Derivation of ε-NTU Method for Heat Exchangers with Heat Leak

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OBJECTIVE:

A powerful and useful method for heat exchanger analysis is the effectiveness-NTU method. The equations for this technique presented in textbooks, however, are limited to the case where all of the heat transfer occurs between the two fluid streams. In an application of interest to us, cryogenic heat exchangers, we wish to consider a heat leak term. Thus, we have derived equations for the ε-NTU method with heat leak involved.

The cases to be studied include evaportators, condensers, and counter-flow, with heat leak both in and out.

DEFINITIONS:

variables:
- A= heat transfer surface area
- C= heat capacity
- Q= overall heat transfer
- Q'= overall heat transfer per unit area
- δq= heat transfer into differential element
- T= temperature
- U= overall heat transfer coefficient

subscripts:
- c= cold side fluid
- h=hot side fluid
- leak= pertaining to heat leak portion of total heat transfer
- hx= pertaining to heat exchanger portion of total heat transfer

ASSUMPTIONS:

1. U is known and is constant
2. Fluid properties are constant
3. Heat leak is constant, spread uniformly throughout exchanger
4. Heat leak entirely added to single phase fluid
5. Steady state

SOLUTION:

I. SINGLE PHASE TO TWO PHASE EXCHANGERS

A EVAPORATOR WITH HEAT LEAK IN

a) Physical description of problem:

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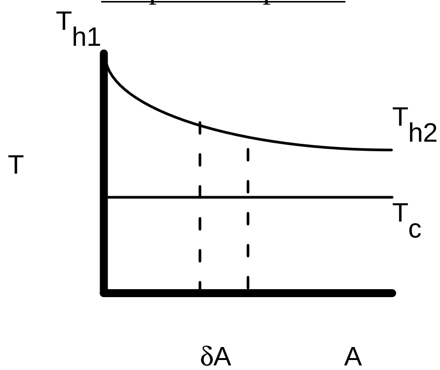
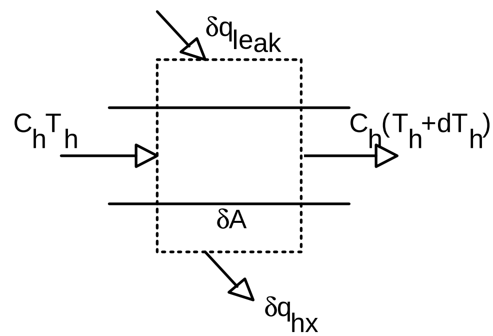
temperature profile



control volume



b) Energy balance: energy in = energy out

$$C_h T_h + \delta q_{leak} = \delta q_{hx} + C_h T_h + C_h \frac{dT_h}{dA}\,\delta A$$

$$\delta q_{leak} - \delta q_{hx} = C_h \frac{dT_h}{dA}\,\delta A \qquad (1A)$$

The heat leak, based on a constant energy per unit area:

$$\delta q_{leak} = Q'_{leak}\,\delta A \qquad (2A)$$

Rate equation for the heat exchanger:

$$\delta q_{hx} = U\delta A(T_h - T_c) \qquad (3A)$$

Combining (1A), (2A), and (3A):

$$Q'_{leak} - U(T_h - T_c) = C_h \frac{dT_h}{dA} \qquad (4A)$$

c) Solve this differential equation:

$$\int_0^A \frac{1}{C_h} dA = \int_{T_{h1}}^{T_{h2}} \frac{dT_h}{Q'_{leak} - U(T_h - T_c)}$$

$$\left.\frac{1}{C_h} A\right|_{A=0}^{A=A} = -\left.\frac{\ln\left[-U(T_h - T_c) + Q'_{leak}\right]}{U}\right|_{T_h = T_{h1}}^{T_h = T_{h2}}$$

$$\frac{UA}{C_h} = -\ln\left[-U(T_{h2} - T_c) + Q'_{leak}\right] + \ln\left[-U(T_{h1} - T_c) + Q'_{leak}\right]$$

$$\frac{UA}{C_h} = \ln\left|\frac{-U(T_{h1} - T_c) + Q'_{leak}}{-U(T_{h2} - T_c) + Q'_{leak}}\right| \qquad (5A)$$

d) By definition, the "number of heat transfer units" for an exchanger is:

$$\int_0^A \frac{U}{C_{\min}} dA = NTU$$

For evaporator with $C_c \approx \infty$, and $C_h = C_{min}$

$$\int_0^A \frac{U}{C_{\min}} dA = \frac{UA}{C_h} = NTU \qquad (6A)$$

e) Combine (5A) and (6A)

$$NTU = \ln \left| \frac{U(T_{h1} - T_c) - Q'_{leak}}{U(T_{h2} - T_c) - Q'_{leak}} \right|$$

$$-NTU = \ln \left| \frac{U(T_{h2} - T_c) - Q'_{leak}}{U(T_{h1} - T_c) - Q'_{leak}} \right|$$

$$-e^{-NTU} = -\frac{U(T_{h2} - T_c) - Q'_{leak}}{U(T_{h1} - T_c) - Q'_{leak}}$$

$$1 - e^{-NTU} = \frac{U(T_{h1} - T_c) - Q'_{leak} - U(T_{h2} - T_c) + Q'_{leak}}{U(T_{h1} - T_c) - Q'_{leak}}$$

$$1 - e^{-NTU} = \frac{U(T_{h1} - T_{h2})}{U(T_{h1} - T_c) - Q'_{leak}}$$

f) Since $Q'_{leak} = \dfrac{Q_{leak}}{A}$ we can say:

$$1 - e^{-NTU} = \frac{(T_{h1} - T_{h2})}{(T_{h1} - T_c) - \dfrac{Q_{leak}}{UA}}$$

