Beam-Beam Interactions

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Abstract

In high energy storage-ring colliders, the nonlinear effect arising from beam-beam interactions is a major source that leads to the emittance growth, the reduction of beam life time, and limits the collider luminosity. In this paper, two models of beam-beam interactions are introduced, which are weak-strong and strong-strong beam-beam interactions. In addition, space-charge model is introduced.

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I. Weak-Strong Beam-Beam Interaction

In a weak-strong (or incoherent) beam-beam interaction model, we assume that one beam is strong and is not affected by the other beam while the other beam is weak and experiences a beam-beam force due to the strong beam during the collision. The force which the strong beam exerts on individual particles of the weak beam moving with a velocity $c\vec{\beta}$ and a charge q is given by

$$\vec{F} = q\left(\vec{E} + c\vec{\beta}\times\vec{B}\right). \tag{1}$$

Here, $\vec{E}$ and $\vec{B}$ are electromagnetic fields induced by the strong beam moving with a velocity $c\vec{\beta}_*$. The magnetic field $\vec{B}$ can be easily obtained in the rest frame as follows: $\vec{B} = \vec{\beta}_*\times\vec{E}/c$. Then, Eq. (1) becomes

$$\begin{aligned}\vec{F} &= q\left(\vec{E} + \vec{\beta}\times\left(\vec{\beta}_*\times\vec{E}\right)\right),\\ &= q\left[\left(1-\vec{\beta}\cdot\vec{\beta}_*\right)\vec{E} + \left(\vec{\beta}\cdot\vec{E}\right)\vec{\beta}_*\right].\end{aligned} \tag{2}$$

Note that the coordinates $\vec{r}$, or (x, y, z), denote the rest frame of the strong beam. If the two beams follow the different trajectories (say, the beams are separated by constant distance $\vec{L}$), the force which the weak beam will experience should be considered as follows:

$$\vec{F}\left(\vec{r}\right) = \vec{F}\left(\vec{r}+\vec{L}\right) - \vec{F}\left(\vec{L}\right). \tag{3}$$

The beam-beam kick $\Delta\vec{r'}$ is obtained from the momentum change of a test particle in the weak beam by integrating the force over the collision time:

$$\begin{aligned}\Delta\vec{r'} &= \frac{\Delta\vec{p}}{p},\\ &= \frac{1}{p}\int_{-\infty}^{\infty} dt\,\vec{F}.\end{aligned} \tag{4}$$

In this paper, we will consider a Gaussian distribution as the strong beam's charge distribution:

$$\rho\left(x,y,z\right) = \frac{N_* q_*}{\left(2\pi\right)^{3/2}\sigma_x\sigma_y\sigma_z}\exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{z^2}{2\sigma_z^2}\right), \tag{5}$$

where N is the total number of particles, q the electric charge, and σ's are the rms beam sizes along x, y , and z respectively. The electrostatic potential of Gaussian distribution bunch can be found from solution of Poisson equation as follows [see Appendix A on page 16]:

$$\Phi\left(x,y,z\right) = \frac{N_* q_*}{4\pi^{3/2}\epsilon_0}\int_0^{\infty} d\zeta\frac{-1+\exp\left(-\frac{x^2}{2\sigma_x^2+\zeta} - \frac{y^2}{2\sigma_y^2+\zeta} - \frac{z^2}{2\sigma_z^2+\zeta}\right)}{\sqrt{\left(2\sigma_x^2+\zeta\right)\left(2\sigma_y^2+\zeta\right)\left(2\sigma_z^2+\zeta\right)}}, \tag{6}$$

where the -1 term in the numerator of the integrand is added to remove the singularity at $x = y = z = 0$.

A. Infinite Longitudinal Bunch

If it is assumed that the bunch length σ_z is much larger than the transverse *rms* length σ_x and σ_y, the charge distribution of the bunch can be given by

$$\rho\left(x,y\right) = \frac{n_* q_*}{2\pi\sigma_x\sigma_y}\exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}\right), \tag{7}$$

which gives the two-dimensional potential:

$$\Phi\left(x,y\right) = \frac{n_* q_*}{4\pi\epsilon_0}\int_0^{\infty} d\zeta\frac{-1+\exp\left(-\frac{x^2}{2\sigma_x^2+\zeta} - \frac{y^2}{2\sigma_y^2+\zeta}\right)}{\sqrt{\left(2\sigma_x^2+\zeta\right)\left(2\sigma_y^2+\zeta\right)}}. \tag{8}$$

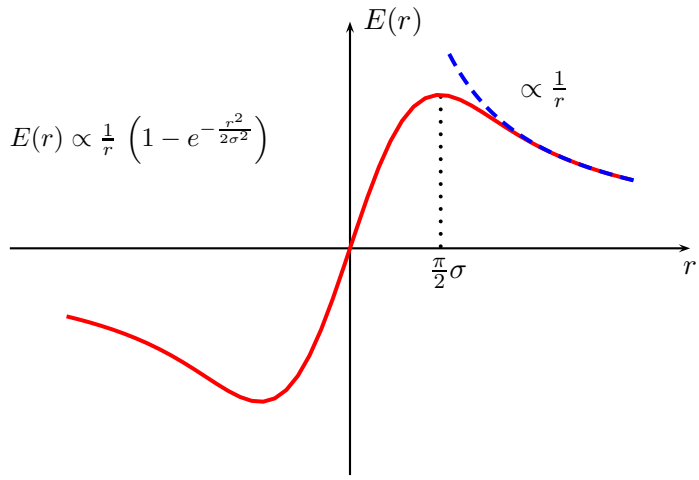

Figure 1. Electric field strength: $E\left(r\right)=\frac{n_* q_*}{2\pi\epsilon_0}\frac{1}{r}\left(1-\exp\left(-\frac{r^2}{2\sigma^2}\right)\right)$.

Note that in case of infinite longitudinal bunch, the transverse lengths σ_x and σ_y are assumed to be constant at near the interaction point, i.e., σ_x and σ_y are not function of s. The electric fields of transverse Gaussian bunch is given by:

1) Round beam with $\sigma_x=\sigma_y=\sigma$:

$$\vec{E}\left(x,y\right)=\frac{n_* q_*}{2\pi\epsilon_0}\frac{x\hat{i}+y\hat{j}}{x^2+y^2}\left(1-\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)\right). \tag{9}$$

or

$$\vec{E}\left(x,y\right)=\frac{n_* q_*}{2\pi\epsilon_0}\frac{\vec{r}}{r^2}\left(1-\exp\left(-\frac{r^2}{2\sigma^2}\right)\right).$$

2) Elliptical beam with $\sigma_x>\sigma_y$:

$$E_y+iE_x=\frac{n_* q_*}{2\epsilon_0\sqrt{2\pi\left(\sigma_x^2-\sigma_y^2\right)}}\left[w\left(\frac{x+iy}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\right)-e^{-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}}w\left(\frac{\frac{x\sigma_y}{\sigma_x}+i\frac{y\sigma_x}{\sigma_y}}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\right)\right], \tag{10}$$

where $w\left(z\right)$ is the complex error function, $w\left(z\right)=e^{-z^2}\left(1+\frac{2i}{\sqrt{\pi}}\int_0^z dt\, e^{t^2}\right)$.

Notice that variables q and q_* contain their electric charge sign.

1) Head-On Beam-Beam Interaction : If two beams are moving in opposite direction without offset and without crossing angle, they will experience head-on collisions. In case of head-on collision without crossing angle, $\vec{\beta}\cdot\vec{\beta}_*=-\left|\vec{\beta}\right|\left|\vec{\beta}_*\right|$ and $\vec{\beta}\cdot\vec{E}=0$. From Eq. (2), the Lorentz force acting on a test particle with charge q_t is given by

$$\vec{F}\left(x,y\right)=q\left(1+\left|\vec{\beta}\right|\left|\vec{\beta}_*\right|\right)\vec{E}. \tag{11}$$

Since the strong bunch is moving with velocity $c\vec{\beta}_*$ and has longitudinal charge distribution, a test particle (which is moving with $c\vec{\beta}$) will experience the force from the bunch as follows:

$$\vec{F}\left(x,y,s:t\right)=\frac{\vec{F}\left(x,y\right)}{\left(2\pi\right)^{1/2}\sigma_z}\exp\left(-\frac{\left(s+c\left(\left|\vec{\beta}\right|+\left|\vec{\beta}_*\right|\right)t\right)^2}{2\sigma_z^2}\right). \tag{12}$$

The beam-beam kick $\Delta\vec{r'}$ is obtained by Eq. (4):

$$\begin{aligned}\Delta\vec{r'} &= \frac{1}{pc\left(\left|\vec{\beta}\right|+\left|\vec{\beta}_*\right|\right)}\vec{F}\left(x,y\right),\\ &= \frac{q\left(1+\left|\vec{\beta}\right|\left|\vec{\beta}_*\right|\right)}{m_0c^2\left|\vec{\beta}\right|\gamma\left(\left|\vec{\beta}\right|+\left|\vec{\beta}_*\right|\right)}\vec{E}\left(x,y\right).\end{aligned} \tag{13}$$

Note that $\int_{-\infty}^{\infty} e^{-at^2} = \sqrt{\frac{\pi}{a}}$. Hence, we get the beam-beam kicks due to Gaussian beams:

1) Round beam with $\sigma_x = \sigma_y = \sigma$:

$$\Delta x' = \frac{2\tilde{n}r_0}{\gamma}\frac{x}{x^2+y^2}\left(1-\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)\right), \tag{14a}$$

$$\Delta y' = \frac{2\tilde{n}r_0}{\gamma}\frac{y}{x^2+y^2}\left(1-\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)\right), \tag{14b}$$

or

$$\Delta\vec{r'} = \frac{2\tilde{n}r_0}{\gamma}\frac{\vec{r}}{r^2}\left(1-\exp\left(-\frac{r^2}{2\sigma^2}\right)\right).$$

2) Elliptical beam with $\sigma_x > \sigma_y$:

$$\Delta x' = \frac{2\tilde{n}_*r_0}{\gamma}\frac{\sqrt{\pi}}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\Im W\left(x,y\right), \tag{15a}$$

$$\Delta y' = \frac{2\tilde{n}_*r_0}{\gamma}\frac{\sqrt{\pi}}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\Re W\left(x,y\right). \tag{15b}$$

where

$$W\left(x,y\right) = w\left(\frac{x+iy}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\right) - e^{-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}}w\left(\frac{\frac{x\sigma_y}{\sigma_x}+i\frac{y\sigma_x}{\sigma_y}}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\right). \tag{16}$$

3) Elliptical beam with $\sigma_x < \sigma_y$:

$$\Delta x' = \frac{2\tilde{n}_*r_0}{\gamma}\frac{\sqrt{\pi}}{\sqrt{2\left(\sigma_y^2-\sigma_x^2\right)}}\Re W\left(x,y\right), \tag{17a}$$

$$\Delta y' = \frac{2\tilde{n}_*r_0}{\gamma}\frac{\sqrt{\pi}}{\sqrt{2\left(\sigma_y^2-\sigma_y^2\right)}}\Im W\left(x,y\right). \tag{17b}$$

where

$$W\left(x,y\right) = w\left(\frac{y+ix}{\sqrt{2\left(\sigma_y^2-\sigma_x^2\right)}}\right) - e^{-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}}w\left(\frac{\frac{y\sigma_x}{\sigma_y}+i\frac{x\sigma_y}{\sigma_x}}{\sqrt{2\left(\sigma_y^2-\sigma_x^2\right)}}\right). \tag{18}$$

Here, new constants are defined as $r_0 \equiv qq_*/4\pi\epsilon_0 m_0c^2$ and $\tilde{n} \equiv n_*\left(\left|\vec{\beta}\right|^{-1}+\left|\vec{\beta}_*\right|\right)/\left(\left|\vec{\beta}\right|+\left|\vec{\beta}_*\right|\right)$. Note that for off-momentum particles with $\delta = \Delta p/p$, the factor $\tilde{n}_*$ should be replaced by

$$\tilde{n}_* = \frac{n_*}{\left|\vec{\beta}\right|\left(1+\delta\right)}\frac{1+\left|\vec{\beta}_*\right|\left|\vec{\beta}_\delta\right|}{\left|\vec{\beta}_\delta\right|+\left|\vec{\beta}_*\right|},$$

where $\left|\vec{\beta}_\delta\right| = \frac{p_t}{\sqrt{1+p_t^2}}$; $p_t = \left|\vec{\beta}\right|\left(1+\delta\right)\gamma$.

2) Long-Range Beam-Beam Interaction: The beam-beam kicks due to long-range beam-beam interaction without crossing angle can be easily obtained by Eq. (3) and the results of Sec. I-A1 on page 3, say Eqs. (14)-(15). When the separation distance between the strong and the weak beams is given by $\vec{L} = (L_x, L_y)$, the beam-beam kicks are given by [1]

$$\Delta x'_{lrbb}(x,y) = \Delta x'(x + L_x, y + L_y), \tag{19a}$$

$$\Delta y'_{lrbb}(x,y) = \Delta y'(x + L_x, y + L_y), \tag{19b}$$

where $\Delta x'$ and $\Delta y'$ are defined in Eq. (14) and Eq. (15) for round and elliptical beams respectively.

3) Linear Beam-Beam Tune Shift: To determine the effect of beam-beam interaction, we apply $\mathcal{M}_{bb}$ the transfer matrix for the beam-beam interaction int the revolution matrix $\mathcal{M}$:

$$\begin{aligned}\mathcal{M} &= \mathcal{M}_L \mathcal{M}_{bb}, \\ &= \begin{pmatrix} \cos 2\pi Q_z & \beta_z \sin 2\pi Q_z \\ \beta_z^{-1} \sin 2\pi Q_z & \cos 2\pi Q_z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}, \\ &= \begin{pmatrix} \cos 2\pi (Q_z + \xi_z) & \beta_z \sin 2\pi (Q_z + \xi_z) \\ \beta_z^{-1} \sin 2\pi (Q_z + \xi_z) & \cos 2\pi (Q_z + \xi_z) \end{pmatrix},\end{aligned} \tag{20}$$

where z is the horizontal (x) or vertical (y) plane, $2\pi Q_z$ the phase advance, β_z the amplitude function at collision point, and δ the beam-beam kick factor. δ is defined as

$$\delta = \left. \frac{d\left(\Delta r'\right)}{dr} \right|_{r=0}, \tag{21}$$

where

$$\delta = \begin{cases} \frac{\tilde{n} r_0}{\gamma \sigma^2} & , \, round\ beam \\ \frac{\tilde{n} r_0}{\gamma \sigma_z (\sigma_x + \sigma_y)/2} & , \, elliptical\ beam \end{cases}. \tag{22}$$

The trace of the map $\mathcal{M}$ yields the perturbed tune:

$$Tr(\mathcal{M}) = 2\cos 2\pi (Q_z + \xi_z) = 2\cos 2\pi Q_z - \delta\, \beta_z \sin 2\pi Q_z. \tag{23}$$

The linear beam-beam tune shift $(\xi_z \ll 1)$ in the z-plane is given by

$$\begin{aligned}\xi_z &= \frac{\beta_z \delta}{4\pi}, \\ &= \begin{cases} \frac{\tilde{n} r_0 \beta_z}{4\pi \gamma \sigma^2} & , \, round\ beam \\ \frac{\tilde{n} r_0 \beta_z}{2\pi \gamma \sigma_z (\sigma_x + \sigma_y)} & , \, elliptical\ beam \end{cases}.\end{aligned} \tag{24}$$

Note that the rms beam size for the elliptical beam is $\sigma^2 = \frac{1}{2}\sigma_{x,y}(\sigma_x + \sigma_y)$.

