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Exact Solutions for the Longitudinal and Transverse Impedances of an Off-Centered Beam in a Rectangular Beampipe

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I. Introduction

In storage rings and accumulators, some sections of the beampipe may be rectangular in cross section and the beam may not be at the center of the beampipe. In this note, through conformal mapping, we try to compute exactly the longitudinal, vertical and horizontal transverse impedances for the beam under these circumstances. For simplicity, we restrict the beam to the horizontal symmetry axis of the beampipe. Both the effects due to the resistivity of the beampipe's wall and space-charge are considered.

II. Conformal Mapping

The rectangular cross section of the beampipe is placed in the upper half of the complex z-plane as shown in Figure 1a. The beam is on the y-axis at a distance y_0 from the origin of the plane. It is rather unfortunate that x and y are used here to denote respectively the vertical and horizontal directions. To avoid confusion, subscripts and superscripts V and H will be used below to distinguish quantities that are vertical and horizontal respectively.

Making a conformal mapping onto the z_1-plane by

$$z_1 = sn\left(2K(k)z/h,\ k\right), \tag{1}$$

the sides of the rectangle $AB_-C_-C_+B_+A$ open up into the x_1-axis $A'B'_-C'_-C'_+B'_+A'$

as shown in Figure 1b. The interior of the beampipe is therefore mapped onto the complete upper z_1-plane while the exterior the complete lower z_1-plane. In Eq. (1), K(k) is the complete elliptic function of the first kind

$$K(k) = \int_0^1 (1-t^2)^{-\frac{1}{2}}(1-k^2t^2)^{-\frac{1}{2}}dt, \tag{2}$$

and k is related to the width w and height h of the rectangle by

$$K'(k)/K(k) = 2w/h, \tag{3}$$

with

$$K'(k) = K(k') \tag{4}$$

and

$$k'^2 = 1 - k^2 \tag{5}$$

An actual method to determine k is by an expansion in terms of the "nome" $q = \exp[-\pi K'(k)/K(k)]$:

$$k' = \left(1 + 2\sum_{n=1}^{\infty} q^{n^2}\right)/\left(1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\right). \tag{6}$$

As shown in Figure 1 and using Eq. (1), the corners of the rectangles $B_\pm$ and $C_\pm$ are mapped into the points $(\pm k^{-1},0)$ and $(\pm 1,0)$ at $B'_\pm$ and $C'_\pm$ while the position of the beam is mapped into

$$\begin{aligned} y_{10} &= \mathrm{Im}\ \mathrm{sn}\left[i\,2K(k)y_0/h,\ k\right] \\ &= \mathrm{sc}\left[2K(k)y_0/h,\ k'\right]. \end{aligned} \tag{7}$$

III. Longitudinal Impedance due to Resistive Wall

Let us first assume the wall of the beampipe to be perfectly conducting. Thus the x_1-axis in the z_1-plane is of zero potential. For a line current I at y_0 in the z-plane or y_{10} z_1-plane, the image current density induced on the x_1-axis is

$$J_1(x_1) = -\frac{I}{\pi}\frac{y_{10}}{x_1^2 + y_{10}^2}. \tag{8}$$

Resistivity ρ is now introduced to the wall of the beampipe. The corresponding skin depth at disturbance circular frequency ω is $\delta = (2\rho/\mu\omega)^{\frac{1}{2}}$, μ being the magnetic permeability of the beampipe material. The average power dissipated at this plane per unit length is therefore

$$P_1 = \frac{1}{2}\frac{\rho}{\delta}\int_{-\infty}^{\infty} J^2 dx_1 . \tag{9}$$

In a conformal mapping the fundamental relation that connects all the physical quantities in the two planes is

$$|E(z)||dz| = |E(z_1)||dz_1| , \tag{10}$$

where E(z) and dz are respectively the electric field and a line element at z in the z-plane while $E(z_1)$ and dz_1 are respectively the electric field and the line element at the transformed point z_1 in the z_1-plane. The image current density J in a resistive wall is essentially proportional to the electric field in the wall. Thus the actual average power lost at the wall of the beampipe per unit length (in the z-plane) is

$$P = \frac{1}{2}\frac{\rho}{\delta}\int_{-\infty}^{\infty} J^2 \left|\frac{dz_1}{dz}\right| dx_1 . \tag{11}$$