Thinking in terms of a transposed maximum possible temperature difference:

$$1 - e^{-NTU} = \frac{(T_{h1} - T_{h2})}{T_{h1} - (T_c + \dfrac{Q_{leak}}{UA})} \qquad (7A)$$

g) Introduce definition of effectiveness: $\varepsilon = \dfrac{Q_{act}}{Q_{\max}}$

Kays (and other sources) show that for an evaporator or condenser: $\varepsilon = 1 - e^{-NTU}$

Thus we see from (7A) that the effect of heat leak is to modify the maximum possible heat transer for the single phase fluid, and consequently, the net cooling heat transfer from the single phase fluid is reduced.

Solving (7A) for the actual net single phase heat tranfer and final single phase temperture yields the following results:

$$\boxed{Q_{act1\phi} = \varepsilon C_h \left| T_{h1} - (T_c + \frac{Q_{leak}}{UA}) \right| \qquad (8A)}$$

$$T_{h2} = -\varepsilon \left| T_{h1} - (T_c + \frac{Q_{leak}}{UA}) \right| + T_{h1} \qquad (9A)$$

Note: The above equation addresses what is happening with the single phase fluid in computing a net single phase heat transfer. If one wishes to consider another Q term, one can go back to the energy balance relationship (1A), with the sign conventions noted in this analysis (positive Q in directions shown in the control volume diagram) and get:

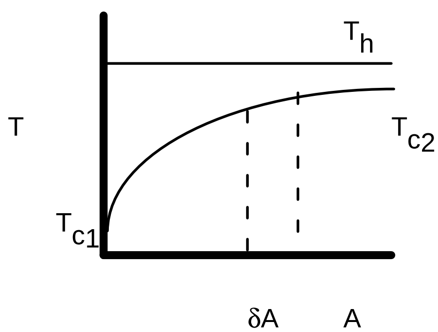
$$Q_{leak} - Q_{hx} = C_h \Delta T_h = -Q_{act1\phi}$$

If the heat leak is in the opposite direction (heat leak out), then the sign of the heat leak term is simply the opposite of that show in all the above derivations.

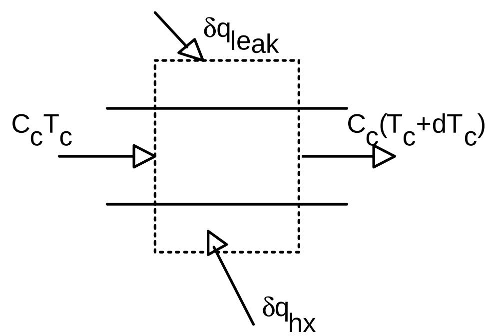
B. CONDENSER WITH HEAT LEAK IN

a) Physical description of problem:

temperature profile



control volume



b) Energy balance: energy in = energy out

$$C_c T_c + \delta q_{leak} + \delta q_{hx} = C_c T_c + C_c \frac{dT_c}{dA} \delta A$$

$$\delta q_{leak} + \delta q_{hx} = C_c \frac{dT_c}{dA} \delta A \qquad (1B)$$

The heat leak, based on a constant energy per unit area:

$$\delta q_{leak} = Q'_{leak} \delta A \qquad (2B)$$

Rate equation for the heat exchanger:

$$\delta q_{hx} = U \delta A (T_h - T_c) \qquad (3B)$$

Combining (1B), (2B), and (3B):

$$Q'_{leak} + U(T_h - T_c) = C_c \frac{dT_c}{dA} \qquad (4B)$$

c) Solve this differential equation:

$$\int_0^A \frac{1}{C_c} dA = \int_{T_{c1}}^{T_{c2}} \frac{dT_c}{Q'_{leak} + U(T_h - T_c)}$$

$$\frac{1}{C_c} A \Big|_{A=0}^{A=A} = -\frac{\ln\left[-U(T_c - T_h) + Q'_{leak}\right]}{U} \Bigg|_{T_c=T_{c1}}^{T_c=T_{c2}}$$

$$\frac{UA}{C_c} = -\ln\left[-U(T_{c2} - T_h) + Q'_{leak}\right] + \ln\left[-U(T_{c1} - T_h) + Q'_{leak}\right]$$

$$\frac{UA}{C_c} = \ln \left|\frac{-U(T_{c1} - T_h) + Q'_{leak}}{-U(T_{c2} - T_h) + Q'_{leak}}\right| \qquad (5B)$$

d) Recall definition for the "number of heat transfer units" of a heat exchanger applied to a condenser with $C_h \approx \infty$, and $C_c = C_{min}$

$$\int_0^A \frac{U}{C_{\min}} dA = \frac{UA}{C_c} = NTU \qquad (6B)$$

e) Combine (5B) and (6B)

$$-NTU = \ln \left|\frac{U(T_{c2} - T_h) - Q'_{leak}}{U(T_{c1} - T_h) - Q'_{leak}}\right|$$

$$-e^{-NTU} = -\frac{U(T_{c2} - T_h) - Q'_{leak}}{U(T_{c1} - T_h) - Q'_{leak}}$$

$$1 - e^{-NTU} = \frac{U(T_{c1} - T_h) - Q'_{leak} - U(T_{c2} - T_h) + Q'_{leak}}{U(T_{c1} - T_h) - Q'_{leak}}$$

$$1 - e^{-NTU} = \frac{U(T_{c1} - T_{c2})}{U(T_{c1} - T_h) - Q'_{leak}}$$

f) Since $Q'_{leak} = \frac{Q_{leak}}{A}$ we can say:

$$1 - e^{-NTU} = \frac{(T_{c1} - T_{c2})}{(T_{c1} - T_h) - \frac{Q_{leak}}{UA}}$$