B. Finite Longitudinal Bunch

The bunch length effect needs to be considered in case of (1) the longitudinal bunch length σ_z is comparable to the transverse lengths σ_x and σ_y, (2) the orbit function β_x and β_y are not constant through beam-beam interactions, and (3) the transverse beta functions are small and comparable to σ_z. The synchro-beam map includes beam-beam interactions due to the longitudinal component of the electric field as well as the transverse components. The

transformation is given by [2]

$$x^{new} = x + S\left(z, z_*\right) \left.\frac{\partial U}{\partial x}\right|_S, \tag{25a}$$

$$p_x^{new} = p_x - \left.\frac{\partial U}{\partial x}\right|_S, \tag{25b}$$

$$y^{new} = y + S\left(z, z_*\right) \left.\frac{\partial U}{\partial y}\right|_S, \tag{25c}$$

$$p_y^{new} = p_y - \left.\frac{\partial U}{\partial y}\right|_S, \tag{25d}$$

$$z^{new} = z, \tag{25e}$$

$$\delta^{new} = \delta - \frac{1}{2} \left.\frac{\partial U}{\partial x}\right|_S \left[p_x - \frac{1}{2} \left.\frac{\partial U}{\partial x}\right|_S\right] - \frac{1}{2} \left.\frac{\partial U}{\partial y}\right|_S \left[p_y - \frac{1}{2} \left.\frac{\partial U}{\partial y}\right|_S\right] - \left.\frac{\partial U}{\partial z}\right|_S. \tag{25f}$$

Here, $S\left(z, z_*\right) = \frac{1}{2}\left(z - z_*\right)$ is the collision location of the test particle , the subscript * stands for the strong beam, $|_S$ represents the evaluation at collision point $S\left(z, z_*\right)$, the transverse coordinates at $S\left(z, z_*\right)$ is $(X, Y) = \left(x + p_x S\left(z, z_*\right), y + p_y S\left(z, z_*\right)\right)$, the number of particles in the strong slice $n_* = N_* \lambda_*\left(z_*\right)$, and the electric scalar potential of the strong slice U. Notice that U is the normalized potential energy, say, $U = q\Phi/E_0$. Consider the strong bunch moving with velocity $c\vec{\beta}_*$ and the test particle of the weak bunch moving with velocity $c\vec{\beta}$. If $\vec{\beta}_*$ and $\vec{\beta}$ are parallel and in opposite direction, the electrostatic potential which the test particle experiences is as follows:

$$\Phi = \frac{\left(1 + \left|\vec{\beta}\right| \left|\vec{\beta}_*\right|\right)}{\left(\left|\vec{\beta}\right| + \left|\vec{\beta}_*\right|\right)} \tilde{\Phi}, \tag{26}$$

where $\tilde{\Phi}$ is the electrostatic potential at the rest frame. The electric potential energy U is given by

$$\begin{aligned} U &= \frac{q\left(1 + \left|\vec{\beta}\right| \left|\vec{\beta}_*\right|\right)}{m_0 c^2 \gamma \left(\left|\vec{\beta}\right| + \left|\vec{\beta}_*\right|\right)} \tilde{\Phi}, \\ &= \frac{q\left(1 + \left|\vec{\beta}\right| \left|\vec{\beta}_*\right|\right)}{m_0 c^2 \gamma \left(\left|\vec{\beta}\right| + \left|\vec{\beta}_*\right|\right)} \int d\vec{r}_* \, G\left(\vec{r} - \vec{r}_*\right) \tilde{\rho}\left(\vec{r}_*; s\right). \end{aligned} \tag{27}$$

The density distribution function of the strong bunch at longitudinal distance s can be described by Gaussian distribution:

$$\tilde{\rho}\left(x, y; \sigma_x\left(s\right), \sigma_y\left(s\right)\right) = \frac{n_* q_*}{2\pi \sigma_x \sigma_y} \exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}\right), \tag{28}$$

which gives the electrostatic potential at the rest frame:

$$\tilde{\Phi}\left(x, y; \sigma_x\left(s\right), \sigma_y\left(s\right)\right) = \frac{n_* q_*}{4\pi\epsilon_0} \int_0^\infty d\zeta \frac{-1 + \exp\left(-\frac{x^2}{2\sigma_x^2 + \zeta} - \frac{y^2}{2\sigma_y^2 + \zeta}\right)}{\sqrt{\left(2\sigma_x^2 + \zeta\right)\left(2\sigma_y^2 + \zeta\right)}}. \tag{29}$$

Note that Eqs. (28)-(29) are similar to Eqs. (7)-(8) except that $\sigma_x\left(s\right)$ and $\sigma_y\left(s\right)$ are function of s.

1) Round Gaussian Distribution: In case of $\sigma_x = \sigma_y = \sigma\left(s\right)$, the potential energy is given by

$$U\left(x, y; \sigma\left(s\right)\right) = \frac{\tilde{n}_* r_0}{\gamma} \int_0^\infty d\zeta \frac{-1 + \exp\left(-\frac{x^2 + y^2}{2\sigma^2 + \zeta}\right)}{2\sigma^2 + \zeta}, \tag{30}$$

where $r_0 \equiv qq_*/4\pi\epsilon_0 m_0 c^2$ and $\tilde{n}_* \equiv n_* \left(\left|\vec{\beta}\right|^{-1} + \left|\vec{\beta}_*\right|\right) / \left(\left|\vec{\beta}\right| + \left|\vec{\beta}_*\right|\right)$. One can get easily the derivative of the potential energy U at collision point $S(z, z_*)$:

$$\left.\frac{\partial U}{\partial x}\right|_S = -\frac{2\tilde{n}_* r_0}{\gamma}\frac{X}{X^2+Y^2}\left(1-\exp\left(-\frac{X^2+Y^2}{2\sigma^2(S)}\right)\right), \tag{31a}$$

$$\left.\frac{\partial U}{\partial y}\right|_S = -\frac{2\tilde{n}_* r_0}{\gamma}\frac{Y}{X^2+Y^2}\left(1-\exp\left(-\frac{X^2+Y^2}{2\sigma^2(S)}\right)\right), \tag{31b}$$

$$\left.\frac{\partial U}{\partial z}\right|_S = -\left[\frac{1}{\sigma^2}\frac{d\sigma^2}{ds}\right]_{s=S(z,z_*)} \times \frac{\tilde{n}_* r_0}{2\gamma}\exp\left(-\frac{X^2+Y^2}{2\sigma^2(S)}\right). \tag{31c}$$

Substituting Eq. (31) into Eq. (25), one get the transformation map for round Gaussian beam.

2) Bi-Gaussian Distribution: The potential energy is given by

$$U(x, y; \sigma_x(s), \sigma_y(s)) = \frac{\tilde{n}_* r_0}{\gamma}\int_0^\infty d\zeta \frac{-1+\exp\left(-\frac{x^2}{2\sigma_x^2+\zeta} - \frac{y^2}{2\sigma_y^2+\zeta}\right)}{\sqrt{\left(2\sigma_x^2+\zeta\right)\left(2\sigma_y^2+\zeta\right)}}, \tag{32}$$

where $r_0 \equiv qq_*/4\pi\epsilon_0 m_0 c^2$ and $\tilde{n}_* \equiv n_* \left(\left|\vec{\beta}\right|^{-1} + \left|\vec{\beta}_*\right|\right) / \left(\left|\vec{\beta}\right| + \left|\vec{\beta}_*\right|\right)$. Since we get the integral of above equation in the infinite longitudinal bunch case, we can get easily the transverse derivative of the potential energy as follows:

$$\left.\frac{\partial U}{\partial x}\right|_S = -\Delta x^{'}(X, Y; S(z, z_*)), \tag{33a}$$

$$\left.\frac{\partial U}{\partial y}\right|_S = -\Delta y^{'}(X, Y; S(z, z_*)), \tag{33b}$$

where $\Delta x^{'}$ and $\Delta y^{'}$ are given by Eq. (15). The longitudinal derivative of the potential energy which is related to the longitudinal beam-beam kicks is expressed by

$$\left.\frac{\partial U}{\partial z}\right|_S = \frac{1}{2}\frac{d\sigma_x^2}{ds}\left.\frac{\partial U}{\partial \sigma_x^2}\right|_{s=S(z,z_*)} + \frac{1}{2}\frac{d\sigma_y^2}{ds}\left.\frac{\partial U}{\partial \sigma_y^2}\right|_{s=S(z,z_*)}, \tag{34}$$

where for $\sigma_x > \sigma_y$, $\frac{\partial U}{\partial \sigma_x^2}$ and $\frac{\partial U}{\partial \sigma_y^2}$ are given by [see Appendix E]

$$\frac{\partial U}{\partial \sigma_x^2} = \frac{1}{2\left(\sigma_x^2-\sigma_y^2\right)}\left[x\Delta x^{'} + y\Delta y^{'} + \frac{2\tilde{n}_* r_0}{\gamma}\left(\frac{\sigma_y}{\sigma_x}e^{-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}} - 1\right)\right], \tag{35a}$$

$$\frac{\partial U}{\partial \sigma_y^2} = \frac{-1}{2\left(\sigma_x^2-\sigma_y^2\right)}\left[x\Delta x^{'} + y\Delta y^{'} + \frac{2\tilde{n}_* r_0}{\gamma}\left(\frac{\sigma_x}{\sigma_y}e^{-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}} - 1\right)\right]. \tag{35b}$$

Substituting Eqs. (33)-(34) into Eq. (25), one get the transformation map for bi-Gaussian beam distribution. For $\sigma_x < \sigma_y$, we have

$$\frac{\partial U}{\partial \sigma_x^2} = \frac{1}{2\left(\sigma_y^2-\sigma_x^2\right)}\left[x\Delta x^{'} + y\Delta y^{'} + \frac{2\tilde{n}_* r_0}{\gamma}\left(\frac{\sigma_x}{\sigma_y}e^{-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}} - 1\right)\right], \tag{36a}$$

$$\frac{\partial U}{\partial \sigma_y^2} = \frac{-1}{2\left(\sigma_y^2-\sigma_x^2\right)}\left[x\Delta x^{'} + y\Delta y^{'} + \frac{2\tilde{n}_* r_0}{\gamma}\left(\frac{\sigma_y}{\sigma_x}e^{-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}} - 1\right)\right]. \tag{36b}$$

C. Beam-Beam interaction with a crossing angle

When there exists a finite crossing angle between colliding two beams at interaction point, the beam-beam force experienced by a test particle will have transverse and longitudinal components because $\vec{\beta}_t \cdot \vec{E}$ term is not zero anymore. The existence of longitudinal force makes it difficult to apply the result of previous sections. A transformation can be used to remedy the difficulty. It transforms a crossing angle collision in the laboratory frame to a head-on collision in the rotated and boosted frame which is called the head-on frame [3], [4]. The

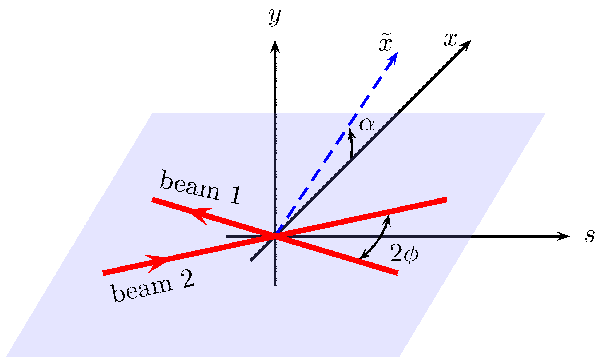

Figure 2. Definition of crossing angles α and ϕ: α is the crossing plane angle in the $x-y$ plane and ϕ is the half crossing angle in the $\tilde{x}-s$ plane.

transformation can be described by a transformation from the accelerator coordinates to Cartesian coordinates, the Lorentz transformation, and again a backward transformation to the accelerator coordinates:

$$\begin{pmatrix} z^*(s^*) \\ x^*(s^*) \\ s^* \\ y^*(s^*) \end{pmatrix} = \mathcal{A}^{-1}\mathcal{L}\mathcal{A}\begin{pmatrix} z(0) \\ x(0) \\ 0 \\ y(0) \end{pmatrix}, \tag{37a}$$

$$\begin{pmatrix} p_z^*(s^*) \\ p_x^*(s^*) \\ h^* \\ p_y^*(s^*) \end{pmatrix} = \frac{p_0}{p_0^*}\mathcal{B}^{-1}\mathcal{L}\mathcal{B}\begin{pmatrix} p_z(0) \\ p_x(0) \\ h \\ p_y(0) \end{pmatrix}, \tag{37b}$$

where $\mathcal{A}$ and $\mathcal{B}$ are the matrices which transform the accelerator coordinates to the Cartesian coordinates, and $\mathcal{L}$ is the Lorentz boost which transform the crossing angle collision to the head-on. They are defined as follows:

$$\mathcal{A} = \mathcal{A}^{-1} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathcal{B} = \mathcal{B}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathcal{L} = \begin{pmatrix} 1/\cos\phi & -\cos\alpha\sin\phi & -\tan\phi\sin\phi & -\sin\alpha\sin\phi \\ -\cos\alpha\tan\phi & 1 & \cos\alpha\tan\phi & 0 \\ 0 & -\cos\alpha\sin\phi & \cos\phi & -\sin\alpha\sin\phi \\ -\sin\alpha\tan\phi & 0 & \sin\alpha\tan\phi & 1 \end{pmatrix},$$

where α is the crossing plane angle in the $x-y$ plane and ϕ is the half crossing angle in the $\tilde{x}-z$ plane. Equation (37) becomes

$$\begin{pmatrix} z^*(s^*) \\ x^*(s^*) \\ s^* \\ y^*(s^*) \end{pmatrix} = \begin{pmatrix} 1/\cos\phi & 0 & 0 & 0 \\ \cos\alpha\tan\phi & 1 & 0 & 0 \\ 0 & -\cos\alpha\sin\phi & \cos\phi & -\sin\alpha\sin\phi \\ \sin\alpha\tan\phi & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} z(0) \\ x(0) \\ 0 \\ y(0) \end{pmatrix}, \tag{39a}$$

$$\begin{pmatrix} p_z^*(s^*) \\ p_x^*(s^*) \\ h^* \\ p_y^*(s^*) \end{pmatrix} = \begin{pmatrix} 1 & -\cos\alpha\tan\phi & \tan^2\phi & -\sin\alpha\tan\phi \\ 0 & 1/\cos\phi & -\cos\alpha\frac{\tan\phi}{\cos\phi} & 0 \\ 0 & 0 & 1/\cos^2\phi & 0 \\ 0 & 0 & -\sin\alpha\frac{\tan\phi}{\cos\phi} & 1/\cos\phi \end{pmatrix}\begin{pmatrix} p_z(0) \\ p_x(0) \\ h \\ p_y(0) \end{pmatrix}. \tag{39b}$$

