\tau_3 M_{4,4}\tau_9 + M_{4,4}\omega_8 + \tau_4 M_{4,4}\tau_8 - 2M_{4,4}\tau_5\tau_7$
$W_{4,3,6,6}$	$-M_{4,3}\omega_{16} + 2M_{4,4}\omega_{14} - 2M_{4,3}\tau_7\tau_9 + M_{4,3}\tau_8^2/2$
$W_{4,4,6,6}$	$-2M_{4,3}\omega_{18} + M_{4,4}\omega_{16} - 2M_{4,4}\tau_7\tau_9 + M_{4,4}\tau_8^2/2$
$W_{5,2,2,2}$	$-\omega_2$
$W_{5,2,3,3}$	$-\tau_3\tau_8 - \omega_7 + \tau_4\tau_7$
$W_{5,2,3,4}$	$-2\tau_3\tau_9 - \omega_8 + 2\tau_5\tau_7$
$W_{5,2,4,4}$	$-\omega_9 - \tau_4\tau_9 + \tau_5\tau_8$
$W_{5,2,2,6}$	$-2\omega_6$
$W_{5,3,3,6}$	$-2\omega_{14}$
$W_{5,3,4,6}$	$-2\omega_{16}$
$W_{5,4,4,6}$	$-2\omega_{18}$
$W_{5,2,6,6}$	$-3\omega_{10}$
$W_{5,6,6,6}$	$-4\omega_{19}$

4.3.6 Transformation to TRANSPORT Coordinates

Let $\mathbf{X}$ denote the TRANSPORT coordinates,

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{pmatrix} = \begin{pmatrix} x - x^d \\ x' - x'^d \\ y \\ y' \\ L - L^d \\ \delta \end{pmatrix} \tag{4.103}$$

We want to work out the transformation between the canonical set (x_1, x_2, x_3, x_4) and the TRANSPORT set (X_1, X_2, X_3, X_4) at the beginning $(B = 0)$ and the end of the entrance fringe region $(B = B_0)$. Here, we ignore the 5^{th} coordinate and the 6^{th} coordinate (δ) is the same for both sets.

Angles x', y' can be related to the mechanical momentum components as follows,

$$\begin{aligned} x' &= \frac{P_x^m}{P_s^m} \\ y' &= \frac{P_y^m}{P_s^m} \end{aligned} \tag{4.104}$$

Here the superscript m denotes "mechanical",

$$P_s^m = \sqrt{P^2 - (P_x^m)^2 - (P_y^m)^2}$$

where

$$\mathbf{P}^m = \mathbf{P} - \frac{e}{c}\mathbf{A}$$

and

$$P = P_0(1 + \delta)$$

We can write in terms of the scaled variables,

$$\begin{aligned} x' &= \frac{p_x - a_x}{\sqrt{(1+\delta)^2 - (p_x - a_x)^2 - p_y^2}} \\ y' &= \frac{p_y}{\sqrt{(1+\delta)^2 - (p_x - a_x)^2 - p_y^2}} \end{aligned} \tag{4.105}$$

These expressions in Eq. 4.105 may be inverted by solving for p_x and p_y :

$$
\begin{aligned}
p_x &= a_x + \frac{(1+\delta)x'}{\sqrt{1+x'^2+y'^2}} \\
p_y &= \frac{(1+\delta)y'}{\sqrt{1+x'^2+y'^2}}
\end{aligned}
\tag{4.106}
$$

We now evaluate Eq. 4.105 and Eq. 4.106 at the final and the initial point of the entrance fringe region respectively.

1. At the final point $s = s_2$, we have

$$B = B_0$$

$$b_1 = b_3 = 0$$

and

$$a_x(s_2) = b_{-1}(s_2) = \frac{1}{\rho}\int_{-\infty}^{s_2} h(s')ds'$$

where,as before,

$$h(s) \equiv \frac{B(s)}{B_0}$$

Recalling that $p_x = x_2 + \sin\beta$ and $g = \sin\beta - b_{-1}$, we get,

$$
\begin{aligned}
X_1^f &= x_1^f \\
X_2^f &= -\frac{g}{n} + \frac{x_2^f + g}{\sqrt{(1+x_6^f)^2 - (x_2^f+g)^2 - (x_4^f)^2}} \\
X_3^f &= x_3^f \\
X_4^f &= \frac{x_4^f}{\sqrt{(1+x_6^f)^2 - (x_2^f+g)^2 - (x_4^f)^2}}
\end{aligned}
\tag{4.107}
$$

2. At the initial point $s = s_1$, we have

$$B = 0$$

$$a_x(s_1) = 0$$

Noting that $x'^d = \tan\beta$ initially, we get

$$\begin{aligned}
x_1 &= X_1 \\
x_2 &= -\sin\beta + \frac{(1+X_6)(X_2+\tan\beta)}{\sqrt{1+(X_2+\tan\beta)^2+X_4^2}} \\
x_3 &= X_3 \\
x_4 &= \frac{(1+X_6)X_4}{\sqrt{1+(X_2+\tan\beta)^2+X_4^2}}
\end{aligned} \tag{4.108}$$

We are now ready to obtain the TRANSPORT transfer matrices.

4.3.7 TRANSPORT Matrices

Given matrices M, Q, W of Eq. 4.101, we can obtain TRANSPORT matrices R, T, U of Eq. 4.63 as follows:

1. Expand Eq. 4.107 in a power series to obtain,

$$X_a^f = \mathcal{R}_{ab}^{x^f \mapsto X^f} x_b^f + \mathcal{T}_{abc}^{x^f \mapsto X^f} x_b^f x_c^f + \mathcal{U}_{abcd}^{x^f \mapsto X^f} x_b^f x_c^f x_d^f + \cdots \tag{4.109}$$

2. Substitute the power expansion of Eq. 4.101 for $\mathbf{x}^f$ in Eq. 4.109 to obtain,

$$X_a^f = \mathcal{R}_{ab}^{x \mapsto X^f} x_b + \mathcal{T}_{abc}^{x \mapsto X^f} x_b x_c + \mathcal{U}_{abcd}^{x \mapsto X^f} x_b x_c x_d + \cdots \tag{4.110}$$

3. Expand Eq. 4.108 in a power series to obtain,

$$x_a = \mathcal{R}_{ab} X_b^{X \mapsto x} + \mathcal{T}_{abc}^{X \mapsto x} X_b X_c + \mathcal{U}_{abcd}^{X \mapsto x} X_b X_c X_d + \cdots \tag{4.111}$$

and substitute Eq. 4.111 into Eq. 4.110 to obtain the desired R_{ab}, T_{abc}, U_{abcd}.