Thinking in terms of a transposed maximum possible temperature difference:

$$1 - e^{-NTU} = \frac{(T_{c2} - T_{c1})}{(T_h + \frac{Q_{leak}}{UA}) - T_{c1}} \qquad (7B)$$

g) Recalling the definition of effectiveness: $\varepsilon = \frac{Q_{act}}{Q_{\max}}$

And recalling that for an evaporator or condenser: $\varepsilon = 1 - e^{-NTU}$

Thus we see from (7B) that the effect of heat leak is to modify the maximum possible heat transer for the single phase fluid, and consequently, the net heat transfer to the single phase fluid is increased.

Solving (7B) for actual net single phase heat tranfer and final single phase temperture yields the following results:

$$Q_{act1\phi} = \varepsilon C_c \left|(T_h + \frac{Q_{leak}}{UA}) - T_{c1}\right| \qquad (8B)$$

$$T_{c2} = \varepsilon \left|(T_h + \frac{Q_{leak}}{UA}) - T_{c1}\right| + T_{c1} \qquad (9B)$$

Note: The above equation addresses what is happening with the single phase fluid in computing a net single phase heat transfer. If one wishes to consider another Q term, one can go back to the energy balance relationship (1B), with the sign conventions noted in this analysis (positive Q in directions shown in the control volume diagram) and get:

$$Q_{leak} + Q_{hx} = C_c \Delta T_c = Q_{act1\phi}$$

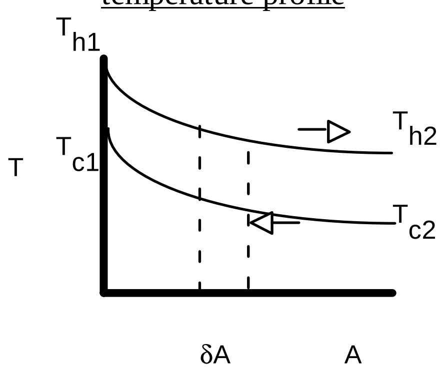
If the heat leak is in the opposite direction (heat leak out), then the sign of the heat leak term is simply the opposite of that show in all the above derivations.

II. COUNTERFLOW EXCHANGERS

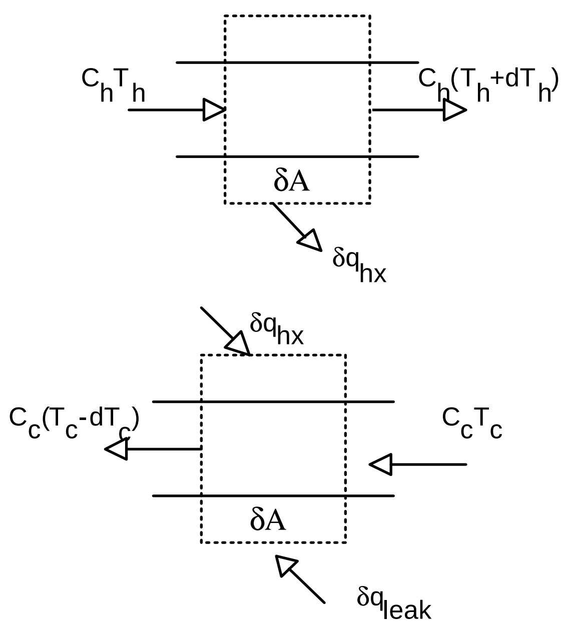
A HOT SIDE MIN CAPACITY/ HEAT LEAK IN COLD SIDE

a) Physical description of problem:

temperature profile



control volumes



Notice that in the sketch for the cold fluid, the cold fluid energy out has a - dT_C term, not +. This is minus because I define the difference, or the differential on the infintesimal scale, to be "section 2" - "section 1" . In the case for the cold fluid, to make the differential "outlet" - "inlet" correspond correctly with "section 2" - "section 1", a negative sign is required.

b) Energy balance

$$C_h T_h = \delta q_{hx} + C_h T_h + C_h \frac{dT_h}{dA}\,\delta A \qquad\qquad C_c T_c + \delta q_{leak} + \delta q_{hx} = C_c T_c - C_c \frac{dT_c}{dA}\,\delta A$$

$$\delta q_{hx} = -C_h \frac{dT_h}{dA}\,\delta A \qquad\qquad \delta q_{hx} = -\delta q_{leak} - C_c \frac{dT_c}{dA}\,\delta A$$

