TM-1267
0200.000

Fermilab

Calculation of Multipole Fields for REC* Configurations with Permeability Not Equal to Unity

Robert Gluckstern†

June 1984

I. Introduction

The analysis of various geometrical configurations of REC* material designed to produce pure multipole field configurations (such as quadrupole, dipole, sextupole, etc.) is greatly simplified by the assumption that the permeability μ is unity. In this case the fields generated by one block of REC penetrate the other blocks of REC as if they are not there, and the total field configuration can be evaluated as a superposition of the fields due to each segment.

When $\mu \neq 1$, the calculation of the fields produced by several REC segments becomes a complex boundary value problem, since the superposition principle no longer applies. Only very simple geometries can be carried through analytically.

In an earlier paper[1], Gluckstern and Holsinger estimated the effect of B, H non-linearity on the multipole structure and quadrupole field strength for a 2-D REC quadrupole ring with continuous rotation of the easy axis. The analysis was carried out to lowest order in $\mu_{\parallel}-1$ and $\mu_{\perp}-\mu_{\parallel}$ where $\mu_{\parallel}$ and $\mu_{\perp}$ are the permeability of the REC material along and perpendicular to the easy axis.

---

*The designation REC stands for rare earth-cobalt, but it is intended to include the larger class of permanent magnet materials such as ceramic ferrites, samarium-cobalt, neodymium-iron, etc. which have permeability close to unity and high remanent field.

†Permanent address: Physics Department, University of Maryland, College Park, MD.

[1] R.L. Gluckstern and R.F. Holsinger, Proceedings of the 1981 Linac Conference, Santa Fe, NM, p. 214.

The purpose of the present note is to provide a general formulation for the analysis for $\mu \neq 1$ which can be adapted for numerical work if so desired.

II. General Analysis

Let us assume that the relation between $\vec{B}$ and $\vec{H}$ is given by

$$\vec{B} = \vec{H} + \vec{M} + \vec{f}(\vec{H}) \tag{2.1}$$

where $\vec{M}$ is a known magnetization distribution supplied by blocks of REC material located in specified positions. Maxwell's equation for $\vec{H}$ is

$$\nabla \times \vec{H} = 0 \tag{2.2}$$

in the absence of current, thus permitting the use of a scalar potential defined by

$$\vec{H} = \nabla\psi \tag{2.3}$$

The Maxwell equation for $\vec{B}$ then becomes

$$\nabla^2\psi = -\nabla\cdot\vec{M} - \nabla\cdot\vec{f}(\vec{H}) \tag{2.4}$$

Clearly $\vec{M} = 0$, $\vec{f} = 0$ outside the REC. (We assume no other permeable material is present.)

Since Eq. (2.4) has the form of Poisson's equation, we can write the solution as

$$\psi(\vec{x}) = (1/4\pi) \int \frac{d\vec{x}' \; \nabla\cdot\vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} + (1/4\pi) \int \frac{d\vec{x}' \; \nabla\cdot\vec{f}(\vec{x}')}{|\vec{x} - \vec{x}'|} \tag{2.5}$$

where the integrals are carried out over all space. Clearly, there is no contribution to the integrals outside the REC. There may, however, be a discontinuity in $\vec{M}$ and/or $\vec{f}$ at the surface of the segments. Recognizing that the discontinuity introduces a surface integral of the normal component of the discontinuous vector function, we can write

$$\psi(\vec{x}) = -(1/4\pi) \int dS' \frac{\vec{n}\cdot\vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} - (1/4\pi) \int dS' \frac{\vec{n}\cdot\vec{f}(\vec{x}')}{|\vec{x} - \vec{x}'|} +$$

$$+ (1/4\pi) \int_{REC} \frac{d\vec{x}'\nabla\cdot\vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} + (1/4\pi) \int \frac{d\vec{x}' \; \nabla\cdot\vec{f}(\vec{x}')}{|\vec{x} - \vec{x}'|} \tag{2.6}$$

where the volume integrals run only over the <u>interior</u> of the REC segments, and where $\vec{n}$ is the outward normal on the surface of each REC segment.

Equation (2.6) is clearly an integral equation, since $\vec{f}(\vec{x}')$ is assumed to be a function of $\vec{H}$, or ψ. But if $\vec{f}$ is small compared to $\vec{M}$, it can be used effectively in a perturbative approach. The first approximation to ψ is obtained by setting $\vec{f} = 0$ on the right side of Eq. (2.6). The value obtained for ψ, and thereby for $\vec{H}$, is then used in a second approximation for $\vec{f}$, leading to a second approximation for ψ in Eq. (2.6). This process can be repeated, and will converge at a rate determined by the ratio $|\vec{f}|/|\vec{M}|$. Thus the method is not expected to converge in the case of materials with $\mu > 2$.