Therefore, the real part of the longitudinal impedance due to beampipe walls is

$$Re\ Z_L\Big|_{wall} = \frac{2P\ell}{I^2} , \tag{12}$$

where ℓ is the length of the section of the rectangular beampipe. Substituting Eqs. (1) and (8) into Eq. (11) and inserting the imaginary part of Eq. (12), we finally arrive at

$$Z_L\Big|_{wall} = (1-i)\frac{2\ell K(k)}{\pi^2 h}\frac{\rho}{\delta}\, 2y_{10}\int_0^{\infty}\frac{|(1-x_1^2)(1-k^2x_1^2)|^{\frac{1}{2}}}{(x_1^2+y_{10}^2)^2}dx_1 , \tag{13}$$

where y_{10}, the position of the beam in the z_1-plane is given by Eq. (7). Equation (13) can be rewritten in a more elegant form; i.e.,

$$Z_L\Big|_{wall} = (1-i)\left(\frac{\ell}{\pi h}\frac{\rho}{\delta}\right)F\left(g, \frac{w}{h}\right),$$

where the first factor is exactly the resistive wall longitudinal impedance of a section of a <u>circular</u> beampipe of radius h/2 with the beam at the center, while the form factor

$$F\left(g, \frac{w}{h}\right) = \frac{4K(k)}{\pi} y_{10}^2 \int_0^\infty \frac{\left|(1-x_1^2)(1-k^2x_1^2)\right|^{\frac{1}{2}}}{(x_1^2+y_{10}^2)^2}\, dx_1 \tag{14}$$

takes into account that the beampipe is rectangular in shape with a width to height ratio w/h and that the beam is displaced from the center of the beampipe gw/2. In terms of g, y_{10} is given by

$$y_{10} = sc\left[K(k)(1-g)w/h\,,\, k'\right]. \tag{15}$$

The form factor F(g, w/h) is plotted in Figure 2 for various values of w/h. We note the following properties:

1. For a square beampipe (w/h = 1), the form factor starts from 1 and increases very rapidly when the beam is displaced from the center of the beampipe. We see that it differs very little from the form factor[1]

$$F_{cir}(g) = \frac{1+g^2}{1-g^2} \tag{16}$$

(the dashed curve) for a displaced beam in a circular beampipe.

2. When w/h is big, the form factor stays flat for quite a wide range of g. For example, w/h = 2, the form factor is unity within 5% up to g = 0.56. As a result, for a flat rectangular beampipe, the form factor

differs not much from unity unless the beam is very near to the wall.

3. With the beam at the center of the rectangular beampipe, the form factor is unity at w/h = 1, then decreases to a minimum of 0.94 at w/h ~1.35 and finally increases to unity again when $w/h \to \infty$. Thus, for a beam at the center, the form factor is $0.94 < F(0, w/h) < 1$; or $F(0, w/h) = 1$ within 6%.

4. When $w/h \to \infty$, the case of two parallel plates, the form factor equals unity exactly. This can be derived by integrating Eq. (14) exactly in the limit of $k \to 0$.

IV. Longitudinal Impedance due to Space Charge

The particle beam is assumed to be circular in cross section with radius a and uniform distribution. When $(\frac{\pi a}{h}) \ll 1$, it can easily be shown that the beam in the transformed z_1-plane is still roughly circular in shape with radius

$$a_1 = a \cdot \frac{2K(k)}{h} \frac{dn[2K(k)y_0/h, k']}{cn^2[2K(k)y_0/h, k']} \tag{17}$$

and uniformly distributed.