Subbing in exchanger rate equation (3A) and leak rate equation (2A) gives:

$$U(T_h - T_c)\frac{dA}{-C_h} = dT_h \qquad (1) \qquad\qquad \frac{Q'_{leak} + U(T_h - T_c)}{-C_c}\,dA = dT_c \qquad (2)$$

c) Subtract (2) from (1)

$$dT_h - dT_c = \left|\frac{U(T_h - T_c)}{-C_h} + \frac{U(T_h - T_c)}{C_c} + \frac{Q'_{leak}}{C_c}\right| dA$$

$$d(T_h - T_c) = \left|\left|\frac{1}{C_c} - \frac{1}{C_h}\right| U(T_h - T_c) + \frac{Q'_{leak}}{C_c}\right| dA$$

d) Integrate this expression

$$\int_{T_{h1}-T_{c1}}^{T_{h2}-T_{c2}} \frac{d(T_h-T_c)}{\left|\frac{1}{C_c}-\frac{1}{C_h}\right|U(T_h-T_c)+\frac{Q'_{leak}}{C_c}} = \int_0^A dA$$

$$\frac{1}{\left|\frac{1}{C_c}-\frac{1}{C_h}\right|U}\ln\left|\frac{Q'_{leak}}{C_c}+\left|\frac{1}{C_c}-\frac{1}{C_h}\right|U(T_h-T_c)\right|\Bigg|_{T_{h1}-T_{c1}}^{T_{h2}-T_{c2}} = A$$

$$\ln\left|\frac{Q'_{leak}}{C_c}+\left|\frac{C_h-C_c}{C_cC_h}\right|U(T_{h2}-T_{c2})\right| - \ln\left|\frac{Q'_{leak}}{C_c}+\left|\frac{C_h-C_c}{C_cC_h}\right|U(T_{h1}-T_{c1})\right| = UA\left|\frac{C_h-C_c}{C_cC_h}\right|$$

$$\ln\left|\frac{Q'_{leak}+\left|1-\frac{C_c}{C_h}\right|U(T_{h2}-T_{c2})}{Q'_{leak}+\left|1-\frac{C_c}{C_h}\right|U(T_{h1}-T_{c1})}\right| = -\frac{UA}{C_h}\left|1-\frac{C_h}{C_c}\right|$$

$$\frac{Q'_{leak}+\left|1-\frac{C_c}{C_h}\right|U(T_{h2}-T_{c2})}{Q'_{leak}+\left|1-\frac{C_c}{C_h}\right|U(T_{h1}-T_{c1})} = e^{-NTU\left|1-\frac{C_h}{C_c}\right|}$$

e) Continue to manipulate by adding (-1) to each side

$$\frac{-Q'_{leak}-\left|1-\frac{C_c}{C_h}\right|U(T_{h1}-T_{c1})+Q'_{leak}+\left|1-\frac{C_c}{C_h}\right|U(T_{h2}-T_{c2})}{Q'_{leak}+\left|1-\frac{C_c}{C_h}\right|U(T_{h1}-T_{c1})} = -1+e^{-NTU\left|1-\frac{C_h}{C_c}\right|}$$

$$\frac{\left|1-\frac{C_c}{C_h}\right|U\left[(T_{h2}-T_{c2})-(T_{h1}-T_{c1})\right]}{Q'_{leak}+\left|1-\frac{C_c}{C_h}\right|U(T_{h1}-T_{c1})} = -1+e^{-NTU\left|1-\frac{C_h}{C_c}\right|}$$

$$\frac{(T_{h1}-T_{c1})-(T_{h2}-T_{c2})}{\frac{Q_{leak}}{\left|1-\frac{C_c}{C_h}\right|UA}+(T_{h1}-T_{c1})} = 1-e^{-NTU\left|1-\frac{C_h}{C_c}\right|} \qquad (3)$$

f) Recalling the energy balance:

$$Q_{hx} = C_h(T_{h1} - T_{h2}) \qquad (4a) \qquad\qquad Q_{hx} + Q_{leak} = C_c(T_{c1} - T_{c2}) \qquad (4b)$$

Thus:

$$(T_{h1} - T_{c1}) - (T_{h2} - T_{c2}) = (T_{h1} - T_{h2}) - (T_{c1} - T_{c2}) = Q_{hx}\left(\frac{1}{C_h} - \frac{1}{C_c}\right) - \frac{Q_{leak}}{C_c} \qquad (5)$$

g) Substitute (5) into (3)

$$\frac{Q_{hx}\left(\frac{C_c - C_h}{C_c C_h}\right) - \frac{Q_{leak}}{C_c}}{\frac{Q_{leak}}{\left(1 - \frac{C_c}{C_h}\right)UA} + (T_{h1} - T_{c1})} = 1 - e^{-NTU\left(1 - \frac{C_h}{C_c}\right)}$$

$$\frac{\frac{1}{C_c}\left(-Q_{hx}\left(1 - \frac{C_c}{C_h}\right) - Q_{leak}\right)}{\frac{Q_{leak}}{\left(1 - \frac{C_c}{C_h}\right)UA} + (T_{h1} - T_{c1})} = 1 - e^{-NTU\left(1 - \frac{C_h}{C_c}\right)}$$

h) We want only inlet temperature terms in denominator, so use (4b) to eliminate T_{c1}.