III. <u>Permeability Not Equal to Unity</u>

If one has REC which satisfies

$$\vec{B} = \vec{M} + \mu\vec{H} \tag{3.1}$$

where μ is a scalar, one can use the previous formulation by setting

$$\vec{f}(\vec{H}) = (\mu - 1)\vec{H} \tag{3.2a}$$

Equation (3.2) then implies

$$\nabla\cdot\vec{f} = (\mu - 1)\, \nabla\cdot\vec{H} = -\frac{\mu - 1}{\mu}\, \nabla\cdot\vec{M} \tag{3.2b}$$

<u>within</u> the REC, thus leading to

$$\psi(x) = -(1/4\pi) \int dS' \frac{\vec{n}\cdot\vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} - \frac{\mu - 1}{4\pi} \int dS' \frac{\vec{n}\cdot\vec{H}(\vec{x}')}{|\vec{x} - \vec{x}'|} +$$

$$+ (1/4\pi\mu) \int_{REC} d\vec{x}' \frac{\nabla\cdot\vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} \tag{3.3}$$

The validity of Eq. (3.3) has been checked by solving the boundary value problem for an REC spherical shell with easy axis defined by

$$\begin{aligned} M_r &= M \cos(\theta) \\ M_\theta &= M \sin(\theta) \end{aligned} \tag{3.4}$$

which produces a uniform field inside the shell of strength

$$\frac{H_z}{M} = \frac{\frac{4}{2\mu+1} \ln(b/a) + \frac{2(\mu-1)}{3(\mu+1)(2\mu+1)} (1 - a^3/b^3)}{1 - \frac{2(\mu-1)^2}{(\mu+2)(2\mu+1)} (a/b)^3} \tag{3.5}$$

Here a, b are the inner, outer radii of the shell, and r, θ, ϕ are the usual spherical coordinate variables. Since the angular dependence of ψ will be proportional to $\cos\theta$, we can write

$$\begin{aligned} \psi(r,\theta,\phi) &= \psi(r) \cos(\theta) \\ H_r &= \psi'(r) \cos(\theta) \end{aligned} \tag{3.6}$$

where we take advantage of the fact that only H_r is needed in Eq. (3.3). Evaluation of Eq. (3.3) leads to

$$\psi(r) = F_1(r) + (\mu-1)F_2(r)\psi'(a+) + (\mu-1)F_3(r)\psi'(b-) + F_4(r)/\mu \tag{3.7}$$

where $F_1(r)$, $\cdots F_4(r)$ are known continuous functions with discontinuous derivatives at r = a, b, and where $\psi'(a+)$, $\psi'(b-)$ are the values of $\psi'(r)$

just within the REC shell. Differentiating Eq. (3.7) enables us to obtain two linear equations for $\psi'(a+)$, $\psi'(b-)$ which can be readily solved, and which lead to exactly the same result as in Eq. (3.5).

If one has a permeability which is different along the easy axis ($\mu_{\parallel}$) than perpendicular to it ($\mu_{\perp}$), one can write

$$\vec{B}_{\parallel} = \mu_{\parallel} \frac{(\vec{H}\cdot\vec{M})\,\vec{M}}{M^2} + \vec{M}$$

$$\vec{B} - \vec{B}_{\parallel} = \mu_{\perp}\{ \vec{H} - \frac{(\vec{H}\cdot\vec{M})\vec{M}}{M^2} \} \tag{3.8}$$

leading to

$$\vec{B} = \vec{M} + \mu_{\perp}\vec{H} + (\mu_{\parallel} - \mu_{\perp}) \frac{(\vec{H}\cdot\vec{M})\vec{M}}{M^2} \tag{3.9}$$

In this case Eq. (2.6) applies with

$$\vec{f} = (\mu_{\perp} - 1)\vec{H} + (\mu_{\parallel} - \mu_{\perp})\frac{(\vec{H}\cdot\vec{M})\vec{M}}{M^2} \tag{3.10}$$

The second of the two terms in Eq. (3.10) has been evaluated[1] for a 2-D quadrupole shell with continuous easy axis rotation, where it is shown that a small non-vanishing value of $\mu_{\parallel}-\mu_{\perp}$ leads to a 2^6 multipole term, and for a 2-D dipole which leads to a 2^3 multipole term (octupole). Clearly an iterative approach using Eq. (2.6) would enable us to obtain the fields to higher order in $\mu_{\parallel}-\mu_{\perp}$, including effects of segmentation.