The results are given below. We have 10 non-zero elements in the first order,

$$\begin{aligned}
R_{1,1} &= 1 \\
R_{1,2} &= M_{1,2}\cos^3\beta
\end{aligned}$$

$$
\begin{aligned}
R_{1,6} &= M_{1,2}\sin\beta + M_{1,6}\\
R_{2,2} &= \cos^3\beta/n^3\\
R_{2,6} &= (\sin\beta - g)/n^3\\
R_{3,3} &= M_{3,3}\\
R_{3,4} &= M_{3,4}\cos\beta\\
R_{4,3} &= M_{4,3}/n\\
R_{4,4} &= M_{4,4}\cos\beta/n\\
R_{6,6} &= 1
\end{aligned}
$$

20 non-zero elements in the second order,

$$
\begin{aligned}
T_{1,2,2} &= -\left(3M_{1,2}\cos^4\beta\sin\beta + 6\tau_1\cos^6\beta\right)/2\\
T_{1,2,6} &= (M_{1,2} - 2\tau_2)\cos^3\beta - 6\tau_1\cos^3\beta\sin\beta\\
T_{1,3,3} &= -\tau_3\\
T_{1,3,4} &= -\tau_4\cos\beta\\
T_{1,4,4} &= -\left(M_{1,2}\cos^2\beta\sin\beta + 2\tau_5\cos^2\beta\right)/2\\
T_{1,6,6} &= -3\tau_1\sin^2\beta - 2\tau_2\sin\beta - \tau_6\\
T_{2,2,2} &= -3\cos^4\beta\left(n^2\sin\beta - g\cos^2\beta\right)/\left(2n^5\right)\\
T_{2,2,6} &= 3g\cos^3\beta\,(\sin\beta - g)/n^5\\
T_{2,3,3} &= M_{4,3}^2 g/\left(2n^3\right)\\
T_{2,3,4} &= M_{4,3}M_{4,4}g\cos\beta/n^3\\
T_{2,4,4} &= -\left(\cos^2\beta\sin\beta - M_{4,4}^2 g\cos^2\beta\right)/\left(2n^3\right)\\
T_{2,6,6} &= \left(3g\sin^2\beta - \left(2 + 4g^2\right)\sin\beta - gn^2 + 3g\right)/\left(2n^5\right)\\
T_{3,2,3} &= (2\tau_3 M_{3,4} - M_{3,3}\tau_4)\cos^3\beta\\
T_{3,2,4} &= (M_{3,4}\tau_4 - 2M_{3,3}\tau_5)\cos^4\beta - M_{3,4}\cos^2\beta\sin\beta\\
T_{3,3,6} &= (2\tau_3 M_{3,4} - M_{3,3}\tau_4)\sin\beta - M_{3,3}\tau_8 + 2M_{3,4}\tau_7
\end{aligned}
$$

$$
\begin{aligned}
T_{3,4,6} &= (M_{3,4}\tau_4 - 2M_{3,3}\tau_5)\cos\beta\sin\beta + (-2M_{3,3}\tau_9 + M_{3,4}\tau_8 + M_{3,4})\cos\beta \\
T_{4,2,3} &= \left((2\tau_3 M_{4,4} - \tau_4 M_{4,3})\,n^2 + M_{4,3}g\right)\cos^3\beta/n^3 \\
T_{4,2,4} &= -\left[M_{4,4}n^2\cos^2\beta\sin\beta + \left((2M_{4,3}\tau_5 - \tau_4 M_{4,4})\,n^2 - M_{4,4}g\right)\cos^4\beta\right]/n^3 \\
T_{4,3,6} &= \left[\left((2\tau_3 M_{4,4} - \tau_4 M_{4,3})\,n^2 + M_{4,3}g\right)\sin\beta + (2M_{4,4}\tau_7 - M_{4,3}\tau_8)\,n^2 - M_{4,3}\right]/n^3 \\
T_{4,4,6} &= -\left[\left((2M_{4,3}\tau_5 - \tau_4 M_{4,4})\,n^2 - M_{4,4}g\right)\cos\beta\sin\beta\right. \\
&\quad \left. + \left((2M_{4,3}\tau_9 - M_{4,4}\tau_8 - M_{4,4})\,n^2 + M_{4,4}\right)\cos\beta\right]/n^3
\end{aligned}
$$