Since $s=0$ is not always transformed to $s^*=0$, a transformation $\vec{x}^*(s^*) \to \vec{x}^*(0^*)$ has to be performed as follows:

$$w_i^*(0^*) = w_i^*(s^*) - h_i^* s^*, \tag{40a}$$

$$= w_i^*(s^*) + h_i^* \sin\phi\left[x(0)\cos\alpha + y(0)\sin\alpha\right], \tag{40b}$$

where $w_i = (x, y, z)$, $h_i^* = \frac{\partial h^*}{\partial p_i^*}$, and $h^* = h\left(p_x^*, p_y^*, p_z^*\right)$. From Eqs (39) and (40), we obtain a transformation $\vec{x}\left(0\right) \to \vec{x}^*\left(0^*\right)$ or $\vec{x}\left(0\right) \to \vec{x}^*\left(s*\right) \to \vec{x}^*\left(0^*\right)$:

$$\begin{aligned}
p_x^* &= \frac{p_x}{\cos\phi} - h\cos\alpha\frac{\tan\phi}{\cos\phi}, \\
p_y^* &= \frac{p_y}{\cos\phi} - h\sin\alpha\frac{\tan\phi}{\cos\phi}, \\
p_z^* &= p_z - p_x\cos\alpha\tan\phi - p_y\sin\alpha\tan\phi + h\tan^2\phi, \\
x^* &= z\cos\alpha\tan\phi + x\left[1 + h_x^*\cos\alpha\sin\phi\right] + yh_x^*\sin\alpha\sin\phi, \\
y^* &= z\sin\alpha\tan\phi + y\left[1 + h_y^*\sin\alpha\sin\phi\right] + xh_y^*\cos\alpha\sin\phi, \\
z^* &= \frac{z}{\cos\phi} + h_z^*\left[x\cos\alpha\sin\phi + y\sin\alpha\sin\phi\right],
\end{aligned} \tag{41}$$

where the Hamiltonian h is $h\left(p_x, p_y, p_z\right) = p_z + 1 - \sqrt{\left(p_z+1\right)^2 - p_x^2 - p_y^2}$. Now, we can get the inverse transform of Eq. (41), i.e., $\vec{x}^*\left(0^*\right) \to \vec{x}\left(0\right)$ or $\vec{x}^*\left(0^*\right) \to \vec{x}^*\left(s*\right) \to \vec{x}\left(0\right)$:

$$\begin{pmatrix} z\left(0\right) \\ x\left(0\right) \\ 0 \\ y\left(0\right) \end{pmatrix} = \left(\mathcal{A}^{-1}\mathcal{L}\mathcal{A}\right)^{-1} \begin{pmatrix} z^*\left(0^*\right) + h_z^* s^* \\ x^*\left(0^*\right) + h_x^* s^* \\ s^* \\ y^*\left(0^*\right) + h_y^* s^* \end{pmatrix}, \tag{42a}$$

$$\begin{pmatrix} p_z\left(0\right) \\ p_x\left(0\right) \\ h \\ p_y\left(0\right) \end{pmatrix} = \left(\mathcal{B}^{-1}\mathcal{L}\mathcal{B}\right)^{-1} \begin{pmatrix} p_z^*\left(0^*\right) \\ p_x^*\left(0^*\right) \\ h^* \\ p_y^*\left(0^*\right) \end{pmatrix}. \tag{42b}$$

Here, $\left(\mathcal{A}^{-1}\mathcal{L}\mathcal{A}\right)^{-1}$ and $\left(\mathcal{B}^{-1}\mathcal{L}\mathcal{B}\right)^{-1}$ are given by

$$\left(\mathcal{A}^{-1}\mathcal{L}\mathcal{A}\right)^{-1} = \begin{pmatrix} \cos\phi & 0 & 0 & 0 \\ -\cos\alpha\sin\phi & 1 & 0 & 0 \\ -\sin\phi\tan\phi & \cos\alpha\tan\phi & \frac{1}{\cos\phi} & \sin\alpha\tan\phi \\ -\sin\alpha\sin\phi & 0 & 0 & 1 \end{pmatrix},$$

$$\left(\mathcal{B}^{-1}\mathcal{L}\mathcal{B}\right)^{-1} = \begin{pmatrix} 1 & \cos\alpha\sin\phi & 0 & \sin\alpha\sin\phi \\ 0 & \cos\phi & \cos\alpha\sin\phi\cos\phi & 0 \\ 0 & 0 & \cos^2\phi & 0 \\ 0 & 0 & \sin\alpha\sin\phi\cos\phi & \cos\phi \end{pmatrix}.$$

Equation (42) can be rewritten by

$$\begin{aligned}
x &= x^* + h_x^* s^* - \left(z^* + h_z^* s^*\right)\cos\alpha\sin\phi, \\
y &= y^* + h_y^* s^* - \left(z^* + h_z^* s^*\right)\sin\alpha\sin\phi, \\
z &= \left(z^* + h_z^* s^*\right)\cos\phi, \\
p_x &= p_x^*\cos\phi + h^*\cos\alpha\sin\phi\cos\phi, \\
p_y &= p_y^*\cos\phi + h^*\sin\alpha\sin\phi\cos\phi, \\
p_z &= p_x^*\cos\alpha\sin\phi + p_y^*\sin\alpha\sin\phi + p_z^*.
\end{aligned} \tag{43}$$

II. Strong-Strong Beam-Beam Interaction

A. Poisson Solver

Poisson equation is

$$\nabla\phi\left(\vec{r}\right) = \frac{1}{4\pi}\rho\left(\vec{r}\right), \tag{44}$$

where ϕ is the electrostatic potential and ρ the density function. The solution can be obtained by

$$\phi\left(\vec{r}\right)=\int G\left(\vec{r},\vec{r^{'}}\right)\rho\left(\vec{r^{'}}\right)d\vec{r^{'}}, \tag{45}$$

where G is the Green's function and is given by, in two dimension and with a boundary condition of $G\left(\infty,y;x^{'},y^{'}\right)=G\left(x,\infty;x^{'},y^{'}\right)=0$,

$$G\left(x,y:x^{'},y^{'}\right)=-\frac{1}{2}\ln\left[\left(x-x^{'}\right)^{2}+\left(y-y^{'}\right)^{2}\right]. \tag{46}$$

Note that $G\left(\vec{r},\vec{r^{'}}\right)=\frac{1}{4\pi}\frac{1}{\left|\vec{r}-\vec{r^{'}}\right|}$ in three dimension and $G\left(\vec{r};\vec{r^{'}}\right)\big|_{|\vec{r}|\to\infty}=0$. Equation (45) can be efficiently calculated using convolution theorem and inverse Fourier transform:

$$\phi\left(\vec{r}\right)=\mathcal{F}^{-1}\left(\hat{G}\left(\vec{\omega}\right)\hat{\rho}\left(\vec{\omega}\right)\right), \tag{47}$$

where $\hat{G}\left(\vec{\omega}\right)=\left(\frac{1}{\sqrt{\pi}}\right)^{2}\int_{\mathbb{R}^{2}}G\left(\vec{r}\right)e^{-i\vec{\omega}\cdot\vec{r}}d\vec{r}$ and $\hat{\rho}\left(\vec{\omega}\right)=\left(\frac{1}{\sqrt{\pi}}\right)^{2}\int_{\mathbb{R}^{2}}\rho\left(\vec{r}\right)e^{-i\vec{\omega}\cdot\vec{r}}d\vec{r}$ in two dimensional space.

III. Space charge

A. Approximate model

We assume that the bunch length σ_z is much larger than the transverse beam radii σ_x and σ_y. (Locally the bunched beam is similar to a coasting beam.) A test particle is inside the bunch and experiences the field generated by the bunch and the test particle itself. The bunch is moving with a velocity $c\vec{\beta}$. The Lorentz force acting on a test particle with charge q is given by

$$\begin{aligned}\vec{F}\left(x,y\right)&=q\left(\vec{E}+c\vec{\beta}\times\vec{B}\right),\\&=q\left(\vec{E}+\vec{\beta}\times\left(\vec{\beta}\times\vec{E}\right)\right),\\&=q\left[\left(1-\vec{\beta}\cdot\vec{\beta}\right)\vec{E}+\left(\vec{\beta}\cdot\vec{E}\right)\vec{\beta}\right],\\&=q\left(1-\beta^{2}\right)\vec{E}_{\perp}+q\vec{E}_{||},\\&=\frac{1}{\gamma^{2}}q\vec{E}_{\perp}+q\vec{E}_{||}.\end{aligned} \tag{48}$$

The beam-beam kick $\Delta\vec{r^{'}}$ is obtained by Eq. (4):

$$\begin{aligned}\Delta\vec{r^{'}}&=\frac{1}{p}\int_{0}^{L/v}\vec{F}\left(x,y,z:t\right)dt\\&=\frac{L}{pv}\vec{F}\left(x,y,z\right),\\&=\frac{qL}{m_{0}c^{2}\beta^{2}\gamma^{3}}\vec{E}_{\perp}+\frac{qL}{m_{0}c^{2}\beta^{2}\gamma}\vec{E}_{||},\end{aligned} \tag{49}$$

where L is the length of accelerator ring. Note that $p=mv=m_0c\beta\gamma$. Since the bunch is moving with velocity $c\vec{\beta}$ and has longitudinal charge distribution, a test particle (which is moving with $c\vec{\beta}$) will experience the force from the bunch as follows:

$$\vec{F}\left(x,y,z:t\right)=\frac{\vec{F}\left(x,y\right)}{\left(2\pi\right)^{1/2}\sigma_{z}}\exp\left(-\frac{z^{2}}{2\sigma_{z}^{2}}\right). \tag{50}$$

The beam-beam kick $\Delta\vec{r^{'}}$ is obtained by Eq. (4):

$$\begin{aligned}\Delta\vec{r^{'}}&=\frac{1}{p}\int_{0}^{L/v}\vec{F}\left(x,y,z:t\right)dt\\&=\frac{1}{pv}\frac{Le^{-z^{2}/2\sigma_{z}^{2}}}{\left(2\pi\right)^{1/2}\sigma_{z}}\vec{F}\left(x,y\right),\\&=\frac{q}{m_{0}c^{2}\beta^{2}\gamma^{3}}\frac{Le^{-z^{2}/2\sigma_{z}^{2}}}{\left(2\pi\right)^{1/2}\sigma_{z}}\vec{E}\left(x,y\right).\end{aligned} \tag{51}$$

Note that $p = mv = m_0 c\beta\gamma$. Now we can write down the explicit formula of beam-beam kicks for transverse kicks:

1) Round beam with $\sigma_x = \sigma_y = \sigma$:

$$\begin{aligned}\Delta\vec{r'} &= \frac{nq^2}{2\pi\epsilon_0 m_0 c^2}\frac{Le^{-z^2/2\sigma_z^2}}{\beta^2\gamma^3\sqrt{2\pi}\sigma_z}\frac{\vec{r}}{r^2}\left(1-\exp\left(-\frac{r^2}{2\sigma^2}\right)\right),\\ &= \frac{2nr_0 Le^{-z^2/2\sigma_z^2}}{\beta^2\gamma^3\sqrt{2\pi}\sigma_z}\frac{\vec{r}}{r^2}\left(1-\exp\left(-\frac{r^2}{2\sigma^2}\right)\right),\\ &= K_{sc}L\frac{\vec{r}}{r^2}\left(1-\exp\left(-\frac{r^2}{2\sigma^2}\right)\right),\end{aligned}$$

where

$$K_{sc} = \frac{2nr_0 e^{-z^2/2\sigma_z^2}}{\beta^2\gamma^3\sqrt{2\pi}\sigma_z}.$$

$$\Delta x' = K_{sc}L\frac{x}{x^2+y^2}\left(1-\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)\right), \tag{52a}$$

$$\Delta y' = K_{sc}L\frac{y}{x^2+y^2}\left(1-\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)\right). \tag{52b}$$

2) Elliptical beam with $\sigma_x > \sigma_y$:

$$\Delta x' = K_{sc}L\frac{\sqrt{\pi}}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\Im W\left(x,y\right), \tag{53a}$$

$$\Delta y' = K_{sc}L\frac{\sqrt{\pi}}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\Re W\left(x,y\right). \tag{53b}$$

where

$$W\left(x,y\right) = w\left(\frac{x+iy}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\right) - e^{-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}}w\left(\frac{\frac{x\sigma_y}{\sigma_x}+i\frac{y\sigma_x}{\sigma_y}}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\right). \tag{54}$$

3) Elliptical beam with $\sigma_x < \sigma_y$:

$$\Delta x' = K_{sc}L\frac{\sqrt{\pi}}{\sqrt{2\left(\sigma_y^2-\sigma_x^2\right)}}\Re W\left(x,y\right), \tag{55a}$$

$$\Delta y' = K_{sc}L\frac{\sqrt{\pi}}{\sqrt{2\left(\sigma_y^2-\sigma_y^2\right)}}\Im W\left(x,y\right). \tag{55b}$$

where

$$W\left(x,y\right) = w\left(\frac{y+ix}{\sqrt{2\left(\sigma_y^2-\sigma_x^2\right)}}\right) - e^{-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}}w\left(\frac{\frac{y\sigma_x}{\sigma_y}+i\frac{x\sigma_y}{\sigma_x}}{\sqrt{2\left(\sigma_y^2-\sigma_x^2\right)}}\right). \tag{56}$$

Here, new constants are defined as $r_0 \equiv qq_*/4\pi\epsilon_0 m_0 c^2$.

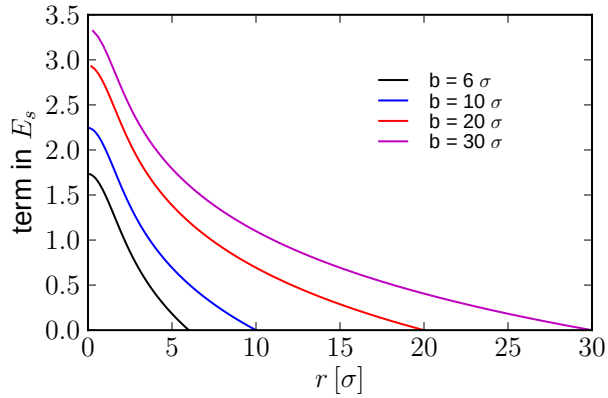

Figure 3. Plot of square bracket term in Eqs.58-59 for different beam pipe radius b.

1) Longitudinal space-charge: Assume that we have a beam with a radial distribution $n(r)$. Then for the linearity of Maxwell's equations, we can calculate its longitudinal space-charge electric field as the superposition of the contributions given by the various rings composing the actual distribution.

$$E_s(r, s-\beta ct) = -\frac{q}{2\pi\epsilon_0\gamma^2}\lambda'(s-\beta ct)\left[\ln\frac{b}{r} - \int_r^b 2\pi r_1 n(r_1)\ln\frac{r_1}{r}dr_1\right], \tag{57}$$

where b is the radius of the beam pipe. Assuming the Gaussian round beam, i.e., $n(r) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right)$, one can simplify the integral result inside square bracket as follows

$$E_s(r,z) = -\frac{q}{2\pi\epsilon_0\gamma^2}\lambda'(z)\left[\ln\frac{b}{r} - \int_r^b \frac{r_1}{\sigma^2}\exp\left(-\frac{r_1^2}{2\sigma^2}\right)\ln\frac{r_1}{r}\,dr_1\right] \tag{58}$$

$$= -\frac{q}{2\pi\epsilon_0\gamma^2}\lambda'(z)\left[\ln\frac{b}{r}\left(1+\exp\left(-\frac{b^2}{2\sigma^2}\right)\right) + \frac{1}{2}\Gamma\left(0,\frac{b^2}{2\sigma^2}\right) - \frac{1}{2}\Gamma\left(0,\frac{r^2}{2\sigma^2}\right)\right], \tag{59}$$

where $\Gamma(a,z) = \int_z^\infty t^{a-1}e^{-t}dt$ is the incomplete Gamma function. The square bracket term in Eqs. 58-59 for different beam pipe radius b is plotted in 3 Fig. .