The contribution to Z_L due to space-charge effect can be obtained by computing the longitudinal electric field at the beam using Ampere's Law:

$$\oint \vec{E} \cdot d\vec{\ell} = - \oint \dot{\vec{B}} \cdot d\vec{S} . \tag{18}$$

The loop is shown in Figure 3. In the z_1-plane the electric field along the y_1 axis is

$$E_y = \begin{cases} \frac{\lambda}{2\pi\epsilon_0}\left(\frac{y_{10}-y_1}{a_1^2} + \frac{1}{y_{10}+y_1}\right) & y_{10}-a_1 \leq y_1 \leq y_{10}, \\ \frac{\lambda}{2\pi\epsilon_0}\left(\frac{1}{y_{10}-y_1} + \frac{1}{y_{10}+y_1}\right) & 0 \leq y_1 \leq y_{10}-a_1 . \end{cases} \tag{19}$$

The magnetic field perpendicular to the loop is

$$B_x = \frac{\mu_0 \epsilon_0 I}{\lambda} E_y , \tag{20}$$

where the disturbed beam line charge density λ is related to the disturbed current I by $I = \lambda\beta_w c$ with $\beta_w c$ equal to the phase velocity of the disturbance.

Putting everything into Eq. (18), we get for the self-field at the beam

$$E_z = \frac{i\omega Z_0 I}{2\pi c\beta_w^2\gamma_w^2}\left\{\frac{1}{2} + \ln\left[\frac{h}{aK(k)}\frac{sn.cn}{dn}\right]\right\} \tag{21}$$

leading to a longitudinal impedance per harmonic ($Z_0 = 120\pi$ ohms)

$$\left(\frac{Z_L}{n}\right)_{sp\,ch} = -\frac{iZ_0\ell}{2\pi R\beta_w\gamma_w^2}\left\{\frac{1}{2} + \ln\left[\frac{h}{aK(k)}\frac{sn.cn}{dn}\right]\right\} \tag{22}$$

where the elliptic functions have arguments $(K(k)(1-g)w/h,\ k)$, $\gamma_w^2 = 1-\beta_w^2$ and R is the average radius of the storage ring. In practice, when $w/h > 1$, $k^2 \ll 1$ and Eq. (22) simplifies to

$$\left(\frac{Z_L}{n}\right)_{sp\,ch} = -i\frac{\ell}{2\pi R}\frac{Z_0}{\beta_w\gamma_w^2}\left\{\frac{1}{2} + \ln\left[\frac{2h}{\pi a}\tanh\frac{\pi W}{2h}(1-g)\right]\right\}. \tag{23}$$

As a comparison, the space-charge contribution to the longitudinal impedance of a beam inside a circular beampipe of radius b is[1]

$$\left(\frac{Z_L}{n}\right)_{sp\,ch} = -i\frac{\ell}{2\pi R}\frac{Z_0}{\beta_w\gamma_w^2}\left\{\frac{1}{2} + \ln\left(\frac{b}{a}\frac{1-g^2}{1+g^2}\right)\right\}, \tag{24}$$

which is valid when $a/b \ll 1$.

V. Transverse Impedances due to Resistive Walls

In order to obtain the transverse impedances, we place the beam at $x_0 + iy_0$ and displace it by an amount Δ in the x or y direction, compute the effects on the beam and then set $x_0 = 0$. In the transformed z_1-plane, the image current of the unperturbed beam on the walls

$$J_1 = \frac{I}{\pi}\frac{y_{10}}{(x_1 - x_{10})^2 + y_{10}^2} \tag{25}$$

will be changed accordingly by

$$\Delta \frac{\partial J_1}{\partial x_0}\bigg|_{x_0=0} = \Delta \left\{ \frac{\partial J_1}{\partial x_{10}} \frac{\partial x_{10}}{\partial x_0} + \frac{\partial J_1}{\partial y_{10}} \frac{\partial y_{10}}{\partial x_0} \right\}_{x_0=0} ,$$

$$\Delta \frac{\partial J_1}{\partial y_0}\bigg|_{x_0=0} = \Delta \left\{ \frac{\partial J_1}{\partial x_{10}} \frac{\partial x_{10}}{\partial y_0} + \frac{\partial J_1}{\partial y_{10}} \frac{\partial y_{10}}{\partial y_0} \right\}_{x_0=0} . \tag{26}$$