and 40 non-zero elements in the third order,

$$
\begin{aligned}
U_{1,2,2,2} &= (18\tau_1\cos^7\beta\sin\beta - 8\omega_1\cos^9\beta - 5M_{1,2}\cos^7\beta + 4M_{1,2}\cos^5\beta)/2 \\
U_{1,2,2,6} &= -((24\omega_1\cos^6\beta + (3M_{1,2} - 6\tau_2)\cos^4\beta)\sin\beta + (6\omega_2 + 30\tau_1)\cos^6\beta - 18\tau_1\cos^4\beta)/2 \\
U_{1,2,3,3} &= -2\omega_3\cos^3\beta \\
U_{1,2,3,4} &= \tau_4\cos^2\beta\sin\beta - 2\omega_4\cos^4\beta \\
U_{1,2,4,4} &= ((6\tau_1\cos^5\beta + 4\tau_5\cos^3\beta)\sin\beta + (-4\omega_5 - 3M_{1,2})\cos^5\beta + 2M_{1,2}\cos^3\beta)/2 \\
U_{1,2,6,6} &= -12\omega_1\cos^3\beta\sin^2\beta + (-6\omega_2 - 6\tau_1)\cos^3\beta\sin\beta + (-2\omega_6 - 2\tau_2)\cos^3\beta \\
U_{1,3,3,6} &= -2\omega_3\sin\beta + \tau_3\tau_8 - \omega_7 - \tau_4\tau_7 \\
U_{1,3,4,6} &= (2\tau_3\tau_9 - \omega_8 - 2\tau_5\tau_7 - \tau_4)\cos\beta - 2\omega_4\cos\beta\sin\beta \\
U_{1,4,4,6} &= (6\tau_1\cos^2\beta\sin^2\beta + (-4\omega_5 + 2\tau_2 - M_{1,2})\cos^2\beta\sin\beta \\
&\quad +(-2\omega_9 + 2\tau_4\tau_9 - 2\tau_5\tau_8 - 4\tau_5)\cos^2\beta)/2 \\
U_{1,6,6,6} &= -4\omega_1\sin^3\beta - 3\omega_2\sin^2\beta - 2\omega_6\sin\beta - \omega_{10} \\
U_{2,2,2,2} &= -(9gn^2\cos^7\beta\sin\beta + (4n^2 - 5)\cos^9\beta + 5n^4\cos^7\beta - 4n^4\cos^5\beta)/(2n^7) \\
U_{2,2,2,6} &= -(((12n^2 - 15)\cos^6\beta - 9gn^2\cos^4\beta)\sin\beta \\
&\quad +(15g - 21gn^2)\cos^6\beta + 9gn^2\cos^4\beta)/(2n^7) \\
U_{2,2,3,3} &= (((4\tau_3 M_{4,3}M_{4,4} - 2\tau_4 M_{4,3}^2)g - 2M_{4,3}^2)n^2 + 3M_{4,3}^2)\cos^3\beta/(2n^5) \\
U_{2,2,3,4} &= -(M_{4,3}M_{4,4}gn^2\cos^2\beta\sin\beta \\
&\quad +(((2M_{4,3}^2\tau_5 - 2\tau_3 M_{4,4}^2)g + 2M_{4,3}M_{4,4})n^2 - 3M_{4,3}M_{4,4})\cos^4\beta)/n^5 \\
U_{2,2,4,4} &= -((3g\cos^5\beta + 2M_{4,4}^2gn^2\cos^3\beta)\sin\beta + (((4M_{4,3}M_{4,4}\tau_5 - 2\tau_4 M_{4,4}^2)g + 2M_{4,4}^2 + 3)n^2 \\
&\quad -3M_{4,4}^2)\cos^5\beta - 2n^2\cos^3\beta)/(2n^5) \\
U_{2,2,6,6} &= ((18gn^2 - 30g)\cos^3\beta\sin\beta + (12n^2 - 15)\cos^5\beta + (6n^4 - 33n^2 + 30)\cos^3\beta)/(2n^7) \\
U_{2,3,3,6} &= ((((4\tau_3 M_{4,3}M_{4,4} - 2\tau_4 M_{4,3}^2)g - 2M_{4,3}^2)n^2 + 3M_{4,3}^2)\sin\beta \\
&\quad +(4M_{4,3}M_{4,4}\tau_7 - 2M_{4,3}^2\tau_8)gn^2 - 3M_{4,3}^2g)/(2n^5) \\
U_{2,3,4,6} &= -((((2M_{4,3}^2\tau_5 - 2\tau_3 M_{4,4}^2)g + 2M_{4,3}M_{4,4})n^2 - 3M_{4,3}M_{4,4})\cos\beta\sin\beta \\
&\quad +((2M_{4,3}^2\tau_9 - 2M_{4,4}^2\tau_7 - M_{4,3}M_{4,4})gn^2 + 3M_{4,3}M_{4,4}g)\cos\beta)/n^5 \\
U_{2,4,4,6} &= -((((4M_{4,3}M_{4,4}\tau_5 - 2\tau_4 M_{4,4}^2)g + 2M_{4,4}^2 + 3)n^2 - 3M_{4,4}^2 - 3)\cos^2\beta\sin\beta - 3g\cos^4\beta
\end{aligned}
$$