$$\frac{\frac{1}{C_c}\left(-Q_{hx}\left(1 - \frac{C_c}{C_h}\right) - Q_{leak}\right)}{\frac{Q_{leak}}{\left(1 - \frac{C_c}{C_h}\right)UA} + T_{h1} - T_{c2} - \frac{Q_{hx}}{C_c} - \frac{Q_{leak}}{C_c}} = 1 - e^{-NTU\left(1 - \frac{C_h}{C_c}\right)}$$

i) Algebraicly simplify:

$$\frac{-Q_{hx}\left(1-\frac{C_c}{C_h}\right)-Q_{leak}}{C_c\left[T_{h1}-T_{c2}+\frac{Q_{leak}}{\left(1-\frac{C_c}{C_h}\right)UA}\right]-Q_{hx}-Q_{leak}}=1-e^{-NTU\left(1-\frac{C_h}{C_c}\right)}$$

$$Q_{hx}\left[\frac{C_c}{C_h}-e^{-NTU\left(1-\frac{C_h}{C_c}\right)}\right]=C_c\left[T_{h1}-T_{c2}+\frac{Q_{leak}}{\left(1-\frac{C_c}{C_h}\right)UA}\right]\left[1-e^{-NTU\left(1-\frac{C_h}{C_c}\right)}\right]+Q_{leak}e^{-NTU\left(1-\frac{C_h}{C_c}\right)}$$

$$Q_{hx}\left[1-\frac{C_h}{C_c}e^{-NTU\left(1-\frac{C_h}{C_c}\right)}\right]=C_h\left[T_{h1}-T_{c2}+\frac{Q_{leak}}{\left(1-\frac{C_c}{C_h}\right)UA}\right]\left[1-e^{-NTU\left(1-\frac{C_h}{C_c}\right)}\right]+\frac{C_h}{C_c}Q_{leak}e^{-NTU\left(1-\frac{C_h}{C_c}\right)}$$

Substitute in for Q_{hx} using (4a)

$$(T_{h1}-T_{h2})\left(1-\frac{C_h}{C_c}e^{-NTU\left(1-\frac{C_h}{C_c}\right)}\right)-\frac{1}{C_c}Q_{leak}e^{-NTU\left(1-\frac{C_h}{C_c}\right)}=\left(T_{h1}-T_{c2}+\frac{Q_{leak}}{\left(1-\frac{C_c}{C_h}\right)UA}\right)\left(1-e^{-NTU\left(1-\frac{C_h}{C_c}\right)}\right)$$

$$\frac{(T_{h1}-T_{h2})\left(1-\frac{C_h}{C_c}e^{-NTU\left(1-\frac{C_h}{C_c}\right)}\right)-\frac{1}{C_c}Q_{leak}e^{-NTU\left(1-\frac{C_h}{C_c}\right)}}{\left(T_{h1}-T_{c2}+\frac{Q_{leak}}{\left(1-\frac{C_c}{C_h}\right)UA}\right)\left(1-\frac{C_h}{C_c}e^{-NTU\left(1-\frac{C_h}{C_c}\right)}\right)}=\frac{1-e^{-NTU\left(1-\frac{C_h}{C_c}\right)}}{1-\frac{C_h}{C_c}e^{-NTU\left(1-\frac{C_h}{C_c}\right)}}$$

$$\frac{(T_{h1}-T_{h2})-\frac{1}{C_c}\frac{Q_{leak}}{\left(e^{NTU\left(1-\frac{C_h}{C_c}\right)}-\frac{C_h}{C_c}\right)}}{T_{h1}-T_{c2}+\frac{Q_{leak}}{\left(1-\frac{C_c}{C_h}\right)UA}}=\frac{1-e^{-NTU\left(1-\frac{C_h}{C_c}\right)}}{1-\frac{C_h}{C_c}e^{-NTU\left(1-\frac{C_h}{C_c}\right)}}$$

$$\frac{(T_{h1}-T_{h2})+\frac{Q_{leak}/C_h}{\left(1-\frac{C_c}{C_h}e^{NTU\left(1-\frac{C_h}{C_c}\right)}\right)}}{T_{h1}-T_{c2}+\frac{Q_{leak}}{\left(1-\frac{C_c}{C_h}\right)UA}}=\frac{1-e^{-NTU\left(1-\frac{C_h}{C_c}\right)}}{1-\frac{C_h}{C_c}e^{-NTU\left(1-\frac{C_h}{C_c}\right)}}$$

j. Recalling definition of hot and cold fluid inlets, and recognizing that the right hand side of this equation is the effectiveness of a counterflow exchanger with the hot fluid being the minimum capacity side:

$$\boxed{\frac{(T_{h,in}-T_{h,out})+\frac{Q_{leak}/C_h}{\left(1-\frac{C_c}{C_h}e^{NTU\left(1-\frac{C_h}{C_c}\right)}\right)}}{T_{h,in}-T_{c,in}+\frac{Q_{leak}}{\left(1-\frac{C_c}{C_h}\right)UA}}=\frac{1-e^{-NTU\left(1-\frac{C_h}{C_c}\right)}}{1-\frac{C_h}{C_c}e^{-NTU\left(1-\frac{C_h}{C_c}\right)}}=\varepsilon_{cf}}$$

B COLD SIDE MIN CAPACITY/ HEAT LEAK IN COLD SIDE

Still take heat leak into the cold side. Now take minimum capacity as cold side.