For longitudinal space-charge kick, see [5]. The longitudinal space-charge force causes both an energy shift and an energy variation along the bunch, which depend on beam size, beam-pipe radius, and bunch population. The space-charge induced energy variation changes with location and thus may affect the chromatic correction. Because it is intensity-dependent, a limit on the bunch intensity exists beyond which the resulting spot-size increase will degrade the final-focus performance.

Let us consider a relativistic charged bunch transversing at the center of a perfectly conducting circular beam pipe which is assumed to be at zero potential. It is convenient to introduce a 'bunch rest system', by which we mean the inertial reference frame moving along the (sufficiently straight) beam line at a velocity equal to the time-averaged velocity of the bunch centroid. In the such-defined rest frame, and for typical parameters, the instantaneous velocity of the bunch centroid is non-relativistic, and thus, in this system, magnetic fields generated by the beam can be neglected. Hence, the tilde quantities will refer to the beam rest frame and the no-tilde quantities to the laboratory system. Suppose now that the bunch passes the location $s=0$ at time $t=0$. Consider a particle on the beam axis $(x,y)=(0,0)$ at a longitudinal distance $z=s-vt$ from the bunch center. Here, v is the velocity of the bunch centroid. In the beam rest frame, the electrostatic potential energy $e\tilde{\Phi}(\tilde{x},\tilde{y},\tilde{z},\tilde{s})$ of this particle is given by

$$q\tilde{\Phi}(0,0,\tilde{z},\tilde{s}) = \frac{q\tilde{\lambda}(\tilde{z})}{4\pi\epsilon_0}\left(g+\ln 2 - 2\ln\left(\frac{\sigma_x(\tilde{s})+\sigma_y(\tilde{s})}{R(\tilde{s})}\right)\right), \tag{60}$$

where $g \approx 0.577215665$ denotes Euler's constant, q the electric charge, $R(s)$ the beam-pipe radius, $\sigma_x(\tilde{s})$ and $\sigma_y(\tilde{s})$ the horizontal and vertical rms beam sizes, respectively, and $\tilde{\lambda}$ the charge line density at longitudinal position $\tilde{z}$ within the bunch.

The line density in the laboratory frame is simply $\lambda(z) = \gamma\tilde{\lambda}(\tilde{z})$. The electric potential transforms as $\Phi = \gamma\tilde{\Phi}$. Thus, in the laboratory system the electric potential at location s and position z within the bunch is

$$q\Phi(0,0,z,s) = \frac{q\lambda(z)}{4\pi\epsilon_0}\left(C + \ln 2 - 2\ln\left(\frac{\sigma_x(s)+\sigma_y(s)}{R(s)}\right)\right). \tag{61}$$

Next, the longitudinal component of the magnetic vector potential in the laboratory frame reads

$$A_s(x,y,z,s) = \beta\gamma\tilde{\Phi}(x,y,\tilde{z},\tilde{s}) = \beta\Phi(x,y,z,s), \tag{62}$$

where β is the particle velocity. Considering the position s and time t as the independent variables, the corresponding potentials are $\hat{\Phi}(x,y,s,t) \equiv \Phi(x,y,s-vt,s)$, $\hat{A}_s(x,y,s,t) \equiv A_s(x,y,s-vt,s)$. The longitudinal electric field in the laboratory frame can be expressed in terms of the derivatives of the electric potential $\Phi(x,y,z,s)$

$$E_s = -\frac{\partial\hat{\Phi}}{\partial s} - \frac{1}{c}\frac{\partial\hat{A}_s}{\partial t} = -\frac{\partial\Phi}{\partial s} - \frac{1}{\gamma^2}\frac{\partial\Phi}{\partial z}. \tag{63}$$

The last term describes the normally considered component of the space-charge force which arises from the non-uniformity of the longitudinal distribution and is very important at lower energies. The first term is the space-charge force due to changes in the transverse beam size or the beam-pipe radius which are encountered as the beam travels along the beam line. In case the first term is negligible, i.e., the transverse beam size and the beam-pipe radius do not change, the longitudinal electric field is

$$E_s = -\frac{g}{4\pi\epsilon_0\gamma^2}\frac{d\lambda}{dz}. \tag{64}$$

Hence, the longitudinal kick is given by

$$\begin{aligned}\Delta\frac{\delta p}{p} &= -\frac{qL}{m_0c^2\beta^2\gamma}\frac{g}{4\pi\epsilon_0\gamma^2}\frac{d\lambda}{dz},\\ &= -\frac{qLg}{4\pi\epsilon_0 m_0c^2\beta^2\gamma^3}\frac{d\lambda}{dz}.\end{aligned}$$

Using Gaussian charge distribution, i.e., $\lambda = \frac{qN_b}{\sqrt{2\pi}\sigma_z}\exp\left(-\frac{z^2}{2\sigma_z^2}\right)$, one can get

$$\Delta\frac{\delta p}{p} = \frac{N_b r_0 g L}{\beta^2\gamma^3}\frac{z}{\sqrt{2\pi}\sigma_z^3}\exp\left(-\frac{z^2}{2\sigma_z^2}\right).$$

B. Three dimensional model

We will consider a Gaussian distribution:

$$\rho(x,y,z) = \frac{Nq}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z}\exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{z^2}{2\sigma_z^2}\right), \tag{65}$$

where N is the total number of particles, q the electric charge, and σ's are the rms beam sizes along x, y , and z respectively. The electrostatic potential of Gaussian distribution bunch can be found from solution of Poisson equation as follows [see Appendix A on page 16]:

$$\Phi(x,y,z) = \frac{Nq}{4\pi^{3/2}\epsilon_0}\int_0^\infty d\zeta\frac{-1+\exp\left(-\frac{x^2}{2\sigma_x^2+\zeta} - \frac{y^2}{2\sigma_y^2+\zeta} - \frac{z^2}{2\sigma_z^2+\zeta}\right)}{\sqrt{\left(2\sigma_x^2+\zeta\right)\left(2\sigma_y^2+\zeta\right)\left(2\sigma_z^2+\zeta\right)}}, \tag{66}$$

where the -1 term in the numerator of the integrand is added to remove the singularity at $x = y = z = 0$. The variable z is defined as $z = s - c\beta t$ with s the longitudinal coordinate, assuming that the centroid of the bunch is

at $s=0$ at time $t=0$. The longitudinal component of the vector potential is, $A_s(x,y,z,s)=\beta\Phi(x,y,z,s)=\beta\Phi(x,y,s-c\beta t,s)$. We can calculate the electric field, [6]

$$E_x(x,y,s,t)=-\frac{\partial\Phi}{\partial x}, \tag{67a}$$

$$E_y(x,y,s,t)=-\frac{\partial\Phi}{\partial y}, \tag{67b}$$

$$E_s(x,y,s,t)=-\frac{\partial\Phi}{\partial s}-\frac{\partial\Phi}{\partial z}\frac{\partial z}{\partial s}-\frac{1}{c}\frac{\partial A_s}{\partial t}. \tag{67c}$$

Using $\frac{\partial A_s}{\partial t}=\beta(-c\beta)\frac{\partial\Phi}{\partial z}$, one gets

$$\begin{aligned}E_s(x,y,s,t)&=-\frac{\partial\Phi}{\partial s}-\frac{1}{\gamma^2}\frac{\partial\Phi}{\partial z},\\&=-\frac{d\sigma_x^2}{ds}\frac{\partial\Phi}{\partial\sigma_x^2}-\frac{d\sigma_y^2}{ds}\frac{\partial\Phi}{\partial\sigma_y^2}-\frac{d\sigma_z^2}{ds}\frac{\partial\Phi}{\partial\sigma_z^2}-\frac{1}{\gamma^2}\frac{\partial\Phi}{\partial z},\\&=e_s(x,y,z,s)+\frac{1}{\gamma^2}e_z(x,y,z,s).\end{aligned} \tag{68}$$

The Lorentz force acting on a test particle with charge q is given by

$$\begin{aligned}\vec{F}&=q\left(\vec{E}+c\vec{\beta}\times\vec{B}\right),\\&=q\left(\vec{E}+\vec{\beta}\times\left(\vec{\beta}\times\vec{E}\right)\right),\\&=q\left[\left(1-\vec{\beta}\cdot\vec{\beta}\right)\vec{E}+\left(\vec{\beta}\cdot\vec{E}\right)\vec{\beta}\right],\\&=q\left(1-\beta^2\right)\vec{E}_\perp+q\vec{E}_{||},\\&=\frac{1}{\gamma^2}q\vec{E}_\perp+q\vec{E}_{||}.\end{aligned} \tag{69}$$

1) Round Gaussian and $\sigma_\perp=\sigma_x=\sigma_y$ is comparable to σ_z

$$\begin{aligned}E_r(r,s,t)&=\frac{Nq}{2\pi^{3/2}\epsilon_0}\frac{r}{2\sigma_\perp^2}\int_0^1 dq\left(\frac{q}{2\sigma_\perp^2+2q\left(\sigma_z^2-\sigma_\perp^2\right)}\right)^{\frac{1}{2}}\exp\left(-\frac{qr^2}{2\sigma_\perp^2}-\frac{qz^2}{2\sigma_\perp^2+2q\left(\sigma_z^2-\sigma_\perp^2\right)}\right),\\
e_z(r,z,s)&=\frac{Nq}{\pi^{3/2}\epsilon_0}\frac{z}{\sqrt{2}\sigma_z}\int_0^1 dq\frac{q^2}{2\sigma_z^2+2q^2\left(\sigma_\perp^2-\sigma_z^2\right)}\exp\left(-\frac{q^2r^2}{2\sigma_z^2+2q^2\left(\sigma_\perp^2-\sigma_z^2\right)}-\frac{q^2z^2}{3\sigma_z^2}\right),\\
\frac{\partial\Phi}{\partial\sigma_\perp^2}&=\frac{Nq}{2\pi^{3/2}\epsilon_0}\frac{1}{\left(2\sigma_\perp^2\right)^2}\int_0^1 dq\frac{q^{3/2}\left(r^2-\frac{2\sigma_\perp^2}{q}\right)}{\left(2\sigma_\perp^2+2q\left(\sigma_z^2-\sigma_\perp^2\right)\right)^{1/2}}\exp\left(-\frac{qr^2}{2\sigma_\perp^2}-\frac{qz^2}{2\sigma_\perp^2+2q\left(\sigma_z^2-\sigma_\perp^2\right)}\right),\\
\frac{\partial\Phi}{\partial\sigma_z^2}&=\frac{Nq}{2\pi^{3/2}\epsilon_0}\frac{1}{\left(2\sigma_3^2\right)^{3/2}}\int_0^1 dq\frac{2z^2q^4-2\sigma_3^2q^2}{2\sigma_z^2+2q^2\left(\sigma_\perp^2-\sigma_z^2\right)}\exp\left(-r^2q^2-\frac{z^2q^2}{2\sigma_z^2}\right).,\\
e_s(r,z,s)&=-\frac{d\sigma_\perp^2}{ds}\frac{\partial\Phi}{\partial\sigma_\perp^2}-\frac{d\sigma_z^2}{ds}\frac{\partial\Phi}{\partial\sigma_z^2},\\
E_s(r,s,t)&=e_s(r,z,s)+\frac{1}{\gamma^2}e_z(r,z,s).\end{aligned}$$

Here the change of variable is applied: $q=\frac{2\sigma_\perp^2}{2\sigma_\perp^2+t}$, $q=\left(\frac{2\sigma_z^2}{2\sigma_z^2+t}\right)^{1/2}$.

$$\Delta\vec{r'}=\frac{qL}{m_0c^2\beta^2\gamma^3}\vec{E}_r,$$

$$\Delta\delta=\frac{qL}{m_0c^2\beta^2\gamma}\vec{E}_s$$

2) Round Gaussian and $\sigma_\perp = \sigma_x = \sigma_y$ is much smaller than σ_z: $2\sigma_z^2 + t \to 2\sigma_z^2$

$$\vec{E}_r = \frac{Nq}{2\pi\epsilon_0}\frac{e^{-z^2/2\sigma_z^2}}{\sqrt{2\pi}\sigma_z}\frac{\vec{r}}{r^2}\left(1-\exp\left(-\frac{r^2}{2\sigma_\perp^2}\right)\right),$$
$$e_z = \frac{Nq}{2\pi\epsilon_0}\frac{e^{-z^2/2\sigma_z^2}}{\sqrt{2\pi}\sigma_z}\frac{z}{2\sigma_z^2}\int_0^1 d\zeta\frac{e^{-\frac{r^2}{2\sigma_\perp^2}\zeta}}{\zeta},$$
$$\frac{\partial\Phi}{\partial\sigma_\perp^2} = \frac{Nq}{2\pi\epsilon_0}\frac{e^{-z^2/2\sigma_z^2}}{\sqrt{2\pi}\sigma_z}\frac{1}{2}\int_0^{\frac{1}{2\sigma_\perp^2}} d\zeta\left(r^2-\zeta\right)\zeta e^{-\zeta r^2},$$
$$\frac{\partial\Phi}{\partial\sigma_z^2} = \frac{Nq}{2\pi\epsilon_0}\frac{e^{-z^2/2\sigma_z^2}}{\sqrt{2\pi}\sigma_z}\frac{1}{2\sigma_z^2}\int_0^{\frac{1}{2\sigma_\perp^2}} d\zeta\left(z^2-\sigma_z^2\right)\frac{e^{-\zeta r^2}}{\zeta},$$

$$\Delta\vec{r'} = K_{sc}L\frac{\vec{r}}{r^2}\left(1-\exp\left(-\frac{r^2}{2\sigma_\perp^2}\right)\right),$$

where

$$K_{sc} = \frac{2Nr_0e^{-z^2/2\sigma_z^2}}{\beta^2\gamma^3\sqrt{2\pi}\sigma_z}.$$