$$
\begin{aligned}
& +((4M_{4,3}M_{4,4}\tau_9 - 2M_{4,4}^2\tau_8 - 2M_{4,4}^2)gn^2 + (3M_{4,4}^2 + 3)g)\cos^2\beta)/(2n^5) \\
U_{2,6,6,6} &= -((4n^2-5)\sin^3\beta + g(15-6n^2)\sin^2\beta - (2n^4+15g^2)\sin\beta - g(3n^2+5))/(2n^7) \\
U_{3,2,2,3} &= ((3M_{3,3}\tau_4 - 6\tau_3 M_{3,4})\cos^4\beta\sin\beta \\
& +(-4\tau_3 M_{3,3}\tau_5 - 2M_{3,3}\omega_4 + M_{3,3}\tau_4^2 + 4\omega_3 M_{3,4})\cos^6\beta)/2 \\
U_{3,2,2,4} &= ((10M_{3,3}\tau_5 - 5M_{3,4}\tau_4)\cos^5\beta\sin\beta + (-4M_{3,3}\omega_5 - 4\tau_3 M_{3,4}\tau_5 \\
& +2M_{3,4}\omega_4 + M_{3,4}\tau_4^2)\cos^7\beta - 3M_{3,4}\cos^5\beta + 2M_{3,4}\cos^3\beta)/2 \\
U_{3,2,3,6} &= (-4\tau_3 M_{3,3}\tau_5 - 2M_{3,3}\omega_4 + M_{3,3}\tau_4^2 + 4\omega_3 M_{3,4})\cos^3\beta\sin\beta \\
& -(2\tau_3 M_{3,3}\tau_9 + M_{3,3}\omega_8 - M_{3,3}\tau_4\tau_8 - 2M_{3,4}\omega_7 + 2M_{3,3}\tau_5\tau_7 + M_{3,3}\tau_4 - 2\tau_3 M_{3,4})\cos^3\beta \\
U_{3,2,4,6} &= ((-4M_{3,3}\omega_5 - 4\tau_3 M_{3,4}\tau_5 + 2M_{3,4}\omega_4 + M_{3,4}\tau_4^2)\cos^4\beta \\
& +(2M_{3,3}\tau_9 - M_{3,4}\tau_8 - M_{3,4})\cos^2\beta)\sin\beta + (-2M_{3,3}\omega_9 - 2\tau_3 M_{3,4}\tau_9 + M_{3,4}\omega_8 \\
& +M_{3,4}\tau_4\tau_8 - 2M_{3,4}\tau_5\tau_7 - 6M_{3,3}\tau_5 + 3M_{3,4}\tau_4)\cos^4\beta + (2M_{3,3}\tau_5 - M_{3,4}\tau_4)\cos^2\beta \\
U_{3,3,3,3} &= 4M_{3,4}\omega_{11} - M_{3,3}\omega_{12} \\
U_{3,3,3,4} &= (3M_{3,4}\omega_{12} - 2M_{3,3}\omega_{13})\cos\beta \\
U_{3,3,4,4} &= ((M_{3,3}\tau_4 - 2\tau_3 M_{3,4})\cos^2\beta\sin\beta + (4M_{3,4}\omega_{13} - 6M_{3,3}\omega_{15})\cos^2\beta)/2 \\
U_{3,3,6,6} &= -((4\tau_3 M_{3,3}\tau_5 + 2M_{3,3}\omega_4 - M_{3,3}\tau_4^2 - 4\omega_3 M_{3,4})\sin^2\beta \\
& +(4\tau_3 M_{3,3}\tau_9 + 2M_{3,3}\omega_8 - 2M_{3,3}\tau_4\tau_8 - 4M_{3,4}\omega_7 + 4M_{3,3}\tau_5\tau_7)\sin\beta \\
& +2M_{3,3}\omega_{16} - 4M_{3,4}\omega_{14} + 4M_{3,3}\tau_7\tau_9 - M_{3,3}\tau_8^2)/2 \\
U_{3,4,4,4} &= ((2M_{3,3}\tau_5 - M_{3,4}\tau_4)\cos^3\beta\sin\beta + (-8M_{3,3}\omega_{17} + 2M_{3,4}\omega_{15} - M_{3,4})\cos^3\beta)/2 \\
U_{3,4,6,6} &= -((4M_{3,3}\omega_5 + 4\tau_3 M_{3,4}\tau_5 - 2M_{3,4}\omega_4 - M_{3,4}\tau_4^2)\cos\beta\sin^2\beta \\
& +(4M_{3,3}\omega_9 + 4\tau_3 M_{3,4}\tau_9 - 2M_{3,4}\omega_8 - 2M_{3,4}\tau_4\tau_8 + 4M_{3,4}\tau_5\tau_7 \\
& +4M_{3,3}\tau_5 - 2M_{3,4}\tau_4)\cos\beta\sin\beta + (4M_{3,3}\omega_{18} - 2M_{3,4}\omega_{16} + (4M_{3,4}\tau_7 + 4M_{3,3})\tau_9 \\
& -M_{3,4}\tau_8^2 - 2M_{3,4}\tau_8)\cos\beta)/2 \\
U_{4,2,2,3} &= -(((6\tau_3 M_{4,4} - 3\tau_4 M_{4,3})n^4 + 3M_{4,3}gn^2)\cos^4\beta\sin\beta \\
& +((4\tau_3 M_{4,3}\tau_5 - 4\omega_3 M_{4,4} + (2\omega_4 - \tau_4^2)M_{4,3})n^4 + ((2\tau_4 M_{4,3} - 4\tau_3 M_{4,4})g + 2M_{4,3})n^2 \\
& -3M_{4,3})\cos^6\beta)/(2n^5) \\
U_{4,2,2,4} &= (((10M_{4,3}\tau_5 - 5\tau_4 M_{4,4})n^4 - 5M_{4,4}gn^2)\cos^5\beta\sin\beta \\
& +((-4M_{4,3}\omega_5 - 4\tau_3 M_{4,4}\tau_5 + (2\omega_4 + \tau_4^2)M_{4,4})n^4 + ((2\tau_4 M_{4,4} - 4M_{4,3}\tau_5)g - 2M_{4,4})n^2 \\
& +3M_{4,4})\cos^7\beta - 3M_{4,4}n^4\cos^5\beta + 2M_{4,4}n^4\cos^3\beta)/(2n^5) \\
U_{4,2,3,6} &= -((((4\tau_3 M_{4,3}\tau_5 - 4\omega_3 M_{4,4} + (2\omega_4 - \tau_4^2)M_{4,3})n^4 + ((2\tau_4 M_{4,3} - 4\tau_3 M_{4,4})g + 2M_{4,3})n^2 \\
& -3M_{4,3})\cos^3\beta\sin\beta + ((2\tau_3 M_{4,3}\tau_9 + M_{4,3}\omega_8 - \tau_4 M_{4,3}\tau_8 - 2M_{4,4}\omega_7 + 2M_{4,3}\tau_5\tau_7 \\
& -2\tau_3 M_{4,4} + \tau_4 M_{4,3})n^4 + ((M_{4,3}\tau_8 - 2M_{4,4}\tau_7 - M_{4,3})g \\
& +2\tau_3 M_{4,4} - \tau_4 M_{4,3})n^2 + 3M_{4,3}g)\cos^3\beta)/n^5 \\
U_{4,2,4,6} &= -((((4M_{4,3}\omega_5 + 4\tau_3 M_{4,4}\tau_5 + (-2\omega_4 - \tau_4^2)M_{4,4})n^4 + ((4M_{4,3}\tau_5 - 2\tau_4 M_{4,4})g + 2M_{4,4})n^2 \\
& -3M_{4,4})\cos^4\beta + ((-2M_{4,3}\tau_9 + M_{4,4}\tau_8 + M_{4,4})n^4 - M_{4,4}n^2)\cos^2\beta)\sin\beta \\
& +((2M_{4,3}\omega_9 + 2\tau_3 M_{4,4}\tau_9 - M_{4,4}\omega_8 - \tau_4 M_{4,4}\tau_8 + 2M_{4,4}\tau_5\tau_7 + 6M_{4,3}\tau_5 - 3\tau_4 M_{4,4})n^4
\end{aligned}
$$