a) Physical description is unchanged

b) Energy balance equations (1) and (2) are unchanged

c) Integration is the same, but algebra grouping different

$$\ln \left| \frac{\dfrac{Q'_{leak}}{C_c} + \left|\dfrac{C_h - C_c}{C_c C_h}\right| U(T_{h2} - T_{c2})}{\dfrac{Q'_{leak}}{C_c} + \left|\dfrac{C_h - C_c}{C_c C_h}\right| U(T_{h1} - T_{c1})} \right| = \frac{UA}{C_c}\left|1 - \frac{C_c}{C_h}\right|$$

$$\frac{Q'_{leak} + \left|1 - \dfrac{C_c}{C_h}\right| U(T_{h1} - T_{c1})}{Q'_{leak} + \left|1 - \dfrac{C_c}{C_h}\right| U(T_{h2} - T_{c2})} = e^{-NTU\left|1 - \frac{C_c}{C_h}\right|}$$

d) Add (-1)

$$\frac{-Q'_{leak} - \left|1 - \dfrac{C_c}{C_h}\right| U(T_{h2} - T_{c2}) + Q'_{leak} + \left|1 - \dfrac{C_c}{C_h}\right| U(T_{h1} - T_{c1})}{Q'_{leak} + \left|1 - \dfrac{C_c}{C_h}\right| U(T_{h2} - T_{c2})} = -1 + e^{-NTU\left|1 - \frac{C_c}{C_h}\right|}$$

$$\frac{\left|1 - \dfrac{C_c}{C_h}\right| U\left[(T_{h1} - T_{c1}) - (T_{h2} - T_{c2})\right]}{Q'_{leak} + \left|1 - \dfrac{C_c}{C_h}\right| U(T_{h2} - T_{c2})} = -1 + e^{-NTU\left|1 - \frac{C_c}{C_h}\right|}$$

$$\frac{(T_{c1} - T_{c2}) - (T_{h1} - T_{h2})}{\dfrac{Q_{leak}}{\left|1 - \dfrac{C_c}{C_h}\right| UA} + (T_{h2} - T_{c2})} = 1 - e^{-NTU\left|1 - \frac{C_c}{C_h}\right|} \qquad (3)$$

e) Recall Energy balance (same as in part A)

$$Q_{hx} = C_h(T_{h1} - T_{h2}) \qquad (4a) \qquad\qquad Q_{hx} + Q_{leak} = C_c(T_{c1} - T_{c2}) \qquad (4b)$$

Thus:

$$(T_{c1} - T_{c2}) - (T_{h1} - T_{h2}) = Q_{hx}\left(\frac{1}{C_c} - \frac{1}{C_h}\right) + \frac{Q_{leak}}{C_c} \quad (5)$$

f) Sustitute (5) into (3) and then eliminate T_{h2} term

$$\frac{Q_{hx}\left(\frac{C_h - C_c}{C_c C_h}\right) + \frac{Q_{leak}}{C_c}}{\frac{Q_{leak}}{\left(1 - \frac{C_c}{C_h}\right)UA} + (T_{h2} - T_{c2})} = 1 - e^{-NTU\left(1 - \frac{C_c}{C_h}\right)}$$

$$\frac{\frac{1}{C_c}\left(Q_{hx}\left(1 - \frac{C_c}{C_h}\right) + Q_{leak}\right)}{\frac{Q_{leak}}{\left(1 - \frac{C_c}{C_h}\right)UA} + T_{h1} - T_{c2} - \frac{Q_{hx}}{C_h}} = 1 - e^{-NTU\left(1 - \frac{C_c}{C_h}\right)}$$

g) Algebraicly simplify:

$$\frac{Q_{hx}\left(1 - \frac{C_c}{C_h}\right) + Q_{leak}}{C_c\left(T_{h1} - T_{c2} + \frac{Q_{leak}}{\left(1 - \frac{C_c}{C_h}\right)UA}\right) - \frac{Q_{hx}C_c}{C_h}} = 1 - e^{-NTU\left(1 - \frac{C_c}{C_h}\right)}$$

$$Q_{hx}\left(1 - \frac{C_c}{C_h}e^{-NTU\left(1 - \frac{C_c}{C_h}\right)}\right) + Q_{leak} = C_c\left(T_{h1} - T_{c2} + \frac{Q_{leak}}{\left(1 - \frac{C_c}{C_h}\right)UA}\right)\left(1 - e^{-NTU\left(1 - \frac{C_c}{C_h}\right)}\right)$$

$$\frac{Q_{hx} + \frac{Q_{leak}}{1 - \frac{C_c}{C_h}e^{-NTU\left(1 - \frac{C_c}{C_h}\right)}}}{C_c\left(T_{h1} - T_{c2} + \frac{Q_{leak}}{\left(1 - \frac{C_c}{C_h}\right)UA}\right)} = \frac{1 - e^{-NTU\left(1 - \frac{C_c}{C_h}\right)}}{1 - \frac{C_c}{C_h}e^{-NTU\left(1 - \frac{C_c}{C_h}\right)}}$$

h) Removing the Q_{hx} term with the energy balance relationship (4b), recognizing on right hand side the effectiveness of a simple counterflow exchanger, and clarifying in and out points...

$$\boxed{\frac{T_{c,out}-T_{c,in}+\dfrac{Q_{leak}}{C_c}\left|\dfrac{1}{1-\dfrac{C_c}{C_h}e^{-NTU\left|1-\frac{C_c}{C_h}\right|}}-1\right|}{T_{h,in}-T_{c,in}+\dfrac{Q_{leak}}{\left|1-\dfrac{C_c}{C_h}\right|UA}}=\frac{1-e^{-NTU\left|1-\frac{C_c}{C_h}\right|}}{1-\dfrac{C_c}{C_h}e^{-NTU\left|1-\frac{C_c}{C_h}\right|}}=\varepsilon_{cf}}$$

C COLD SIDE MIN CAPACITY/ HEAT LEAK IN HOT SIDE

a) Physical description of problem: Same as before except now the heat leak is into the warm side instead of the cold side.