3) Elliptical Gaussian and transverse beam size is much smaller than σ_z: $2\sigma_z^2 + t \to 2\sigma_z^2$

$$E_x = \frac{q\lambda(z)}{2\epsilon_0\sqrt{2\pi\left(\sigma_x^2-\sigma_y^2\right)}}\Im W(x,y),$$
$$E_y = \frac{q\lambda(z)}{2\epsilon_0\sqrt{2\pi\left(\sigma_x^2-\sigma_y^2\right)}}\Re W(x,y),$$
$$e_z = \frac{Nq}{4\pi\epsilon_0}2z\lambda(z)\int_0^1 dq\,\frac{\exp\left(-\frac{qx^2}{2\sigma_z^2+2q(\sigma_x^2-\sigma_z^2)}\right)\exp\left(-\frac{qy^2}{2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)}\right)}{q\sqrt{2\sigma_z^2+2q\left(\sigma_x^2-\sigma_z^2\right)}\sqrt{2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)}},$$
$$\frac{\partial\Phi}{\partial\sigma_x^2} = \frac{Nq}{4\pi\epsilon_0}\lambda(z)\int_0^1 dq\,2q\sigma_z^2\frac{\exp\left(-\frac{qx^2}{2\sigma_z^2+2q(\sigma_x^2-\sigma_z^2)}\right)\exp\left(-\frac{qy^2}{2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)}\right)}{\left(2\sigma_z^2+2q\left(\sigma_x^2-\sigma_z^2\right)\right)^{5/2}\sqrt{2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)}}$$
$$\frac{\partial\Phi}{\partial\sigma_y^2} = \frac{Nq}{4\pi\epsilon_0}\lambda(z)\int_0^1 dq\,2q\sigma_z^2\frac{\exp\left(-\frac{qx^2}{2\sigma_z^2+2q(\sigma_x^2-\sigma_z^2)}\right)\exp\left(-\frac{qy^2}{2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)}\right)}{\sqrt{2\sigma_z^2+2q\left(\sigma_x^2-\sigma_z^2\right)}\left(2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)\right)^{5/2}}$$
$$\frac{\partial\Phi}{\partial\sigma_z^2} = \frac{Nq}{4\pi\epsilon_0}\lambda(z)\int_0^1 dq\left(2\frac{qz^2}{2\sigma_z^2}-1\right)\frac{\exp\left(-\frac{qx^2}{2\sigma_z^2+2q(\sigma_x^2-\sigma_z^2)}\right)\exp\left(-\frac{qy^2}{2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)}\right)}{q^2\sqrt{2\sigma_z^2+2q\left(\sigma_x^2-\sigma_z^2\right)}\sqrt{2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)}},$$

where

$$\lambda(z) = \frac{1}{\sqrt{2\pi}\sigma_z}\exp\left(-\frac{z^2}{2\sigma_z^2}\right),$$
$$W(x,y) = w\left(\frac{x+iy}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\right) - e^{-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}}w\left(\frac{\frac{x\sigma_y}{\sigma_x}+i\frac{y\sigma_x}{\sigma_y}}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\right).$$

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APPENDIX

A. Derivation of Electrostatic Potential

The Fourier transform of Poisson equation $\nabla^2\Phi = -\frac{1}{\epsilon_0}\rho$ is given by

$$\left(k_x^2 + k_y^2 + k_z^2\right)\tilde{\Phi}\left(k_x, k_y, k_z\right) = \frac{1}{\epsilon_0}\tilde{\rho}\left(k_x, k_y, k_z\right),$$

where

$$\tilde{\Phi} = \frac{1}{(2\pi)^{3/2}}\int dxdydz\,\Phi\left(x,y,z\right)e^{i(k_x x + k_y y + k_z z)}, \quad \tilde{\rho} = \frac{1}{(2\pi)^{3/2}}\int dxdydz\,\rho\left(x,y,z\right)e^{i(k_x x + k_y y + k_z z)}.$$

Since, for a Gaussian beam distribution, $\tilde{\rho} = \frac{Nq}{(2\pi)^{3/2}}e^{-\frac{1}{2}\sigma_x^2 k_x^2 - \frac{1}{2}\sigma_y^2 k_y^2 - \frac{1}{2}\sigma_z^2 k_z^2}$, $\tilde{\Phi}$ is given by

$$\tilde{\Phi} = \frac{Nq}{(2\pi)^{3/2}\,\epsilon_0\left(k_x^2 + k_y^2 + k_z^2\right)}e^{-\frac{1}{2}\sigma_x^2 k_x^2 - \frac{1}{2}\sigma_y^2 k_y^2 - \frac{1}{2}\sigma_z^2 k_z^2}.$$

Using $\frac{1}{k_x^2+k_y^2+k_z^2} = \frac{1}{4}\int_0^\infty d\zeta\, e^{-\frac{1}{4}\zeta\left(k_x^2+k_y^2+k_z^2\right)}$, $\tilde{\Phi}$ becomes

$$\tilde{\Phi} = \frac{Nq}{(2\pi)^{3/2}\,\epsilon_0}\frac{1}{4}\int_0^\infty d\zeta\, e^{-\frac{1}{4}\left((2\sigma_x^2+\zeta)k_x^2 + \left(2\sigma_y^2+\zeta\right)k_y^2 + (2\sigma_z^2+\zeta)k_z^2\right)}.$$

The electrostatic potential Φ is obtained by inverse Fourier transform of $\tilde{\Phi}$.

$$\begin{aligned}
\Phi\left(x,y,z\right) &= \mathcal{F}^{-1}\left(\tilde{\Phi}\right), \\
&= \frac{Nq}{(2\pi)^{3/2}\,\epsilon_0}\frac{1}{4}\int_0^\infty d\zeta\,\mathcal{F}^{-1}\left(e^{-\frac{1}{4}(2\sigma_x^2+\zeta)k_x^2}\right)\mathcal{F}^{-1}\left(e^{-\frac{1}{4}\left(2\sigma_y^2+\zeta\right)k_y^2}\right)\mathcal{F}^{-1}\left(e^{-\frac{1}{4}(2\sigma_z^2+\zeta)k_z^2}\right), \\
&= \frac{Nq}{4\pi^{3/2}\epsilon_0}\int_0^\infty d\zeta\frac{-1+\exp\left(-\frac{x^2}{2\sigma_x^2+\zeta} - \frac{y^2}{2\sigma_y^2+\zeta} - \frac{z^2}{2\sigma_z^2+\zeta}\right)}{\sqrt{\left(2\sigma_x^2+\zeta\right)\left(2\sigma_y^2+\zeta\right)\left(2\sigma_z^2+\zeta\right)}},
\end{aligned}$$

where the Fourier transform $\mathcal{F}\left(e^{-\alpha x^2}\right) = \frac{1}{\sqrt{2\alpha}}e^{-\frac{k^2}{4\alpha}}$ is used, and the -1 term in the numerator of the integrand is added to remove the singularity at $x = y = z = 0$.

For one-dimensional case,

$$\Phi\left(z\right) = \frac{Nq}{4\pi^{1/2}\epsilon_0}\int_0^\infty d\zeta\frac{-1+\exp\left(-\frac{z^2}{2\sigma_z^2+\zeta}\right)}{\sqrt{\left(2\sigma_z^2+\zeta\right)}}.$$

The electric field is given by

$$
\begin{aligned}
E = -\frac{d\Phi}{dz} &= \frac{Nq}{4\sqrt{\pi}\epsilon_0}\int_0^\infty d\zeta\, 2z \frac{\exp\left(-\frac{z^2}{2\sigma_z^2+\zeta}\right)}{\left(2\sigma_z^2+\zeta\right)^{3/2}}, \\
&= \frac{Nq}{2\epsilon_0}\frac{2}{\sqrt{\pi}}\int_0^{\frac{z}{\sqrt{2}\sigma_z}} dt\, e^{-t^2}, \\
&= \frac{Nq}{2\epsilon_0} erf\left(\frac{z}{\sqrt{2}\sigma_z}\right).
\end{aligned}
$$

B. Integral of Eq. (12)

$$
\begin{aligned}
\int_{-\infty}^{\infty} dt\, \vec{F}\left(x,y,z:t\right) &= \frac{\vec{F}\left(x,y\right)}{\left(2\pi\right)^{1/2}\sigma_z}\int_{-\infty}^{\infty} dt\, \exp\left(-\frac{\left(c\left(\left|\vec{\beta}\right| \mp \left|\vec{\beta}_*\right|\right)t\right)^2}{2\sigma_z^2}\right), \\
&= \frac{\vec{F}\left(x,y\right)}{\left(2\pi\right)^{1/2}\sigma_z}\int_{-\infty}^{\infty} dz\, \exp\left(-\frac{\left(\left(\left|\vec{\beta}\right| \mp \left|\vec{\beta}_*\right|\right)z\right)^2}{2\sigma_z^2}\right), \\
&= \frac{\vec{F}\left(x,y\right)}{\left|\vec{\beta}\right| \mp \left|\vec{\beta}_*\right|}\frac{1}{\left(2\pi\right)^{1/2}\sigma_z}\int_{-\infty}^{\infty} d\zeta\, \exp\left(-\frac{\zeta^2}{2\sigma_z^2}\right), \\
&= \frac{\vec{F}\left(x,y\right)}{\left|\vec{\beta}\right| \mp \left|\vec{\beta}_*\right|}.
\end{aligned}
$$

C. Derivation of Finite Beam-Beam Map (Eq. (25)*)*

The map from $\mathbf{x}_{ip}$ to $\mathbf{x}_{ip}^{new}$ (note: the collision takes place at the collision point cp, but the map is represented at ip) is given by

$$
\begin{aligned}
\mathbf{x}_{ip}^{new} &= \exp\left(: -\hat{U}\left(\mathbf{x}\right) :\right)\mathbf{x}_{ip}, \\
&= \exp\left(: -D\left(S\right) :\right)\exp\left(: -U\left(\mathbf{X}\right) :\right)\exp\left(: D\left(S\right) :\right)\mathbf{x}_{ip},
\end{aligned}
$$

Here, $\hat{U}\left(\mathbf{x}\right)$ is the normalize potential energy defined by $\hat{U}\left(\mathbf{x}\right) \equiv q\Phi/E_0$, where q is electric charge of particle, E_0 total energy of particle, and Φ electrostatic potential at $\mathbf{x}$. A new canonical variable $\mathbf{X} = [X, P_X, Y, P_Y, Z, P_Z]$ is defined by

$$
\begin{aligned}
X &= x + p_x S\left(z, z^*\right), \qquad P_X = p_x, \\
Y &= y + p_y S\left(z, z^*\right), \qquad P_Y = p_y, \\
Z &= z, \qquad P_Z = \delta + \frac{p_x^2 + p_y^2}{4},
\end{aligned}
$$

and can be related to the original variable $\mathbf{x}$ using Lie operator:

$$
\begin{aligned}
\mathbf{X} &= \exp\left(: D\left(S\right) :\right)\mathbf{x}, \\
&= \begin{pmatrix} M_t & 0 & 0 \\ 0 & M_t & 0 \\ 0 & 0 & M_\tau \end{pmatrix}, \quad M_t = \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}, \quad M_\tau = \begin{pmatrix} 1 & 0 \\ 0 & 1 + sign\left(S\right)\left(\frac{p_x^2+p_y^2}{4}\right)^\dagger \end{pmatrix},
\end{aligned}
$$

where $D(s) = \frac{p_x^2+p_y^2}{2}s$. The Lie operator is simply defined as

$$\begin{aligned}
\exp\left(:f:\right) &= \sum_{k=0}^{\infty} \frac{:f:^k}{k!}, \\
:f:^k g &= \left[f, :f:^{k-1} g\right], \\
:f:g \equiv [f,g] &= \sum_i \left(\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x^i}\right).
\end{aligned}$$

Hence, we get

$$\exp\left(:-U\left(\mathbf{X}\right):\right)\mathbf{X} \simeq \left(I + :-U:\right)\mathbf{X} = \begin{pmatrix} X \\ P_X - \frac{\partial U}{\partial X} \\ Y \\ P_Y - \frac{\partial U}{\partial Y} \\ Z \\ P_Z - \frac{\partial U}{\partial Z} \end{pmatrix}.$$

Since the drift map is

$$\begin{aligned}
\exp\left(:D\left(-S\right):\right)\left(P_Z - \frac{\partial U}{\partial Z}\right) &\simeq \left(I + :D\left(-S\right):\right)\left(P_Z - \frac{\partial U}{\partial Z}\right) \\
&= P_Z - \frac{\partial U}{\partial Z} + \frac{\partial D}{\partial Z}\frac{\partial}{\partial P_Z}\left(P_Z - \frac{\partial U}{\partial Z}\right) \\
&\quad - \frac{\partial D}{\partial P_X}\frac{\partial}{\partial X}\left(P_Z - \frac{\partial U}{\partial Z}\right) - \frac{\partial D}{\partial P_Y}\frac{\partial}{\partial Y}\left(P_Z - \frac{\partial U}{\partial Z}\right), \\
&= P_Z - \frac{\partial U}{\partial Z} - \frac{P_X^2 + P_Y^2}{4} + P_X S\frac{\partial^2 U}{\partial X \partial Z} + P_Y S\frac{\partial^2 U}{\partial Y \partial Z},
\end{aligned}$$

we can get the map at ip:

$$\begin{aligned}
\mathbf{x}^{new} &= \exp\left(:-\hat{U}\left(\mathbf{x}\right):\right)\mathbf{x}, \\
&= \exp\left(:D\left(-S\right):\right)\exp\left(:-U\left(\mathbf{X}\right):\right)\exp\left(:D\left(S\right):\right)\mathbf{x}, \\
&= \exp\left(:D\left(-S\right):\right)\exp\left(:-U\left(\mathbf{X}\right):\right)\mathbf{X}, \\
&= \begin{pmatrix} X - S\left(P_X - \frac{\partial U}{\partial X}\right) \\ P_X - \frac{\partial U}{\partial X} \\ Y - S\left(P_Y - \frac{\partial U}{\partial Y}\right) \\ P_Y - \frac{\partial U}{\partial Y} \\ Z \\ P_Z - \frac{\partial U}{\partial Z} - \frac{p_X^2+p_Y^2}{4} \end{pmatrix}, \\
&= \begin{pmatrix} x + S\frac{\partial U}{\partial X} \\ p_x - \frac{\partial U}{\partial X} \\ y + S\frac{\partial U}{\partial Y} \\ p_y - \frac{\partial U}{\partial Y} \\ z \\ p_z - \frac{\partial U}{\partial Z} \end{pmatrix}.
\end{aligned}$$

D. Derivatives of the Potential of Round Gaussian

From Eq. (30), one can get

$$\left.\frac{\partial U}{\partial x}\right|_{s=S(z,z_*)} = -\frac{2\tilde{n}_* r_0}{\gamma} X \int_0^\infty d\zeta \frac{\exp\left(-\frac{X^2+Y^2}{2\sigma^2(S)+\zeta}\right)}{\left(2\sigma^2\left(S\right)+\zeta\right)^2}.$$

By applying change of variables $t=\frac{1}{2\sigma^2+\zeta}$, one get get

$$\begin{aligned}\left.\frac{\partial U}{\partial x}\right|_{s=S(z,z_*)} &= \frac{2\tilde{n}_* r_0}{\gamma} X \int_{\frac{1}{2\sigma^2}}^{0} dt\, e^{-(X^2+Y^2)t},\\ &= -\frac{2\tilde{n}_* r_0}{\gamma}\frac{X}{X^2+Y^2}\left(1-\exp\left(-\frac{X^2+Y^2}{2\sigma^2(S)}\right)\right),\\ \left.\frac{\partial U}{\partial y}\right|_{s=S(z,z_*)} &= -\frac{2\tilde{n}_* r_0}{\gamma}\frac{Y}{X^2+Y^2}\left(1-\exp\left(-\frac{X^2+Y^2}{2\sigma^2(S)}\right)\right).\end{aligned}$$

Finally, $\left.\frac{\partial U}{\partial z}\right|_{s=S(z,z_*)}$ can be obtained as follows:

$$\begin{aligned}\left.\frac{\partial U}{\partial z}\right|_{s=S(z,z_*)} &= \frac{1}{2}\left.\frac{d\sigma^2}{ds}\right|_{s=S}\left.\frac{\partial U}{\partial \sigma^2}\right|_{s=S},\\ &= \left.\frac{d\sigma^2}{ds}\right|_{s=S}\frac{\tilde{n}_* r_0}{2\gamma}\int_0^\infty d\zeta\, 2\exp\left(-\frac{X^2+Y^2}{2\sigma^2(S)+\zeta}\right)\left[\frac{X^2+Y^2}{(2\sigma^2(S)+\zeta)^3}-\frac{2}{(2\sigma^2(S)+\zeta)^2}\right],\\ &= -\left[\frac{1}{\sigma^2}\frac{d\sigma^2}{ds}\right]_{s=S(z,z_*)}\times\frac{\tilde{n}_* r_0}{2\gamma}\exp\left(-\frac{X^2+Y^2}{2\sigma^2(S)}\right).\end{aligned}$$

Note that the transverse coordinates follow the sign convention of the strong beam: in the head-on collisions, it does not matter, but it is important in case of long-range interactions.