$$
\begin{aligned}
& +((2M_{4,3}\tau_9 - M_{4,4}\tau_8 - 3M_{4,4})g - 2M_{4,3}\tau_5 + \tau_4 M_{4,4})n^2 + 3M_{4,4}g)\cos^4\beta \\
& +((\tau_4 M_{4,4} - 2M_{4,3}\tau_5)n^4 + M_{4,4}gn^2)\cos^2\beta)/n^5 \\
U_{4,3,3,3} &= -((2M_{4,3}\omega_{12} - 8M_{4,4}\omega_{11})n^2 - M_{4,3}^3)/(2n^3) \\
U_{4,3,3,4} &= -((4M_{4,3}\omega_{13} - 6M_{4,4}\omega_{12})n^2 - 3M_{4,3}^2 M_{4,4})\cos\beta/(2n^3) \\
U_{4,3,4,4} &= -(((2\tau_3 M_{4,4} - \tau_4 M_{4,3})n^2 + M_{4,3}g)\cos^2\beta\sin\beta \\
& +((6M_{4,3}\omega_{15} - 4M_{4,4}\omega_{13})n^2 - 3M_{4,3}M_{4,4}^2)\cos^2\beta)/(2n^3) \\
U_{4,3,6,6} &= -(((4\tau_3 M_{4,3}\tau_5 - 4\omega_3 M_{4,4} + (2\omega_4 - \tau_4^2)M_{4,3})n^4 + ((2\tau_4 M_{4,3} - 4\tau_3 M_{4,4})g + 2M_{4,3})n^2 \\
& -3M_{4,3})\sin^2\beta + ((4\tau_3 M_{4,3}\tau_9 + 2M_{4,3}\omega_8 - 2\tau_4 M_{4,3}\tau_8 - 4M_{4,4}\omega_7 + 4M_{4,3}\tau_5\tau_7)n^4 \\
& +((2M_{4,3}\tau_8 - 4M_{4,4}\tau_7)g + 4\tau_3 M_{4,4} - 2\tau_4 M_{4,3})n^2 + 6M_{4,3}g)\sin\beta \\
& +(2M_{4,3}\omega_{16} - 4M_{4,4}\omega_{14} + 4M_{4,3}\tau_7\tau_9 - M_{4,3}\tau_8^2)n^4 \\
& +(-2M_{4,3}\tau_8 + 4M_{4,4}\tau_7 + M_{4,3})n^2 - 3M_{4,3})/(2n^5) \\
U_{4,4,4,4} &= (((2M_{4,3}\tau_5 - \tau_4 M_{4,4})n^2 - M_{4,4}g)\cos^3\beta\sin\beta \\
& +((-8M_{4,3}\omega_{17} + 2M_{4,4}\omega_{15} - M_{4,4})n^2 + M_{4,4}^3)\cos^3\beta)/(2n^3) \\
U_{4,4,6,6} &= -(((4M_{4,3}\omega_5 + 4\tau_3 M_{4,4}\tau_5 + (-2\omega_4 - \tau_4^2)M_{4,4})n^4 + ((4M_{4,3}\tau_5 - 2\tau_4 M_{4,4})g + 2M_{4,4})n^2 \\
& -3M_{4,4})\cos\beta\sin^2\beta + ((4M_{4,3}\omega_9 + 4\tau_3 M_{4,4}\tau_9 - 2M_{4,4}\omega_8 - 2\tau_4 M_{4,4}\tau_8 + 4M_{4,4}\tau_5\tau_7 \\
& +4M_{4,3}\tau_5 - 2\tau_4 M_{4,4})n^4 + ((4M_{4,3}\tau_9 - 2M_{4,4}\tau_8 - 2M_{4,4})g - 4M_{4,3}\tau_5 + 2\tau_4 M_{4,4})n^2 \\
& +6M_{4,4}g)\cos\beta\sin\beta + ((4M_{4,3}\omega_{18} - 2M_{4,4}\omega_{16} + (4M_{4,4}\tau_7 + 4M_{4,3})\tau_9 - M_{4,4}\tau_8^2 \\
& -2M_{4,4}\tau_8)n^4 + (-4M_{4,3}\tau_9 + 2M_{4,4}\tau_8 + 3M_{4,4})n^2 - 3M_{4,4})\cos\beta)/(2n^5)
\end{aligned}
$$

4.4 Discussion of Results

The second and third order elements given in the previous section provide a more general result for the combined nonlinear map than those given in Chapter 3 in that they are not expanded in ϵ. To compare the two, we must expand all the integrals to $O(\epsilon)$. In the end, all the integrals appearing in the matrix elements computed in this Chapter are functions of the first-order matrix $\mathbf{M}$ which itself depends on the field line integral

$$b_{-1}(\bar{s}) = \epsilon \int_{\bar{s}_1}^{\bar{s}} h(s)ds$$

and its products with the field derivatives.

Needless to say, the expansion procedure is quite tedious and involves juggling a great deal of integrals. Although the relation between the two sets of matrix elements is not obvious, after long algebraic manipulations we do get the same answer to the right order in ϵ. We have verified all the terms through the second order and selected terms in the third order, and they indeed do agree.

Hamiltonian methods give a much more elegant approach of solving the problem, but in the end we must pay the price of dealing with a more complicated answer. Although there is some inter-relationship between the different terms, it seems to be buried in the double, triple and quadruple field integrals. If an analytic form of the field is known or can be fitted from the measurements as some function of $\bar{s}$, all the integrals can be evaluated numerically and the matrix elements can be obtained as pure functions of the rotation angle β. In fact, this is a general approach taken in [8]. A single-parameter function such as

$$h(\bar{s}) = \frac{1}{2}\left[1 + \frac{\bar{s}}{\sqrt{\bar{s}^2 + \epsilon^2}}\right]$$

can be used to model the field by fitting the "field extent" ϵ. All the matrices can then be obtained by direct integration.