b) Energy balance

$$C_hT_h+\delta q_{leak}=\delta q_{hx}+C_hT_h+C_h\frac{dT_h}{dA}\delta A \qquad\qquad C_cT_c+\delta q_{hx}=C_cT_c-C_c\frac{dT_c}{dA}\delta A$$

$$\delta q_{hx}-\delta q_{leak}=-C_h\frac{dT_h}{dA}\delta A \qquad\qquad \delta q_{hx}=-C_c\frac{dT_c}{dA}\delta A$$

Subbing in exchanger rate equation (3A) and leak rate equation (2A) gives:

$$\left|\frac{U(T_h-T_c)-Q'_{leak}}{-C_h}\right|dA=dT_h \qquad (1) \qquad\qquad \frac{U(T_h-T_c)}{-C_c}dA=dT_c \qquad (2)$$

c) Subtract (2) from (1)

$$dT_h-dT_c=\left|\frac{U(T_h-T_c)}{-C_h}+\frac{U(T_h-T_c)}{C_c}+\frac{Q'_{leak}}{C_h}\right|dA$$

d) Integrate this expression

$$\int_{T_{h1}-T_{c1}}^{T_{h2}-T_{c2}} \frac{d(T_h - T_c)}{\left|\frac{1}{C_c} - \frac{1}{C_h}\right| U(T_h - T_c) + \frac{Q'_{leak}}{C_h}} = \int_0^A dA$$

$$\frac{1}{\left|\frac{1}{C_c} - \frac{1}{C_h}\right| U} \ln \left|\frac{Q'_{leak}}{C_h} + \left|\frac{1}{C_c} - \frac{1}{C_h}\right| U(T_h - T_c)\right| \Bigg|_{T_{h1}-T_{c1}}^{T_{h2}-T_{c2}} = A$$

$$\ln \left| \frac{Q'_{leak} + \left|\frac{C_h}{C_c} - 1\right| U(T_{h2} - T_{c2})}{Q'_{leak} + \left|\frac{C_h}{C_c} - 1\right| U(T_{h1} - T_{c1})} \right| = \frac{UA}{C_c} \left|1 - \frac{C_c}{C_h}\right|$$

$$\frac{Q'_{leak} + \left|\frac{C_h}{C_c} - 1\right| U(T_{h1} - T_{c1})}{Q'_{leak} + \left|\frac{C_h}{C_c} - 1\right| U(T_{h2} - T_{c2})} = e^{-NTU\left|1 - \frac{C_c}{C_h}\right|}$$

e) Add (-1) to both sides and simplify:

$$\frac{(T_{h2} - T_{c2}) - (T_{h1} - T_{c1})}{\frac{Q_{leak}}{\left|\frac{C_h}{C_c} - 1\right| UA} + (T_{h2} - T_{c2})} = 1 - e^{-NTU\left|1 - \frac{C_c}{C_h}\right|} \qquad (3)$$

f) Recall energy balance:

$$Q_{hx} - Q_{leak} = C_h(T_{h1} - T_{h2}) \qquad (4a) \qquad\qquad Q_{hx} = C_c(T_{c1} - T_{c2}) \qquad (4b)$$

Thus:

$$(T_{c1} - T_{c2}) - (T_{h1} - T_{h2}) = Q_{hx}\left|\frac{1}{C_c} - \frac{1}{C_h}\right| + \frac{Q_{leak}}{C_h} \qquad (5)$$

g) Subbing (5) into (3) and eliminating T_{h2} term gives:

$$\frac{Q_{hx}\left(\frac{C_h - C_c}{C_c C_h}\right) + \frac{Q_{leak}}{C_h}}{\frac{Q_{leak}}{\left(\frac{C_h}{C_c} - 1\right)UA} + (T_{h2} - T_{c2})} = 1 - e^{-NTU\left(1 - \frac{C_c}{C_h}\right)}$$

$$\frac{\frac{1}{C_h}\left(Q_{hx}\left(\frac{C_h}{C_c} - 1\right) + Q_{leak}\right)}{\frac{Q_{leak}}{\left(\frac{C_h}{C_c} - 1\right)UA} + T_{h1} - \frac{Q_{hx}}{C_h} + \frac{Q_{leak}}{C_h} - T_{c2}} = 1 - e^{-NTU\left(1 - \frac{C_c}{C_h}\right)}$$

g) Algebraicly simplify:

$$\frac{Q_{hx}\left(\frac{C_h}{C_c} - 1\right) + Q_{leak}}{C_h\left(T_{h1} - T_{c2} + \frac{Q_{leak}}{\left(\frac{C_h}{C_c} - 1\right)UA}\right) - Q_{hx} + Q_{leak}} = 1 - e^{-NTU\left(1 - \frac{C_c}{C_h}\right)}$$

$$Q_{hx}\left(\frac{C_h}{C_c} - e^{-NTU\left(1 - \frac{C_c}{C_h}\right)}\right) = C_h\left(T_{h1} - T_{c2} + \frac{Q_{leak}}{\left(\frac{C_h}{C_c} - 1\right)UA}\right)\left(1 - e^{-NTU\left(1 - \frac{C_c}{C_h}\right)}\right) - Q_{leak}e^{-NTU\left(1 - \frac{C_c}{C_h}\right)}$$