E. Derivatives of the Potential of Elliptical Gaussian

From Eq. (32), we have

$$\frac{\partial U}{\partial \sigma_x^2}=\frac{1}{4\left(\sigma_x^2-\sigma_y^2\right)}\frac{\partial^2 U}{\partial a^2},\qquad \frac{\partial U}{\partial \sigma_y^2}=\frac{1}{4\left(\sigma_x^2-\sigma_y^2\right)}\frac{\partial^2 U}{\partial b^2},$$

where $a=\frac{x}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}$ and $y=\frac{y}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}$. Since $\frac{\partial U}{\partial a}=\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}\frac{\partial U}{\partial x}$, we can get

$$\frac{\partial^2 U}{\partial a^2}=-\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}\frac{\partial}{\partial a}\Delta x^{'}(a,b).$$

Here, $\Delta y^{'}(a,b)+i\Delta x^{'}(a,b)$ is given by

$$\Delta y^{'}+i\Delta x^{'}=\frac{2\tilde{n}_* r_0}{\gamma}\frac{\sqrt{\pi}}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\left[w(a+ib)-e^{-\left(\sigma_x^2-\sigma_y^2\right)\left(\frac{a^2}{\sigma_x^2}+\frac{b^2}{\sigma_y^2}\right)}w\left(\frac{\sigma_y}{\sigma_x}a+i\frac{\sigma_x}{\sigma_y}b\right)\right].$$

To proceed, we need the derivative of complex error function:

$$\begin{aligned}\frac{\partial}{\partial a}w(a+ib) &= -2\left[(a+ib)\,w-\frac{i}{\sqrt{\pi}}\right],\\ \frac{\partial}{\partial b}w(a+ib) &= -2i\left[(a+ib)\,w-\frac{i}{\sqrt{\pi}}\right],\\ \frac{\partial}{\partial a}w\left(\frac{\sigma_y}{\sigma_x}a+i\frac{\sigma_x}{\sigma_y}b\right) &= -2\left[\left(\frac{\sigma_y^2}{\sigma_x^2}a+ib\right)w-\frac{\sigma_y}{\sigma_x}\frac{i}{\sqrt{\pi}}\right],\\ \frac{\partial}{\partial b}w\left(\frac{\sigma_y}{\sigma_x}a+i\frac{\sigma_x}{\sigma_y}b\right) &= -2i\left[\left(a+i\frac{\sigma_x^2}{\sigma_y^2}b\right)w-\frac{\sigma_x}{\sigma_y}\frac{i}{\sqrt{\pi}}\right].\end{aligned}$$

The term $\frac{\partial}{\partial a}\Delta x^{'}(a,b)$ can be easily obtained by

$$\begin{aligned}\frac{\partial}{\partial a}\Delta x^{'}(a,b) &= \Im\frac{\partial}{\partial a}\left(\Delta y^{'}+i\Delta x^{'}\right),\\ &= \frac{2\tilde{n}_* r_0}{\gamma}\frac{\sqrt{\pi}}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\Im\frac{\partial}{\partial a}\left[w(a+ib)-e^{-\left(\sigma_x^2-\sigma_y^2\right)\left(\frac{a^2}{\sigma_x^2}+\frac{b^2}{\sigma_y^2}\right)}w\left(\frac{\sigma_y}{\sigma_x}a+i\frac{\sigma_x}{\sigma_y}b\right)\right],\end{aligned}$$

where

$$\frac{\partial}{\partial a} w\left(a+ib\right) = -2\left[\left(a+ib\right)w - \frac{i}{\sqrt{\pi}}\right],$$

$$\frac{\partial}{\partial a} e^{-\left(\sigma_x^2-\sigma_y^2\right)\left(\frac{a^2}{\sigma_x^2}+\frac{b^2}{\sigma_y^2}\right)} w\left(\frac{\sigma_y}{\sigma_x}a + i\frac{\sigma_x}{\sigma_y}b\right) = e^{-\left(\sigma_x^2-\sigma_y^2\right)\left(\frac{a^2}{\sigma_x^2}+\frac{b^2}{\sigma_y^2}\right)}\left[-2a\left(1-\frac{\sigma_y^2}{\sigma_x^2}\right)w - 2\left(\left(\frac{\sigma_y^2}{\sigma_x^2}a+ib\right)w - \frac{\sigma_y}{\sigma_x}\frac{i}{\sqrt{\pi}}\right)\right],$$

$$= -2\left[\left(a+ib\right)w - \frac{\sigma_y}{\sigma_x}\frac{i}{\sqrt{\pi}}\right] e^{-\left(\sigma_x^2-\sigma_y^2\right)\left(\frac{a^2}{\sigma_x^2}+\frac{b^2}{\sigma_y^2}\right)}$$

Then, we can get

$$\begin{aligned}
\frac{\partial}{\partial a}\Delta x^{'}\left(a,b\right) &= -2\frac{2\tilde{n}_* r_0}{\gamma}\frac{\sqrt{\pi}}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\Im\left[\left(a+ib\right)\left(w\left(a+ib\right) - e^{-\left(\sigma_x^2-\sigma_y^2\right)\left(\frac{a^2}{\sigma_x^2}+\frac{b^2}{\sigma_y^2}\right)} w\left(\frac{\sigma_y}{\sigma_x}a + i\frac{\sigma_x}{\sigma_y}b\right)\right)\right. \\
&\qquad \left. -\frac{i}{\sqrt{\pi}}\left(1-\frac{\sigma_y}{\sigma_x}e^{-\left(\sigma_x^2-\sigma_y^2\right)\left(\frac{a^2}{\sigma_x^2}+\frac{b^2}{\sigma_y^2}\right)}\right)\right] \\
&= -2\Im\left[\left(a+ib\right)\left(\Delta y^{'} + i\Delta x^{'}\right) + \frac{2\tilde{n}_* r_0}{\gamma}\frac{\sqrt{\pi}}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\frac{i}{\sqrt{\pi}}\left(\frac{\sigma_y}{\sigma_x}e^{-\left(\sigma_x^2-\sigma_y^2\right)\left(\frac{a^2}{\sigma_x^2}+\frac{b^2}{\sigma_y^2}\right)} - 1\right)\right], \\
&= \frac{-2}{\sqrt{2\left(\sigma_x^2-\sigma_y^2\right)}}\left[x\Delta x^{'} + y\Delta y^{'} + \frac{2\tilde{n}_* r_0}{\gamma}\left(\frac{\sigma_y}{\sigma_x}e^{-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}} - 1\right)\right].
\end{aligned}$$

Hence, we have

$$-\frac{\partial U}{\partial \sigma_x^2} = \frac{-1}{2\left(\sigma_x^2-\sigma_y^2\right)}\left[x\Delta x^{'} + y\Delta y^{'} + \frac{2\tilde{n}_* r_0}{\gamma}\left(\frac{\sigma_y}{\sigma_x}e^{-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}} - 1\right)\right].$$

Similarly, we have

$$-\frac{\partial U}{\partial \sigma_y^2} = \frac{1}{2\left(\sigma_x^2-\sigma_y^2\right)}\left[x\Delta x^{'} + y\Delta y^{'} + \frac{2\tilde{n}_* r_0}{\gamma}\left(\frac{\sigma_x}{\sigma_y}e^{-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}} - 1\right)\right].$$

In case of $\sigma_x < \sigma_y$, we have

$$-\frac{\partial U}{\partial \sigma_x^2} = \frac{-1}{2\left(\sigma_y^2-\sigma_x^2\right)}\left[x\Delta x^{'} + y\Delta y^{'} + \frac{2\tilde{n}_* r_0}{\gamma}\left(\frac{\sigma_x}{\sigma_y}e^{-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}} - 1\right)\right],$$

$$-\frac{\partial U}{\partial \sigma_y^2} = \frac{1}{2\left(\sigma_y^2-\sigma_x^2\right)}\left[x\Delta x^{'} + y\Delta y^{'} + \frac{2\tilde{n}_* r_0}{\gamma}\left(\frac{\sigma_y}{\sigma_x}e^{-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}} - 1\right)\right].$$

F. Derivation of space charge formula

$$\Phi\left(x,y,z\right)=\frac{Nq}{4\pi^{3/2}\epsilon_0}\int_0^\infty d\zeta\frac{-1+\exp\left(-\frac{x^2}{2\sigma_x^2+\zeta}-\frac{y^2}{2\sigma_y^2+\zeta}-\frac{z^2}{2\sigma_z^2+\zeta}\right)}{\sqrt{\left(2\sigma_x^2+\zeta\right)\left(2\sigma_y^2+\zeta\right)\left(2\sigma_z^2+\zeta\right)}},$$

1) σ_x and σ_y are comparable to σ_z

$$\begin{aligned}
e_z=-\frac{\partial\Phi}{\partial z}&=\frac{Nq}{4\pi^{3/2}\epsilon_0}\int_0^\infty d\zeta\,2z\frac{\exp\left(-\frac{x^2}{2\sigma_x^2+\zeta}-\frac{y^2}{2\sigma_y^2+\zeta}-\frac{z^2}{2\sigma_z^2+\zeta}\right)}{\sqrt{\left(2\sigma_x^2+\zeta\right)\left(2\sigma_y^2+\zeta\right)\left(2\sigma_z^2+\zeta\right)^3}},\\
&=\frac{Nq}{4\pi^{3/2}\epsilon_0}2z\int_0^1 dq\,\frac{2\sigma_z^2}{q^2}\frac{\exp\left(-\frac{qx^2}{2\sigma_z^2+2q(\sigma_x^2-\sigma_z^2)}\right)\exp\left(-\frac{qy^2}{2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)}\right)\exp\left(-\frac{qz^2}{2\sigma_z^2}\right)}{q^{-5/2}\sqrt{2\sigma_z^2+2q\left(\sigma_x^2-\sigma_z^2\right)}\sqrt{2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)}\left(2\sigma_z^2\right)^{3/2}}.\\
&=\frac{Nq}{4\pi^{3/2}\epsilon_0}\int_0^1 dq\,2z\sqrt{q}\frac{\exp\left(-\frac{qx^2}{2\sigma_z^2+2q(\sigma_x^2-\sigma_z^2)}\right)\exp\left(-\frac{qy^2}{2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)}\right)\exp\left(-\frac{qz^2}{2\sigma_z^2}\right)}{\sqrt{2\sigma_z^2+2q\left(\sigma_x^2-\sigma_z^2\right)}\sqrt{2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)}\sqrt{2\sigma_z^2}}.
\end{aligned}$$

2) σ_x and σ_y are much smaller than σ_z

$$\begin{aligned}
e_z=-\frac{\partial\Phi}{\partial z}&=\frac{Nqz}{2\pi^{3/2}\epsilon_0}\int_0^\infty d\zeta\frac{\exp\left(-\frac{x^2}{2\sigma_x^2+\zeta}-\frac{y^2}{2\sigma_y^2+\zeta}-\frac{z^2}{2\sigma_z^2+\zeta}\right)}{\sqrt{\left(2\sigma_x^2+\zeta\right)\left(2\sigma_y^2+\zeta\right)\left(2\sigma_z^2+\zeta\right)^3}},\\
&=\frac{Nq}{2\pi^{3/2}\epsilon_0}\frac{z\,\exp\left(-\frac{z^2}{2\sigma_z^2}\right)}{2^{3/2}\sigma_z^3}\int_0^\infty d\zeta\frac{\exp\left(-\frac{x^2}{2\sigma_x^2+\zeta}-\frac{y^2}{2\sigma_y^2+\zeta}\right)}{\sqrt{\left(2\sigma_x^2+\zeta\right)\left(2\sigma_y^2+\zeta\right)}},\\
&=\frac{Nq}{4\pi\epsilon_0}\frac{z\,\lambda\left(z\right)}{\sigma_z^2}\int_0^1 dq\,\frac{2\sigma_z^2}{q^2}\frac{\exp\left(-\frac{qx^2}{2\sigma_z^2+2q(\sigma_x^2-\sigma_z^2)}\right)\exp\left(-\frac{qy^2}{2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)}\right)}{\frac{1}{q}\sqrt{2\sigma_z^2+2q\left(\sigma_x^2-\sigma_z^2\right)}\sqrt{2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)}},\\
&=\frac{Nq}{4\pi\epsilon_0}2z\lambda\left(z\right)\int_0^1 dq\,\frac{\exp\left(-\frac{qx^2}{2\sigma_z^2+2q(\sigma_x^2-\sigma_z^2)}\right)\exp\left(-\frac{qy^2}{2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)}\right)}{q\sqrt{2\sigma_z^2+2q\left(\sigma_x^2-\sigma_z^2\right)}\sqrt{2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)}}.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\Phi}{\partial\sigma_x^2}&=\frac{Nq}{4\pi^{3/2}\epsilon_0}\int_0^\infty d\zeta\left(2x^2-2\sigma_x^2-\zeta\right)\frac{\exp\left(-\frac{x^2}{2\sigma_x^2+\zeta}-\frac{y^2}{2\sigma_y^2+\zeta}-\frac{z^2}{2\sigma_z^2+\zeta}\right)}{\sqrt{\left(2\sigma_x^2+\zeta\right)^5\left(2\sigma_y^2+\zeta\right)\left(2\sigma_z^2+\zeta\right)}},\\
&=\frac{Nq}{4\pi\epsilon_0}\frac{\exp\left(-\frac{z^2}{2\sigma_z^2}\right)}{\sqrt{2\pi}\sigma_z}\int_0^\infty d\zeta\left(2x^2-2\sigma_x^2-\zeta\right)\frac{\exp\left(-\frac{x^2}{2\sigma_x^2+\zeta}-\frac{y^2}{2\sigma_y^2+\zeta}\right)}{\sqrt{\left(2\sigma_x^2+\zeta\right)^5\left(2\sigma_y^2+\zeta\right)}},\\
&=\frac{Nq}{4\pi\epsilon_0}\lambda\left(z\right)\int_0^1 dq\,\frac{2\sigma_z^2}{q^2}\frac{\exp\left(-\frac{qx^2}{2\sigma_z^2+2q(\sigma_x^2-\sigma_z^2)}\right)\exp\left(-\frac{qy^2}{2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)}\right)}{q^{-3}\left(2\sigma_z^2+2q\left(\sigma_x^2-\sigma_z^2\right)\right)^{5/2}\sqrt{2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)}},\\
&=\frac{Nq}{4\pi\epsilon_0}\lambda\left(z\right)\int_0^1 dq\,2q\sigma_z^2\frac{\exp\left(-\frac{qx^2}{2\sigma_z^2+2q(\sigma_x^2-\sigma_z^2)}\right)\exp\left(-\frac{qy^2}{2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)}\right)}{\left(2\sigma_z^2+2q\left(\sigma_x^2-\sigma_z^2\right)\right)^{5/2}\sqrt{2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)}},
\end{aligned}$$

$$\frac{\partial\Phi}{\partial\sigma_y^2}=\frac{Nq}{4\pi\epsilon_0}\lambda\left(z\right)\int_0^1 dq\,2q\sigma_z^2\frac{\exp\left(-\frac{qx^2}{2\sigma_z^2+2q\left(\sigma_x^2-\sigma_z^2\right)}\right)\exp\left(-\frac{qy^2}{2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)}\right)}{\sqrt{2\sigma_z^2+2q\left(\sigma_x^2-\sigma_z^2\right)}\left(2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)\right)^{5/2}},$$

$$\begin{aligned}\frac{\partial\Phi}{\partial\sigma_z^2}&=\frac{Nq}{4\pi^{3/2}\epsilon_0}\int_0^\infty d\zeta\left(2z^2-2\sigma_z^2-\zeta\right)\frac{\exp\left(-\frac{x^2}{2\sigma_x^2+\zeta}-\frac{y^2}{2\sigma_y^2+\zeta}-\frac{z^2}{2\sigma_z^2+\zeta}\right)}{\sqrt{\left(2\sigma_x^2+\zeta\right)\left(2\sigma_y^2+\zeta\right)\left(2\sigma_z^2+\zeta\right)^5}},\\
&=\frac{Nq}{4\pi^{3/2}\epsilon_0}\frac{\exp\left(-\frac{z^2}{2\sigma_z^2}\right)}{\left(2\sigma_z^2\right)^{5/2}}\int_0^1 dq\,\frac{\left(2\sigma_z^2\right)^2}{q^3}\left(2\frac{qz^2}{2\sigma_z^2}-1\right)\frac{\exp\left(-\frac{qx^2}{2\sigma_z^2+2q\left(\sigma_x^2-\sigma_z^2\right)}\right)\exp\left(-\frac{qy^2}{2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)}\right)}{q^{-1}\sqrt{2\sigma_z^2+2q\left(\sigma_x^2-\sigma_z^2\right)}\sqrt{2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)}},\\
&=\frac{Nq}{4\pi\epsilon_0}\lambda\left(z\right)\int_0^1 dq\,\left(2\frac{qz^2}{2\sigma_z^2}-1\right)\frac{\exp\left(-\frac{qx^2}{2\sigma_z^2+2q\left(\sigma_x^2-\sigma_z^2\right)}\right)\exp\left(-\frac{qy^2}{2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)}\right)}{q^2\sqrt{2\sigma_z^2+2q\left(\sigma_x^2-\sigma_z^2\right)}\sqrt{2\sigma_z^2+2q\left(\sigma_y^2-\sigma_z^2\right)}},\end{aligned}$$