Since the mapping is conformal,

$$\frac{\partial y_{10}}{\partial x_0}\bigg|_{x_0=0} = \frac{\partial x_{10}}{\partial y_0}\bigg|_{x_0=0} = 0 , \tag{27}$$

while using Eq. (1),

$$\frac{\partial x_{10}}{\partial x_0}\bigg|_{x_0=0} = \frac{\partial y_{10}}{\partial y_0}\bigg|_{x_0=0} = \frac{2K(k)}{h} \frac{dn\left[2K(k)y_0/h,\ k'\right]}{cn^2\left[2K(k)y_0/h,\ k'\right]} . \tag{28}$$

Similar to Eq. (11) in the longitudinal case, using the fundamental concept of Eq. (10), the actual average power losses due to these dipole motions of the beam are

$$P_{V,H} = \frac{1}{2} \frac{\rho}{\delta} \int_{-\infty}^{\infty} dx_1\ \Delta^2 \left| \frac{dz_1}{dz} \right| \begin{Bmatrix} (\partial J_1/\partial x_0)^2 \\ (\partial J_1/\partial y_0)^2 \end{Bmatrix}_{x_0=0} , \tag{29}$$

which can also be written as

$$P_{V,H} = -\frac{1}{2} \mathcal{R}e\ I\Delta \begin{Bmatrix} \partial E_s(z_1, z_0)/\partial x \\ \partial E_s(z_1, z_0)/\partial y \end{Bmatrix}_{z=z_0=iy_0}$$

$$= -\frac{1}{2} \mathcal{R}e\ I\Delta \begin{Bmatrix} -i\omega B_y \\ i\omega B_x \end{Bmatrix} , \tag{30}$$

where $E_s(z, z_0)$ is the longitudinal electric field at z due to a current I

at z_0 while, through Ampere's Law, B_x and B_y are the corresponding magnetic field in the x and y directions. The transverse impedances are defined by

$$Z_{V,H} = -\frac{i}{\beta I \Delta} \int_{\ell} \left(\vec{E} + \vec{v} \times \vec{B} \right)_{V,H} ds \tag{31}$$

$$= \pm \frac{i c \ell}{I \Delta} B_{y,x} ,$$

where $v = \beta c$ is the velocity of the particles in the beam.

Putting Eq. (29) into Eq. (30) and then into Eq. (31) we get

$$\mathcal{R}e\ Z_{V,H}\Big|_{wall} = \frac{2c\ell}{\omega (I\Delta)^2} P_{V,H}$$

$$= \frac{\rho}{\delta} \frac{c\ell}{\omega} \int_{-\infty}^{\infty} dx_1 \left| \frac{dz_1}{dz} \right| I^{-2} \left\{ \begin{matrix} \left(\partial J_1 / \partial x_0 \right)^2 \\ \left(\partial J_1 / \partial y_0 \right)^2 \end{matrix} \right\}_{x_0 = 0} \tag{32}$$

Using Eqs. (1), (25), (26), (27) and (28) and including the imaginary part, we finally arrive at

$$Z_{V,H}\Big|_{wall} = (1-i) \frac{c\rho\ell}{\pi\omega\delta} \left(\frac{2}{h}\right)^3 F_{V,H}\left(g, \frac{w}{h}\right), \tag{33}$$

where the first factor represents the wall resistive part of the transverse impedance for a section of a circular beampipe of length ℓ and radius h/2 with the beam at the center. The form factors $F_{V,H}(g, w/h)$ take into account that the beampipe is rectangular and that the beam is displaced by gw/2 from the center. These form factors are given by

$$F_V\left(g, \frac{w}{h}\right) = \frac{8K^3(k)}{\pi} \frac{dn^2[K(k)(1-g)w/h, k']}{cn^4[K(k)(1-g)w/h, k']} y_{10}^2 \int_0^{\infty} \frac{x_1^2 \left| (1-x_1^2)(1-k^2x_1^2) \right|^{\frac{1}{2}}}{(x_1^2 + y_{10}^2)^4} dx_1 \tag{34}$$

and

$$F_H\left(g,\frac{w}{h}\right) = \frac{2K^3(k)}{\pi}\,\frac{dn^2\left[K(k)(1-g)w/h,\,k'\right]}{cn^4\left[K(k)(1-g)w/h,\,k'\right]}\int_0^\infty \frac{(x_1^2-y_{10}^2)^2}{(x_1^2+y_{10}^2)^4}\left|(1-x_1^2)(1-k^2x_1^2)\right|^{\frac{1}{2}}dx_1 \tag{35}$$