Appendix A

Effect of a Curved Boundary

A.1 Introduction

It is possible for a boundary of a dipole to have some curvature in addition to being inclined with respect to the reference trajectory. Even for a geometrically straight dipole edge, the fringe field may possess an *effective* curvature due to the finite width of the magnet and the extended nature of the field. The effect of the curvature must be incorporated into the dipole fringe field optics.

Working in the sharp-cutoff approximation, one can show [13] that the optical effects due to the boundary curvature first manifest itself only in the second order. In fact, just three second-order terms are affected; they are the terms describing the dependence of the angular variables x', y' on the beam displacements x_0, y_0.

In this section, we investigate the effect of the extended field of a curved boundary on the optics up to the third order. Using the mathematical machinery of Chapter 3, we obtain the ϵ expanded matrix elements. We calculate the linear terms to $O(\epsilon^2)$ and the higher order terms to $O(\epsilon)$.

The midplane fringe field of a curved boundary is mathematically equivalent to the field of an inclined boundary with some dependence on the transverse coordinate u. We do not need to recalculate pure drift and bend transformations and must focus attention only on the fringe field region.

A.2 Equations of Motion

For the boundary with an effective curvature K, the midplane vertical component of the magnetic field may be written as follows,

$$B_y(s,u,0) = B_0\left(h + \frac{K}{2}\dot{h}u^2\right) \tag{A.1}$$

where all the quantities are as defined in Chapter 3.

We can get all the components of the full three-dimensional magnetic field from Eq. A.1 using Maxwell's equations and the midplane symmetry. We obtain the following relation,

$$\begin{aligned} B_s(s,u,y) &= \left[y - \frac{1}{6}y^3\nabla^2 + \cdots\right]\frac{\partial}{\partial s}B_y(s,u,0) \\ B_u(s,u,y) &= \left[y - \frac{1}{6}y^3\nabla^2 + \cdots\right]\frac{\partial}{\partial u}B_y(s,u,0) \\ B_y(s,u,y) &= \left[1 - \frac{1}{2}y^2\nabla^2 + \cdots\right]B_y(s,u,0) \end{aligned} \tag{A.2}$$

where

$$\nabla^2 = \left(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial u^2}\right)$$

Applying Eq. A.3 to Eq. A.1, we get the result

$$\begin{aligned} B_s(s,u,y) &= B_0\left[\frac{dh}{ds} + \frac{K}{2}\frac{d^2h}{ds^2}u^2\right]y + \cdots \\ B_u(s,u,y) &= B_0K\frac{dh}{ds}uy + \cdots \\ B_y(s,u,y) &= B_0\left[h - \frac{1}{2}\frac{d^2h}{ds^2}y^2 + \frac{K}{2}\frac{dh}{ds}(u^2 - y^2) - \frac{K}{4}\frac{d^3h}{ds^3}u^2y^2\right] + \cdots \end{aligned} \tag{A.3}$$

The equations of motion can be written as follows,

$$\begin{aligned} \ddot{u} &= -T\left[f\left(1+\dot{u}^2\right) + g\dot{u}\dot{y}\right] \\ \ddot{y} &= -T\left[g\left(1+\dot{y}^2\right) + f\dot{u}\dot{y}\right] \end{aligned} \tag{A.4}$$

Here, T is given by

$$T = \left[1 + \dot{u}^2 + \dot{y}^2\right]^{\frac{1}{2}}$$

and

$$f = f^{K=0} + \frac{K}{2\rho(1+\delta)}\left[\dot{h}(u^2 - y^2) - \frac{1}{2}\dddot{h}\, u^2y^2 - \ddot{h}u^2y\dot{y}\right]$$

$$g = g^{K=0} - \frac{K}{\rho(1+\delta)}\left[\dot{h}uy - \frac{\ddot{h}}{2}u^2\dot{u}y - \frac{y^3}{6}(\dddot{h}\, u - \ddot{h}\dot{u})\right]$$

where $f^{K=0}$ and $g^{K=0}$ are given right after Eq. 3.25 in Chapter 3.

By putting $u(s) = \Delta(s)$, $y = \dot{y} = \delta = 0$ in Eq. A.4, we can obtain the equation of motion for the reference trajectory. We can write it as follows in the dimensionless form,

$$\begin{aligned}
\frac{d}{d\bar{s}}\bar{\Delta} &= \epsilon\dot{\Delta} \\
\frac{d}{d\bar{s}}\dot{\Delta} &= (1+\dot{\Delta}^2)^{\frac{3}{2}}\left(\epsilon h + \frac{k}{2}\bar{\Delta}^2\frac{dh}{d\bar{s}}\right)
\end{aligned} \tag{A.5}$$

with the initial conditions

$$\bar{\Delta}(\bar{s}_1) = \epsilon\bar{s}_1\tan\beta \qquad \dot{\Delta}(\bar{s}_1) = \tan\beta$$

where k is the scaled curvature strength,

$$k = K\rho$$

We can solve Eq. A.5 by iteration to $O(\epsilon^2)$ to obtain the result

$$\begin{aligned}
\dot{\Delta}(\bar{s}) &= \dot{\Delta}^{K=0}(\bar{s}) - \epsilon^2\frac{k}{2}\bar{s}_1^2\sec^3\beta\tan^2\beta h(\bar{s}) \\
\bar{\Delta}(\bar{s}) &= \bar{\Delta}^{K=0}(\bar{s})
\end{aligned} \tag{A.6}$$

where $\dot{\Delta}^{K=0}$ and $\bar{\Delta}^{K=0}$ are given in Eq. 3.29 and Eq. 3.30.