G. Collision scheme of sliced beams

1) Collision between 3-sliced beam and 1-sliced beam: The collision scheme of sliced beams is demonstrated in the following. We assume that the beams are sliced by an equal length of which is Δz. First we consider the collision of the last slice of a test beam and the other beam. The coordinates of the last slice is $\left(\bar{x},\bar{x}^{'},\bar{y},\bar{y}^{'},-\Delta z,\bar{\delta}\right)$.

1) Move the slice to the IP and find the transverse coordinates at IP:

$$\begin{aligned}x&\leftarrow\bar{x}-\left(-\Delta z\right)\bar{x}^{'}=\bar{x}+\Delta z\,\bar{x}^{'},\\
x^{'}&\leftarrow\bar{x}^{'}.\end{aligned}$$

2) Calculate the beam-beam kicks from the other beam.

$$\begin{aligned}S_z&=\frac{1}{2}\left(-\Delta z-0\right)=-\frac{1}{2}\Delta z,\\
x_{new}&=x+S_z x^{'}=\bar{x}+\Delta z\,\bar{x}^{'}-\frac{1}{2}\Delta z\,\bar{x}^{'}=\bar{x}+\frac{1}{2}\Delta z\,\bar{x}^{'},\\
&\left(calculate\ \Delta x_1^{'}\right)\\
x&\leftarrow x-S_z\Delta x_1^{'}=\bar{x}+\Delta z\left(\bar{x}^{'}+\frac{1}{2}\Delta x_1^{'}\right),\\
x^{'}&\leftarrow x^{'}+\Delta x_1^{'}=\bar{x}^{'}+\Delta x_1^{'}.\end{aligned}$$

3) Move the slice to the original location and find the transverse coordinate at the original location:

$$\begin{aligned}x&\leftarrow x+\left(-\Delta z\right)x^{'},\\
&=\bar{x}+\Delta z\left(\bar{x}^{'}+\frac{1}{2}\Delta x_1^{'}\right)-\Delta z\left(\bar{x}^{'}+\Delta x_1^{'}\right),\\
&=\bar{x}-\frac{1}{2}\Delta z\,\Delta x_1^{'},\\
x^{'}&\leftarrow\bar{x}^{'}+\Delta x_1^{'}.\end{aligned}$$

2) Collision between 3-sliced beam and 3-sliced beam: The collision scheme of sliced beams is demonstrated in the following. We assume that the beams are sliced by an equal length of which is Δz. First we consider the collision of the last slice of a test beam and the other beam. The coordinates of the last slice is $\left(\bar{x}, \bar{x}^{'}, \bar{y}, \bar{y}^{'}, -\Delta z, \bar{\delta}\right)$.

1) Move the slice to the IP and find the transverse coordinates at IP:

$$\begin{aligned} x &\leftarrow \bar{x} - (-\Delta z)\,\bar{x}^{'} = \bar{x} + \Delta z\,\bar{x}^{'}, \\ x^{'} &\leftarrow \bar{x}^{'}. \end{aligned}$$

2) Calculate the beam-beam kicks from the other beam.

a) Collision with the first slice of the other beam.

$$\begin{aligned} S_z &= \frac{1}{2}\left(-\Delta z - \Delta z\right) = -\Delta z, \\ X &= x + S_z x^{'} = \bar{x} + \Delta z\,\bar{x}^{'} - \Delta z\,\bar{x}^{'} = \bar{x}, \\ X^{'} &= x^{'} = \bar{x}^{'}, \\ &\left(calculate\ \Delta x_1^{'}\right) \\ X_{new} &= X = \bar{x}, \\ X_{new}^{'} &= X^{'} + \Delta x_1^{'} = \bar{x}^{'} + \Delta x_1^{'}, \\ x &\leftarrow X_{new} - S_z X_{new}^{'} = \bar{x} + \Delta z\left(\bar{x}^{'} + \Delta x_1^{'}\right), \\ x^{'} &\leftarrow X_{new}^{'} = \bar{x}^{'} + \Delta x_1^{'}. \end{aligned}$$

b) Collision with the second slice of the other beam.

$$\begin{aligned} S_z &= \frac{1}{2}\left(-\Delta z - 0\right) = -\frac{1}{2}\Delta z, \\ X &= x + S_z x^{'} = \bar{x} + \Delta z\left(\bar{x}^{'} + \Delta x_1^{'}\right) - \frac{1}{2}\Delta z\left(\bar{x}^{'} + \Delta x_1^{'}\right) = \bar{x} + \frac{1}{2}\Delta z\left(\bar{x}^{'} + \Delta x_1^{'}\right), \\ X^{'} &= x^{'} = \bar{x}^{'} + \Delta x_1^{'}, \\ &\left(calculate\ \Delta x_2^{'}\right) \\ X_{new} &= X = \bar{x} + \frac{1}{2}\Delta z\left(\bar{x}^{'} + \Delta x_1^{'}\right), \\ X_{new}^{'} &= X^{'} + \Delta x_2^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'}, \\ x &\leftarrow X_{new} - S_z X_{new}^{'} = \bar{x} + \Delta z\left(\bar{x}^{'} + \Delta x_1^{'} + \frac{1}{2}\Delta x_2^{'}\right), \\ x^{'} &\leftarrow X_{new}^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'}. \end{aligned}$$

c) Collision with the third slice of the other beam.

$$
\begin{aligned}
S_z &= \frac{1}{2}\left(-\Delta z - (-\Delta z)\right) = 0, \\
X &= x + S_z x^{'} = \bar{x} + \Delta z \left(\bar{x}^{'} + \Delta x_1^{'} + \frac{1}{2}\Delta x_2^{'}\right), \\
X^{'} &= x^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'}, \\
&\left(calculate\ \Delta x_3^{'}\right) \\
X_{new} &= X = \bar{x} + \Delta z \left(\bar{x}^{'} + \Delta x_1^{'} + \frac{1}{2}\Delta x_2^{'}\right), \\
X_{new}^{'} &= X^{'} + \Delta x_3^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'} + \Delta x_3^{'}, \\
x &\leftarrow X_{new} - S_z X_{new}^{'} = \bar{x} + \Delta z \left(\bar{x}^{'} + \Delta x_1^{'} + \frac{1}{2}\Delta x_2^{'}\right), \\
x^{'} &\leftarrow X_{new}^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'} + \Delta x_3^{'}.
\end{aligned}
$$

3) Move the slice to the original location and find the transverse coordinate at the original location:

$$
\begin{aligned}
x &\leftarrow x + (-\Delta z)\,\bar{x}^{'} \\
&= \bar{x} + \Delta z \left(\bar{x}^{'} + \Delta x_1^{'} + \frac{1}{2}\Delta x_2^{'}\right) - \Delta z \bar{x}^{'}, \\
&= \bar{x} + \Delta z \left(\Delta x_1^{'} + \frac{1}{2}\Delta x_2^{'}\right), \\
x^{'} &\leftarrow \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'} + \Delta x_3^{'}.
\end{aligned}
$$

Second we consider the collision of the middle slice of a test beam and the other beam. The coordinates of the slice is $\left(\bar{x}, \bar{x}^{'}, \bar{y}, \bar{y}^{'}, 0, \bar{\delta}\right)$.

1) Move the slice to the IP and find the transverse coordinates at IP:

$$
\begin{aligned}
x &\leftarrow \bar{x} - (0)\,\bar{x}^{'} = \bar{x}, \\
x^{'} &\leftarrow \bar{x}^{'}.
\end{aligned}
$$

2) Calculate the beam-beam kicks from the other beam.
   a) Collision with the first slice of the other beam.

$$
\begin{aligned}
S_z &= \frac{1}{2}\left(0 - \Delta z\right) = -\frac{1}{2}\Delta z, \\
X &= x + S_z x^{'} = \bar{x} - \frac{1}{2}\Delta z\,\bar{x}^{'}, \\
X^{'} &= x^{'} = \bar{x}^{'}, \\
&\left(calculate\ \Delta x_1^{'}\right) \\
X_{new} &= X = \bar{x} - \frac{1}{2}\Delta z\,\bar{x}^{'}, \\
X_{new}^{'} &= X^{'} + \Delta x_1^{'} = \bar{x}^{'} + \Delta x_1^{'}, \\
x &\leftarrow X_{new} - S_z X_{new}^{'} = \bar{x} - \frac{1}{2}\Delta z\,\bar{x}^{'} + \frac{1}{2}\Delta z \left(\bar{x}^{'} + \Delta x_1^{'}\right) = \bar{x} + \frac{1}{2}\Delta z\,\Delta x_1^{'}, \\
x^{'} &\leftarrow X_{new}^{'} = \bar{x}^{'} + \Delta x_1^{'}.
\end{aligned}
$$

   b) Collision with the second slice of the other beam.

$$
\begin{aligned}
S_z &= \frac{1}{2}\left(0-0\right) = 0 \\
X &= x + S_z x^{'} = \bar{x} + \frac{1}{2}\Delta z\, \Delta x_1^{'}, \\
X^{'} &= x^{'} = \bar{x}^{'} + \Delta x_1^{'}, \\
&\left(calculate\ \Delta x_2^{'}\right) \\
X_{new} &= X = \bar{x} + \frac{1}{2}\Delta z\, \Delta x_1^{'}, \\
X_{new}^{'} &= X^{'} + \Delta x_2^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'}, \\
x &\leftarrow X_{new} - S_z X_{new}^{'} = \bar{x} + \frac{1}{2}\Delta z\, \Delta x_1^{'}, \\
x^{'} &\leftarrow X_{new}^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'}.
\end{aligned}
$$

c) Collision with the third slice of the other beam.

$$
\begin{aligned}
S_z &= \frac{1}{2}\left(0-\left(-\Delta z\right)\right) = \frac{1}{2}\Delta z \\
X &= x + S_z x^{'} = \bar{x} + \frac{1}{2}\Delta z\, \Delta x_1^{'} + \frac{1}{2}\Delta z\left(\bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'}\right) = \bar{x} + \Delta z\left(\frac{1}{2}\bar{x}^{'} + \Delta x_1^{'} + \frac{1}{2}\Delta x_2^{'}\right), \\
X^{'} &= x^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'}, \\
&\left(calculate\ \Delta x_3^{'}\right) \\
X_{new} &= X = \bar{x} + \Delta z\left(\frac{1}{2}\bar{x}^{'} + \Delta x_1^{'} + \frac{1}{2}\Delta x_2^{'}\right), \\
X_{new}^{'} &= X^{'} + \Delta x_3^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'} + \Delta x_3^{'}, \\
x &\leftarrow X_{new} - S_z X_{new}^{'} = \bar{x} + \Delta z\left(\frac{1}{2}\Delta x_1^{'} - \frac{1}{2}\Delta x_3^{'}\right), \\
x^{'} &\leftarrow X_{new}^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'} + \Delta x_3^{'}.
\end{aligned}
$$

3) Move the slice to the original location and find the transverse coordinate at the original location:

$$
\begin{aligned}
x &\leftarrow x + (0)\,\bar{x}^{'} \\
&= \bar{x} + \Delta z\left(\frac{1}{2}\Delta x_1^{'} - \frac{1}{2}\Delta x_3^{'}\right), \\
x^{'} &\leftarrow \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'} + \Delta x_3^{'}.
\end{aligned}
$$

Third we consider the collision of the first slice of a test beam and the other beam. The coordinates of the first slice is $\left(\bar{x}, \bar{x}^{'}, \bar{y}, \bar{y}^{'}, +\Delta z, \bar{\delta}\right)$.