IV. Numerical Approach for REC Blocks

Some simplification occurs if the REC configuration consists of blocks in each of which $\vec{M}$ is uniform. Specifically $\nabla\cdot\vec{M} = 0$ within each block.

For constant μ, Eq. (3.3) then reduces to two surface integrals, which correspond to calculation of a magnetic scalar potential from two surface distributions of magnetic charge. If one divides the surface into N small segments, each of which is planar or can be assumed so, then one can rewrite Eq. (3.3) as follows: The field $\vec{H}$ is obtained by differentiation, leading to

$$\vec{H}(\vec{x}) = \frac{1}{4\pi} \int dS \frac{\vec{n}\cdot\vec{M}(\vec{x}-\vec{x}')}{|\vec{x} - \vec{x}'|^3} + \frac{\mu - 1}{4\pi} \int dS' \frac{\vec{n}\cdot\vec{H}(\vec{x}')(\vec{x}-\vec{x}')}{|\vec{x} - \vec{x}'|^3} \tag{4.1}$$

Replacing each surface integral by the sum over surface segments, and evaluating the normal field

$$h_i = \vec{n}_i\cdot\vec{H}(\vec{x}_i) \tag{4.2}$$

at each segment, one can write

$$h_i = (1/4\pi) \sum_j \frac{\Delta S_j\ \vec{n}_j\cdot\vec{M}_j(\vec{n}_i\cdot\vec{x}_i - \vec{n}_i\cdot\vec{x}_j)}{|\vec{x}_i - \vec{x}_j|^3} +$$

$$+ \frac{\mu-1}{4\pi} \sum_j \frac{\Delta S_j\ h_j(\vec{n}_i\cdot\vec{x}_i - \vec{n}_i\cdot\vec{x}_j)}{|\vec{x}_i - \vec{x}_j|^3} \tag{4.3}$$

Some care is needed when i = j, but it is well known that a surface charge density σ produces a field $\pm\ \sigma/2$ on each side of the surface. Thus the contribution to the sums in Eq. (4.2) at i = j on the <u>interior</u> side of the surface segment is

$$- \frac{\vec{n}_i \cdot \vec{M}_i}{2} - \frac{\mu - 1}{2} h_i \tag{4.4}$$

We therefore can write

$$\frac{\mu + 1}{2} h_i = \frac{1}{4\pi} \sum_{j \neq i} \frac{\Delta S_j (\vec{n}_j \cdot \vec{M}_j)(\vec{n}_i \cdot \vec{x}_i - \vec{n}_i \cdot \vec{x}_j)}{|\vec{x}_i - \vec{x}_j|^3} +$$

$$- \frac{\vec{n}_i \cdot \vec{M}_i}{2} + \frac{\mu - 1}{4\pi} \sum_{j \neq i} \frac{\Delta S_j h_j (\vec{n}_i \cdot \vec{x}_i - \vec{n}_i \cdot \vec{x}_j)}{|\vec{x}_i - \vec{x}_j|^3} \tag{4.5}$$

Equation (4.5) is therefore a set of N equations for the N unknowns h_i. Once these are found numerically, Eq. (4.1) can be used in the sum form to obtain the field at any location x_i. Note that the discontinuity in $\vec{n} \cdot \vec{H}$ at an REC surface is taken care of by the discontinuity of the normal field as one crosses a charged surface.

If $\mu_{\parallel} \neq \mu_{\perp}$, or if $\vec{f}(\vec{H})$ represents a non-linear relationship, we must return to Eq. (2.6) which now includes a volume integral. This requires dividing the segments into sub-blocks, and evaluating $\nabla \cdot \vec{f}$ approximately in each sub-block. One then can once again obtain a soluble system of linear equations for the field in each sub-unit (surface or volume) into which the region of integration has been divided.

V. Summary

We have obtained an integral equation for the magnetic field produced by a configuration of REC segments in which the relationship between $\vec{B}$ and $\vec{H}$ is either non-linear or corresponds to a permeability different from unity. A computational procedure is then outlined in which successive iterations will

be rapidly convergent if the permeability is close to unity, or if the non-linear part of the relationship between $\vec{B}$ and $\vec{H}$ is small. In those cases in which the REC segments have uniform magnetization and a boundary made up of planar surfaces, it is possible to write a set of self-consistent linear equations for the field at a finite number of points on the surface of, and within the REC segments, which can be readily solved.

VI. Acknowledgement

The author would like to thank Fermilab for its hospitality during the sabbatical semester in which this work was carried out.