Next, we would like to write the full third-order equation similar to Eq. 3.43. Denoting coordinates $\bar{w}$, $\dot{w}$, $\bar{y}$, $\dot{y}$, δ with vector components ζ_i, we can schematically represent the differential equations expanded from Eq. A.4 as follows,

$$\begin{aligned}
\zeta_1' &= \epsilon\zeta_2 \\
\zeta_2' &= \sum_j \Lambda_{2j}\zeta_j + \sum_j\sum_k \Pi_{2jk}\zeta_j\zeta_k + \sum_j\sum_k\sum_l \Phi_{2jkl}\zeta_j\zeta_k\zeta_l \\
\zeta_3' &= \epsilon\zeta_4 \\
\zeta_4' &= \sum_j \Lambda_{4j}\zeta_j + \sum_j\sum_k \Pi_{4jk}\zeta_j\zeta_k + \sum_j\sum_k\sum_l \Phi_{4jkl}\zeta_j\zeta_k\zeta_l
\end{aligned} \tag{A.7}$$

where the prime denotes $d/d\bar{s}$ and matrices Λ_{ij}, Π_{ijk}, Φ_{ijkl} all depend on $\bar{s}$ and can be split into k-dependent and k-independent parts,

$$\begin{aligned} \Lambda_{ij} &= \Lambda^0_{ij} + \lambda_{ij} \\ \Pi_{ijk} &= \Pi^0_{ijk} + \pi_{ijk} \\ \Phi_{ijkl} &= \Phi^0_{ijkl} + \phi_{ijkl} \end{aligned} \tag{A.8}$$

The superscript 0 refers to the $k = 0$ terms, which are all functions of $\dot{\Delta}^{K=0}$ and appear in Eq. 3.43.

The solution to the third order may be represented by the usual power series,

$$\zeta_i = \sum_j R_{ij}\zeta^0_j + \sum_j \sum_k T_{ijk}\zeta^0_j\zeta^0_k + \sum_j \sum_k \sum_l U_{ijkl}\zeta^0_j\zeta^0_k\zeta^0_l \tag{A.9}$$

where ζ^0_i's represent the initial conditions at $\bar{s} = \bar{s}_1$.

By putting Eq. A.9 into Eq. A.8 and collecting terms of the same initial conditions, we obtain equation for the matrix elements which can be solved using the procedure of Chapter 3. We will look for the matrix elements which are affected by the curvature term in the field expansion.

A.3 First-Order Matrix Elements

We will show that the curvature effects on the first-order elements are of $O(\epsilon^2)$. Nonzero contributions λ_{ij} are as follows,

$$\begin{aligned} \lambda_{21} &= k\Big[\epsilon \tan\beta \sec^3\beta\bar{s} \\ &\quad +\epsilon^2\left(3\tan^2\beta\sec^4\beta\bar{s}\int_{\bar{s}_1}^{\bar{s}} h(s)ds + \sec^5\beta\int_{\bar{s}_1}^{\bar{s}}\int_{\bar{s}_1}^{s} h(s')ds'ds\right)\Big]h' \\ \lambda_{22} &= -\frac{3}{2}k\epsilon^2\tan^3\beta\sec\beta\bar{s}^2h' \\ \lambda_{26} &= \frac{1}{2}k\epsilon^2\tan^2\beta\sec^3\beta\bar{s}^2h' \\ \lambda_{43} &= -k\Big[\epsilon\left(\frac{1}{2}\tan^3\beta\sec\beta\bar{s}^2h'' - \tan\beta\sec\beta\bar{s}h'\right) \\ &\quad \epsilon^2\left(\frac{1}{2}(1+\sin^2\beta)\tan^2\beta\sec^4\beta(\bar{s}_1^2hh' + \bar{s}^2h''\int_{\bar{s}_1}^{\bar{s}} h(s)ds)\right. \end{aligned}$$

$$
\begin{aligned}
& + \tan^2\beta \sec^3\beta \bar{s} h'' \int_{\bar{s}_1}^{\bar{s}} \int_{\bar{s}_1}^{s} h(s')ds'ds \\
& - \tan^2\beta \sec^2\beta \bar{s} h \int_{\bar{s}_1}^{\bar{s}} h(s)ds - \sec^3\beta h' \int_{\bar{s}_1}^{\bar{s}} \int_{\bar{s}_1}^{s} h(s')ds'ds \Big) \Big] \\
\lambda_{44} \quad = \quad & -\frac{1}{2} k\epsilon^2 \tan^3\beta \sec\beta \bar{s}^2 h'
\end{aligned}
$$

Obtaining equations for R_{ij} from Eq. A.8 and solving them by iteration, we can identify the curvature contribution. Counting the orders of ϵ at each iteration step, we get

$$
\begin{aligned}
R_{11}(\bar{s}) &= R_{11}^0(\bar{s}) + \epsilon \int_{\bar{s}_1}^{\bar{s}} \int_{\bar{s}_1}^{s} \lambda_{21}(s')ds'ds \\
R_{12}(\bar{s}) &= R_{12}^0(\bar{s}) \\
R_{21}(\bar{s}) &= R_{21}^0(\bar{s}) + \int_{\bar{s}_1}^{\bar{s}} \lambda_{21} ds \\
&\quad + \left(\int_{\bar{s}_1}^{\bar{s}} \lambda_{22}(s) \int_{\bar{s}_1}^{s} \Lambda_{21}^0(s')ds'ds + \int_{\bar{s}_1}^{\bar{s}} \Lambda_{22}^0(s) \int_{\bar{s}_1}^{s} \lambda_{21}(s')ds'ds \right) \\
&\quad + \epsilon \left(\int_{\bar{s}_1}^{\bar{s}} \lambda_{21}(s) \int_{\bar{s}_1}^{\bar{s}} \int_{\bar{s}_1}^{s'} \Lambda_{21}^0(s'')ds''ds'ds + \int_{\bar{s}_1}^{\bar{s}} \Lambda_{21}^0(s) \int_{\bar{s}_1}^{\bar{s}} \int_{\bar{s}_1}^{s'} \lambda_{21}(s'')ds''ds'ds \right) \\
R_{22}(\bar{s}) &= R_{22}^0(\bar{s}) + \int_{\bar{s}_1}^{\bar{s}} \lambda_{22}(s)ds + \epsilon \int_{\bar{s}_1}^{\bar{s}} (s - \bar{s}_1)\lambda_{21}(s)ds \\
R_{16}(\bar{s}) &= R_{16}^0(\bar{s}) \\
R_{26}(\bar{s}) &= R_{26}^0(\bar{s}) + \int_{\bar{s}_1}^{\bar{s}} \lambda_{26}(s)ds
\end{aligned}
$$

Here, as above, superscript 0 refers to the $k = 0$ elements. By putting $1 \mapsto 3$ and $2 \mapsto 4$ in the first four of the above expressions, we obtain the curvature contributions to the terms R_{33}, R_{34}, R_{43}, and R_{44}.