1) Move the slice to the IP and find the transverse coordinates at IP:

$$
\begin{aligned}
x &\leftarrow \bar{x} - (+\Delta z)\,\bar{x}^{'} = \bar{x} - \Delta z\,\bar{x}^{'}, \\
x^{'} &\leftarrow \bar{x}^{'}.
\end{aligned}
$$

2) Calculate the beam-beam kicks from the other beam.
   a) Collision with the first slice of the other beam.

$$
\begin{aligned}
S_z &= \frac{1}{2}\left(\Delta z - \Delta z\right) = 0, \\
X &= x + S_z x^{'} = \bar{x} - \Delta z\, \bar{x}^{'}, \\
X^{'} &= x^{'} = \bar{x}^{'}, \\
&\left(calculate\ \Delta x_1^{'}\right) \\
X_{new} &= X = \bar{x} - \Delta z\, \bar{x}^{'}, \\
X_{new}^{'} &= X^{'} + \Delta x_1^{'} = \bar{x}^{'} + \Delta x_1^{'}, \\
x &\leftarrow X_{new} - S_z X_{new}^{'} = \bar{x} - \Delta z\, \bar{x}^{'}, \\
x^{'} &\leftarrow X_{new}^{'} = \bar{x}^{'} + \Delta x_1^{'}.
\end{aligned}
$$

b) Collision with the second slice of the other beam.

$$
\begin{aligned}
S_z &= \frac{1}{2}\left(\Delta z - 0\right) = \frac{1}{2}\Delta z, \\
X &= x + S_z x^{'} = \bar{x} - \Delta z\, \bar{x}^{'} + \frac{1}{2}\Delta z\left(\bar{x}^{'} + \Delta x_1^{'}\right) = \bar{x} - \frac{1}{2}\Delta z\left(\bar{x}^{'} - \Delta x_1^{'}\right), \\
X^{'} &= x^{'} = \bar{x}^{'} + \Delta x_1^{'}, \\
&\left(calculate\ \Delta x_2^{'}\right) \\
X_{new} &= X = \bar{x} - \frac{1}{2}\Delta z\left(\bar{x}^{'} - \Delta x_1^{'}\right), \\
X_{new}^{'} &= X^{'} + \Delta x_2^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'}, \\
x &\leftarrow X_{new} - S_z X_{new}^{'} = \bar{x} - \Delta z\left(\bar{x}^{'} + \frac{1}{2}\Delta x_2^{'}\right), \\
x^{'} &\leftarrow X_{new}^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'}.
\end{aligned}
$$

c) Collision with the third slice of the other beam.

$$
\begin{aligned}
S_z &= \frac{1}{2}\left(\Delta z - (-\Delta z)\right) = \Delta z, \\
X &= x + S_z x^{'} = \bar{x} + \Delta z\left(\Delta x_1^{'} + \frac{1}{2}\Delta x_2^{'}\right), \\
X^{'} &= x^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'}, \\
&\left(calculate\ \Delta x_3^{'}\right) \\
X_{new} &= X = \bar{x} + \Delta z\left(\Delta x_1^{'} + \frac{1}{2}\Delta x_2^{'}\right), \\
X_{new}^{'} &= X^{'} + \Delta x_3^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'} + \Delta x_3^{'}, \\
x &\leftarrow X_{new} - S_z X_{new}^{'} = \bar{x} - \Delta z\left(\bar{x}^{'} + \frac{1}{2}\Delta x_2^{'} + \Delta x_3^{'}\right), \\
x^{'} &\leftarrow X_{new}^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'} + \Delta x_3^{'}.
\end{aligned}
$$

3) Move the slice to the original location and find the transverse coordinate at the original location:

$$
\begin{aligned}
x &\leftarrow x + (+\Delta z)\,\bar{x}^{'} \\
&= \bar{x} - \Delta z\left(\bar{x}^{'} + \frac{1}{2}\Delta x_2^{'} + \Delta x_3^{'}\right) + \Delta z \bar{x}^{'}, \\
&= \bar{x} - \Delta z\left(\frac{1}{2}\Delta x_2^{'} + \Delta x_3^{'}\right), \\
x^{'} &\leftarrow \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'} + \Delta x_3^{'}.
\end{aligned}
$$

3) Collision between 1-sliced beam and 5-sliced beam: The coordinates of the slice is $\left(\bar{x}, \bar{x}^{'}, \bar{y}, \bar{y}^{'}, 0, \bar{\delta}\right)$.

1) Move the slice to the IP and find the transverse coordinates at IP:

$$
\begin{aligned}
x &\leftarrow \bar{x} - (0)\,\bar{x}^{'} = \bar{x}, \\
x^{'} &\leftarrow \bar{x}^{'}.
\end{aligned}
$$

2) Calculate the beam-beam kicks from the other beam.
   a) Collision with the first slice of the other beam.

$$
\begin{aligned}
S_z &= \frac{1}{2}\left(0 - 2\Delta z\right) = -\Delta z, \\
X &= x + S_z x^{'} = \bar{x} - \Delta z\,\bar{x}^{'}, \\
X^{'} &= x^{'} = \bar{x}^{'}, \\
&\left(calculate\ \Delta x_1^{'}\right) \\
X_{new} &= X = \bar{x} - \Delta z\,\bar{x}^{'}, \\
X_{new}^{'} &= X^{'} + \Delta x_1^{'} = \bar{x}^{'} + \Delta x_1^{'}, \\
x &\leftarrow X_{new} - S_z X_{new}^{'} = \bar{x} - \Delta z\,\bar{x}^{'} + \Delta z\left(\bar{x}^{'} + \Delta x_1^{'}\right) = \bar{x} + \Delta z\,\Delta x_1^{'}, \\
x^{'} &\leftarrow X_{new}^{'} = \bar{x}^{'} + \Delta x_1^{'}.
\end{aligned}
$$

   b) Collision with the second slice of the other beam.

$$
\begin{aligned}
S_z &= \frac{1}{2}\left(0 - \Delta z\right) = -\frac{1}{2}\Delta z, \\
X &= x + S_z x^{'} = \bar{x} + \Delta z\,\Delta x_1^{'} - \frac{1}{2}\Delta z\left(\bar{x}^{'} + \Delta x_1^{'}\right) = \bar{x} - \frac{1}{2}\Delta z\left(\bar{x}^{'} - \Delta x_1^{'}\right), \\
X^{'} &= x^{'} = \bar{x}^{'} + \Delta x_1^{'}, \\
&\left(calculate\ \Delta x_2^{'}\right) \\
X_{new} &= X = \bar{x} - \frac{1}{2}\Delta z\left(\bar{x}^{'} - \Delta x_1^{'}\right), \\
X_{new}^{'} &= X^{'} + \Delta x_2^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'}, \\
x &\leftarrow X_{new} - S_z X_{new}^{'} = \bar{x} - \frac{1}{2}\Delta z\left(\bar{x}^{'} - \Delta x_1^{'}\right) + \frac{1}{2}\Delta z\left(\bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'}\right) = \bar{x} + \Delta z\left(\Delta x_1^{'} + \frac{1}{2}\Delta x_2^{'}\right), \\
x^{'} &\leftarrow X_{new}^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'}.
\end{aligned}
$$

   c) Collision with the third slice of the other beam.

$$
\begin{aligned}
S_z &= \frac{1}{2}(0-0) = 0, \\
X &= x + S_z x^{'} = \bar{x} + \Delta z \left(\Delta x_1^{'} + \frac{1}{2}\Delta x_2^{'}\right), \\
X^{'} &= x^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'}, \\
&\left(calculate\ \Delta x_3^{'}\right) \\
X_{new} &= X = \bar{x} + \Delta z \left(\Delta x_1^{'} + \frac{1}{2}\Delta x_2^{'}\right), \\
X_{new}^{'} &= X^{'} + \Delta x_3^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'} + \Delta x_3^{'}, \\
x &\leftarrow X_{new} - S_z X_{new}^{'} = \bar{x} + \Delta z \left(\Delta x_1^{'} + \frac{1}{2}\Delta x_2^{'}\right), \\
x^{'} &\leftarrow X_{new}^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'} + \Delta x_3^{'}.
\end{aligned}
$$

d) Collision with the fourth slice of the other beam.

$$
\begin{aligned}
S_z &= \frac{1}{2}(0-(-\Delta z)) = \frac{1}{2}\Delta z, \\
X &= x + S_z x^{'} = \bar{x} + \frac{1}{2}\Delta z \left(\bar{x}^{'} + 3\Delta x_1^{'} + 2\Delta x_2^{'} + \Delta x_3^{'}\right), \\
X^{'} &= x^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'} + \Delta x_3^{'}, \\
&\left(calculate\ \Delta x_4^{'}\right) \\
X_{new} &= X = \bar{x} + \frac{1}{2}\Delta z \left(\bar{x}^{'} + 3\Delta x_1^{'} + 2\Delta x_2^{'} + \Delta x_3^{'}\right), \\
X_{new}^{'} &= X^{'} + \Delta x_4^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'} + \Delta x_3^{'} + \Delta x_4^{'}, \\
x &\leftarrow X_{new} - S_z X_{new}^{'} = \bar{x} + \Delta z \left(\Delta x_1^{'} + \frac{1}{2}\Delta x_2^{'} - \frac{1}{2}\Delta x_4^{'}\right) \\
x^{'} &\leftarrow X_{new}^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'} + \Delta x_3^{'} + \Delta x_4^{'}.
\end{aligned}
$$

e) Collision with the fourth slice of the other beam.

$$
\begin{aligned}
S_z &= \frac{1}{2}(0-(-2\Delta z)) = \Delta z, \\
X &= x + S_z x^{'} = \bar{x} + \Delta z \left(\bar{x}^{'} + 2\Delta x_1^{'} + \frac{3}{2}\Delta x_2^{'} + \Delta x_3^{'} + \frac{1}{2}\Delta x_4^{'}\right), \\
X^{'} &= x^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'} + \Delta x_3^{'} + \Delta x_4^{'}, \\
&\left(calculate\ \Delta x_5^{'}\right) \\
X_{new} &= X = \bar{x} + \Delta z \left(\bar{x}^{'} + 2\Delta x_1^{'} + \frac{3}{2}\Delta x_2^{'} + \Delta x_3^{'} + \frac{1}{2}\Delta x_4^{'}\right), \\
X_{new}^{'} &= X^{'} + \Delta x_5^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'} + \Delta x_3^{'} + \Delta x_4^{'} + \Delta x_5^{'}, \\
x &\leftarrow X_{new} - S_z X_{new}^{'} = \bar{x} + \Delta z \left(\Delta x_1^{'} + \frac{1}{2}\Delta x_2^{'} - \frac{1}{2}\Delta x_4^{'} - \Delta x_5^{'}\right) \\
x^{'} &\leftarrow X_{new}^{'} = \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'} + \Delta x_3^{'} + \Delta x_4^{'} + \Delta x_5^{'}.
\end{aligned}
$$

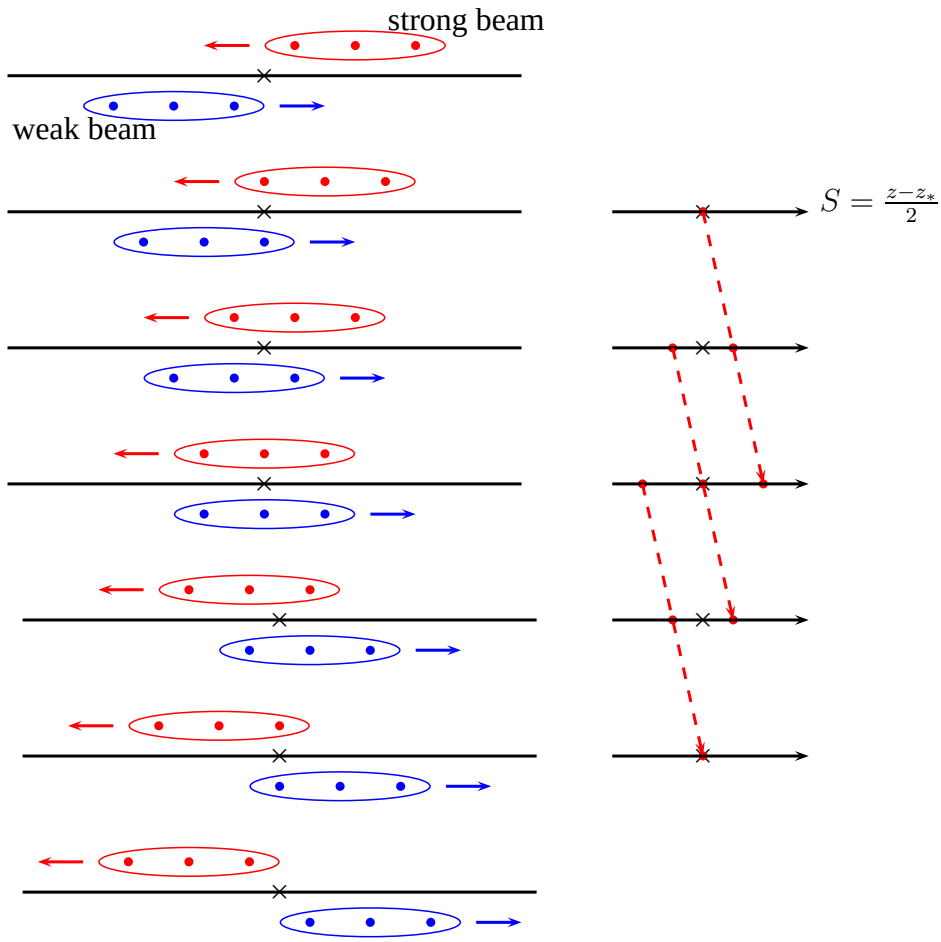

Figure 4. Diagram of beam-beam collisions of sliced beams. Each beam is sliced into three chunks. $\times$ stands for the IP (interaction point).

3) Move the slice to the original location and find the transverse coordinate at the original location:

$$\begin{aligned} x &\leftarrow x + (0)\,\bar{x}^{'} \\ &= \bar{x} + \Delta z \left(\Delta x_1^{'} + \frac{1}{2}\Delta x_2^{'} - \frac{1}{2}\Delta x_4^{'} - \Delta x_5^{'} \right), \\ x^{'} &\leftarrow \bar{x}^{'} + \Delta x_1^{'} + \Delta x_2^{'} + \Delta x_3^{'} + \Delta x_4^{'} + \Delta x_5^{'}. \end{aligned}$$

H. Complex error function

$$\begin{aligned} erf\,(x) &= \frac{2}{\sqrt{\pi}} \int_0^x dt\, e^{-t^2}, \\ erfc\,(x) &= 1 - erf\,(x)\,, \\ &= \frac{2}{\sqrt{\pi}} \int_x^\infty dt\, e^{-t^2}, \\ w\,(z) &= e^{-z^2} erfc\,(-iz)\,, \end{aligned}$$

$w\,(z)$ is the complex error function and $z = x + iy$, $w\,(z) = e^{-z^2}\left(1 + \frac{2i}{\sqrt{\pi}} \int_0^z dt\, e^{t^2}\right)$.

- Property of $erf\,(z)$ with a complex z
  - $erf\,(-z) = -erf\,(z)$: odd function
  - $erf\,(\bar{z}) = \overline{erf\,(z)}$

- $erf\left(-\bar{z}\right) = -erf\left(\bar{z}\right) = -\overline{erf\left(z\right)}$
- Let $z = |x| + i\,|y|$, then

$$\begin{aligned} erf\left(-|x| - i\,|y|\right) &= -erf\left(|x| + i\,|y|\right) = -\Re erf\left(z\right) - \Im erf\left(z\right), \\ erf\left(|x| - i\,|y|\right) &= erf\left(\bar{z}\right) = \overline{erf\left(z\right)} = \Re erf\left(z\right) - \Im erf\left(z\right), \\ erf\left(-|x| + i\,|y|\right) &= erf\left(-\bar{z}\right) = -\Re erf\left(z\right) + \Im erf\left(z\right) \end{aligned}$$

- Property of complex error function $w\left(z\right)$
  - $w\left(-z\right) = 2e^{-z^2} - w\left(z\right)$
  - $w\left(\bar{z}\right) = \overline{w\left(-z\right)}$
  - $w\left(-\bar{z}\right) = \overline{w\left(z\right)}$
- Property of function $F\left(z\right)$
  - $F\left(-z\right) = -F\left(z\right)$
  - $F\left(\bar{z}\right) = -\overline{F\left(z\right)}$
  - $F\left(-\bar{z}\right) = \overline{F\left(z\right)}$
  - Let $z = |x| + i\,|y|$, then

$$\begin{aligned} F\left(-|x| - i\,|y|\right) &= -F\left(|x| + i\,|y|\right) = -\Re F\left(z\right) - \Im F\left(z\right), \\ F\left(|x| - i\,|y|\right) &= -\overline{F\left(z\right)} = -\Re F\left(z\right) + \Im F\left(z\right), \\ F\left(-|x| + i\,|y|\right) &= \overline{F\left(z\right)} = \Re F\left(z\right) - \Im F\left(z\right) \end{aligned}$$