with

$$y_{10} = sc\left[K(k)(1-g)w/h\,,\,k'\right]. \tag{36}$$

When $w/h \geqslant 1$ (or $k^2/16 < 0.00184$), Eqs. (34) and (35) can be readily simplified to

$$F_V\left(g,\frac{w}{h}\right) = \pi^2\,\frac{\sinh^2\left[\pi(1-g)w/2h\right]}{\operatorname{sech}^2\left[\pi(1-g)w/2h\right]}\int_0^\infty \frac{x_1^2\left|(1-x_1^2)(1-k^2x_1^2)\right|^{\frac{1}{2}}}{(x_1^2+y_{10}^2)^4}dx_1\,, \tag{37}$$

$$F_H\left(g,\frac{w}{h}\right) = \frac{\pi^2}{4}\cosh^2\left[\pi(1-g)w/2h\right]\int_0^\infty \frac{(x_1^2-y_{10}^2)^2}{(x_1^2+y_{10}^2)^4}\left|(1-x_1^2)(1-k^2x_1^2)\right|^{\frac{1}{2}}dx_1\,. \tag{38}$$

These form factors are plotted in Figure 4 for various values of w/h. We note the following properties:

1. For a square beampipe (w/h = 1), $F_V(g,1) = F_H(g,1)$. They start from 0.8594 when the beam is at the center and increase rather rapidly when the beam is displaced from the center and approach the form factors[1]

$$F_{V,H}^{cir}(g) = \frac{1+g^2}{(1-g^2)^2} \tag{39}$$

(plotted in dash-dotted curve) for a displaced beam in a circular beampipe when g is large.

2. When $w/h \to \infty$ the form factors can be integrated exactly to give

$$F_V(g,\infty) = \pi^2/12 , \tag{40}$$

$$F_H(g,\infty) = \pi^2/24 , \tag{41}$$

agreeing with those for two parallel plates[2]. We see that even when $w/h = \infty$ Z_H is <u>not</u> zero but is of the same magnitude as Z_V. This is because the transverse impedance is defined as the effect of the oscillating beam images on the center of the <u>unperturbed</u> beam and only the effects of the oscillating beam images on the perturbed or oscillating beam itself is zero when $w/h = \infty$.

We further note that $F_V(0, w/h)$ and $F_H(0, w/h)$ approach the above limits very rapidly (to within 2% when $w/h > 1.10$ for F_V and $w/h > 1.55$ for F_H). However, the change in F_V is very small ~5% but the change in F_H is much bigger ~52%. This is because for Z_V the contribution comes from the image currents on the top and bottom of the beampipe which do not change by very much when w/h increases. For Z_H the contribution comes from the image currents on the two vertical sides of the beampipe when $w/h = 1$. As w/h increases these image currents move into the top and bottom of the beampipe and this change is big.

3. The form factors increase when g increases. Although they stay flat for a range of g when w/h is big, as a whole, the variation is much bigger than that for the longitudinal form factor in Section III. In general, the variation of F_H as a function of g is bigger than that of F_V for the reason stated in the above paragraph.

VI. Transverse Impedances due to Space-Charge

In this section, we first compute the electric image coefficients $\varepsilon_{V,H}$ and $\xi_{V,H}$ for incoherent and coherent tuneshifts defined by Zotter[2]. Then the transverse impedances due to the electric effects of the images are derived. Finally, the transverse impedances due to the complete space-charge effects of the images and the source are constructed.

When a line charge of density λ is placed at (x_0, y_0) in the z-plane or (x_{10}, y_{10}) in the z_1-plane, the potential at a point $z = (x, y)$ due to the source and the wall reflections of the beampipe is

$$\phi(z) = -\frac{\lambda}{4\pi\epsilon_0} \ln \frac{(x_1 - x_{10})^2 + (y_1 - y_{10})^2}{(x_1 - x_{10})^2 + (y_1 + y_{10})^2} . \tag{42}$$