Performing the integrations, we get the "curvature corrections" at the fringe region's endpoint $\bar{s} = \bar{s}_1$. Then, we combine them with the linear elements describing transformations through the drift and the bend. Let us define

$$
r_{ij} \equiv R_{ij} - R_{ij}^0
$$

as the change in matrix elements due to the curvature. Then the nonzero terms to $O(\epsilon^2)$ are

$$
r_{11} \quad = \quad -4k\epsilon^2 \frac{\sin\beta}{\beta^4} I_1
$$

$$
\begin{aligned}
r_{21} &= -k\epsilon^2 \frac{1}{\cos^5\beta}\left[(1+\sin^2\beta)I_1 - 3\sin^2\beta(I_3+I_5)\right] \\
r_{22} &= 2k\epsilon^2 \frac{\sin\beta}{\cos^4\beta}\left[\sin^2\beta I_1 + 2(I_3+I_4)\right] \\
r_{26} &= -2k\epsilon^2 \frac{\sin^2\beta}{\cos^3\beta} I_1 \\
r_{33} &= 6k\epsilon^2 \frac{\sin^3\beta}{\cos^4\beta} I_1 \\
r_{43} &= -k\epsilon^2 \frac{1}{\cos^5\beta}\left[\sin\beta(2+\sin^2\beta)I_1 - 3\sin^2\beta(I_3+I_4)\right] \\
r_{44} &= k\epsilon^2 \frac{\sin\beta}{\cos^2\beta} I_1
\end{aligned}
$$

where integrals I_n are given in Eq. 3.41.

A.4 Nonlinear Matrix Elements

We notice from Eq. A.6 that to the first order in ϵ the reference trajectory remains unaffected. In particular, this fact means that Γ_n's introduced in Chapter 3 and appearing in the equation of motion are unchanged to $O(\epsilon)$. The second-order nonzero "curvature corrections" of Eq. A.9 can be written as follows,

$$
\begin{aligned}
\pi_{211} &= k\frac{dh}{d\bar{s}}\frac{\Gamma_1^3}{2} \\
\pi_{212} &= -3k\frac{dh}{d\bar{s}}\bar{\Delta}\Gamma_1^2\Gamma_2 \\
\pi_{216} &= k\frac{dh}{d\bar{s}}\bar{\Delta}\Gamma_1^3 \\
\pi_{222} &= -\frac{3}{4}k\frac{dh}{d\bar{s}}\bar{\Delta}^2\Gamma_3 \\
\pi_{226} &= \frac{3}{2}k\frac{dh}{d\bar{s}}\bar{\Delta}^2\Gamma_1^2\Gamma_2 \\
\pi_{233} &= k\frac{dh}{d\bar{s}}\frac{\Gamma_1^3}{2} + k\frac{d^3h}{\epsilon^2 d\bar{s}^3}\bar{\Delta}^2\frac{\Gamma_1^3}{4} \\
\pi_{234} &= k\frac{dh}{d\bar{s}}\bar{\Delta}\Gamma_1^2\Gamma_2 + \frac{k}{2}\frac{d^2h}{\epsilon d\bar{s}^2}\bar{\Delta}^2\Gamma_1 \\
\pi_{244} &= -k\frac{dh}{d\bar{s}}\bar{\Delta}^2\frac{\Gamma_1}{4}
\end{aligned}
$$

$$
\begin{aligned}
\pi_{266} &= -k\frac{dh}{d\bar{s}}\bar{\Delta}^2\frac{\Gamma_1^3}{2} \\
\pi_{413} &= k\frac{dh}{d\bar{s}}\Gamma_1^2 - k\frac{d^2h}{\epsilon d\bar{s}^2}\bar{\Delta}\Gamma_1^2\Gamma_2 \\
\pi_{414} &= -k\frac{dh}{d\bar{s}}\bar{\Delta}\Gamma_1^2\Gamma_2 \\
\pi_{423} &= k\frac{dh}{d\bar{s}}\bar{\Delta}\Gamma_2 - k\frac{d^2h}{\epsilon d\bar{s}^2}\bar{\Delta}^2(\Gamma_1\Gamma_2^2 + \frac{\Gamma_1^{-1}}{2}) \\
\pi_{424} &= -k\frac{dh}{d\bar{s}}\bar{\Delta}^2\frac{\Gamma_3}{2} \\
\pi_{436} &= -k\frac{dh}{d\bar{s}}\bar{\Delta}\Gamma_1 + k\frac{d^2h}{\epsilon d\bar{s}^2}\frac{\bar{\Delta}^2}{2}\Gamma_1^2\Gamma_2 \\
\pi_{446} &= k\frac{dh}{d\bar{s}}\frac{\bar{\Delta}^2}{2}\Gamma_1^2\Gamma_2
\end{aligned}
$$

Similar formulas can be obtained for the third-order "curvature corrections" ϕ_{ijk}.

The elements π_{ij} and ϕ_{ijk} go into driving terms f_{ijk} for the second order and g_{ijkl} for the third order, respectively. We just obtain additional terms in Eq. 3.45 and Eq. 3.47. The ϵ-expansion methods are still valid and the order-by-order solution method can be used. Other than performing additional integrations, there is no new work involved in obtaining the nonlinear matrix elements. In the end, the k dependent terms will contain no new form factor integrals. Just like in the zero curvature case, the higher order solution requires only the knowledge of the linear matrix elements, which were obtained in the previous section.

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