$$\frac{Q_{hx} + \frac{\frac{C_c}{C_h}Q_{leak}e^{-NTU\left(1 - \frac{C_c}{C_h}\right)}}{1 - \frac{C_c}{C_h}e^{-NTU\left(1 - \frac{C_c}{C_h}\right)}}}{C_c\left(T_{h1} - T_{c2} + \frac{Q_{leak}}{\left(\frac{C_h}{C_c} - 1\right)UA}\right)} = \frac{1 - e^{-NTU\left(1 - \frac{C_c}{C_h}\right)}}{1 - \frac{C_c}{C_h}e^{-NTU\left(1 - \frac{C_c}{C_h}\right)}}$$

h) Substituting for Q_{hx}, recalling definition of hot and cold fluid inlets, and recognizing that the right hand side of this equation is the effectiveness of a counterflow exchanger with the cold fluid being the minimum capacity side:

$$\frac{(T_{c,out}-T_{c,in})+\dfrac{Q_{leak}/C_c}{\left(\dfrac{C_h}{C_c}e^{NTU\left(1-\frac{C_h}{C_c}\right)}-1\right)}}{T_{h,in}-T_{c,in}+\dfrac{Q_{leak}}{\left(\dfrac{C_h}{C_c}-1\right)UA}}=\frac{1-e^{-NTU\left(1-\frac{C_c}{C_h}\right)}}{1-\dfrac{C_c}{C_h}e^{-NTU\left(1-\frac{C_c}{C_h}\right)}}=\varepsilon_{cf}$$

D HOT SIDE MIN CAPACITY/ HEAT LEAK IN HOT SIDE

a) Physical description: same as in IIC
b) energy balance: same as in IIC
c) integration same as in IIC, but now group a little differently

$$\ln\left(\frac{Q'_{leak}+\left(\dfrac{C_h}{C_c}-1\right)U(T_{h2}-T_{c2})}{Q'_{leak}+\left(\dfrac{C_h}{C_c}-1\right)U(T_{h1}-T_{c1})}\right)=\frac{UA}{C_h}\left(\frac{C_h}{C_c}-1\right)$$

$$\frac{Q'_{leak}+\left(\dfrac{C_h}{C_c}-1\right)U(T_{h2}-T_{c2})}{Q'_{leak}+\left(\dfrac{C_h}{C_c}-1\right)U(T_{h1}-T_{c1})}=e^{-NTU\left(1-\frac{C_h}{C_c}\right)}$$

d) Add (-1) to both sides and simplify:

$$\frac{(T_{h1}-T_{c1})-(T_{h2}-T_{c2})}{\dfrac{Q_{leak}}{\left(\dfrac{C_h}{C_c}-1\right)UA}+(T_{h1}-T_{c1})}=1-e^{-NTU\left(1-\frac{C_h}{C_c}\right)} \qquad (3)$$

e) Recall energy balance (same as in IIC):

$$Q_{hx}-Q_{leak}=C_h(T_{h1}-T_{h2}) \qquad (4a) \qquad\qquad Q_{hx}=C_c(T_{c1}-T_{c2}) \qquad (4b)$$

Thus:

$$(T_{h1}-T_{h2})-(T_{c1}-T_{c2})=Q_{hx}\left(\frac{1}{C_h}-\frac{1}{C_c}\right)-\frac{Q_{leak}}{C_h} \qquad (5)$$

f) Subbing (5) into (3) and eliminating T_{c1} term gives:

$$\frac{Q_{hx}\left(\frac{C_c-C_h}{C_cC_h}\right)-\frac{Q_{leak}}{C_h}}{\frac{Q_{leak}}{\left(\frac{C_h}{C_c}-1\right)UA}+(T_{h1}-T_{c1})}=1-e^{-NTU\left(1-\frac{C_h}{C_c}\right)}$$

$$\frac{\frac{1}{C_h}\left(Q_{hx}\left(1-\frac{C_h}{C_c}\right)-Q_{leak}\right)}{\frac{Q_{leak}}{\left(\frac{C_h}{C_c}-1\right)UA}+T_{h1}-\frac{Q_{hx}}{C_c}-T_{c2}}=1-e^{-NTU\left(1-\frac{C_h}{C_c}\right)}$$

g) Algebraicly simplify:

$$\frac{Q_{hx}-\frac{Q_{leak}}{1-\frac{C_h}{C_c}e^{-NTU\left(1-\frac{C_h}{C_c}\right)}}}{C_h\left(T_{h1}-T_{c2}+\frac{Q_{leak}}{\left(\frac{C_h}{C_c}-1\right)UA}\right)}=\frac{1-e^{-NTU\left(1-\frac{C_h}{C_c}\right)}}{1-\frac{C_h}{C_c}e^{-NTU\left(1-\frac{C_h}{C_c}\right)}}$$

h) Substituting for Q_{hx}, recalling definition of hot and cold fluid inlets, and recognizing that the right hand side of this equation is the effectiveness of a counterflow exchanger with the hot fluid being the minimum capacity side:

$$\boxed{\frac{(T_{h,in}-T_{h,out})+\frac{Q_{leak}}{C_h}\left(\frac{1}{1-\frac{C_h}{C_c}e^{-NTU\left(1-\frac{C_h}{C_c}\right)}}+1\right)}{T_{h,in}-T_{c,in}+\frac{Q_{leak}}{\left(\frac{C_h}{C_c}-1\right)UA}}=\frac{1-e^{-NTU\left(1-\frac{C_h}{C_c}\right)}}{1-\frac{C_h}{C_c}e^{-NTU\left(1-\frac{C_h}{C_c}\right)}}=\varepsilon_{cf}}$$