The actual electric field is given by

$$E_x = -\left(\frac{\partial\phi}{\partial x_1}\frac{\partial x_1}{\partial x} + \frac{\partial\phi}{\partial y_1}\frac{\partial y_1}{\partial x}\right) , \tag{43}$$

$$E_y = -\left(\frac{\partial\phi}{\partial x_1}\frac{\partial x_1}{\partial y} + \frac{\partial\phi}{\partial y_1}\frac{\partial y_1}{\partial y}\right) . \tag{44}$$

Since we are interested in the image effects at the moment, the field due to the source itself must be subtracted away, thus giving

$$E_x^{im} = \frac{\lambda}{2\pi\epsilon_0}\left\{\frac{(x_1 - x_{10})\partial x_1/\partial x + (y_1 - y_{10})\partial y_1/\partial x}{(x_1 - x_{10})^2 + (y_1 - y_{10})^2} - \frac{1}{[(x - x_0)^2 + (y - y_0)^2]^{1/2}}\right.$$

$$\left. - \frac{(x_1 - x_{10})\partial x_1/\partial x + (y_1 + y_{10})\partial y_1/\partial x}{(x_1 - x_{10})^2 + (y_1 + y_{10})^2}\right\} , \tag{45}$$

$$E_y^{im} = \frac{\lambda}{2\pi\epsilon_0}\left\{\frac{(x_1 - x_{10})\partial x_1/\partial y + (y_1 - y_{10})\partial y_1/\partial y}{(x_1 - x_{10})^2 + (y_1 - y_{10})^2} - \frac{1}{[(x - x_0)^2 + (y - y_0)^2]^{1/2}}\right.$$

$$\left. - \frac{(x_1 - x_{10})\partial x_1/\partial y + (y_1 + y_{10})\partial y_1/\partial y}{(x_1 - x_{10})^2 + (y_1 + y_{10})^2}\right\} . \tag{46}$$

The electric image coefficients for incoherent and coherent tune-shifts are defined as[2]

$$\mathcal{E}_1^V = \frac{\pi\epsilon_0 h^2}{4\lambda}\frac{\partial E_x^{im}}{\partial x}\Big|_{\substack{x=x_0=0\\ y=y_0}}, \tag{47}$$

$$\mathcal{E}_1^H = \frac{\pi\epsilon h^2}{4\lambda}\frac{\partial E_y^{im}}{\partial y}\Big|_{\substack{x=x_0=0\\ y=y_0}}, \tag{48}$$

$$\xi_1^V = \frac{\pi\epsilon_0 h^2}{4\lambda}\left(\frac{\partial E_x^{im}}{\partial x} + \frac{\partial E_x^{im}}{\partial x_0}\right)_{\substack{x=x_0=0\\ y=y_0}}, \tag{49}$$

$$\xi_1^H = \frac{\pi\epsilon_0 h^2}{4\lambda}\left(\frac{\partial E_y^{im}}{\partial y} + \frac{\partial E_y^{im}}{\partial y_0}\right)_{\substack{x=x_0=0\\ y=y_0}}, \tag{50}$$

for a beam at a point y_0 on the horizontal symmetry axis of the rectangular beampipe. The computation is straightforward by using the transformation Equation (1), which leads to

$$x_1 = \frac{sn\cdot dn_1}{cn_1^2 + k^2 sn^2 sn_1^2}, \tag{51}$$

$$y_1 = \frac{cn\cdot dn\cdot sn_1\cdot cn_1}{cn_1^2 + k^2 sn^2 sn_1^2}, \tag{52}$$

where sn, cn, dn have arguments (2K(k)x/h, k) while sn, cn, dn have arguments (2K(k)y/h,k'). However, care must be taken to cancel the fictitious poles at the source location before taking the limit $x = x_0$ and $y = y_0$. One way to do this is to expand Eqs. (45) and (46) as Laurent series and pick out the suitable terms for the image coefficients. Our results are

$$\mathcal{E}_1^V = -\mathcal{E}_1^H = \frac{K^2(k)}{4}\left\{\frac{k'^4 sn_{10}^2 cn_{10}^2}{2dn_{10}^2} - \frac{k'^2(1-2sn_{10}^2)}{3} - \frac{dn_{10}^2(3-4sn_{10}^2-4sn_{10}^4)}{6sn_{10}^2 cn_{10}^2}\right\}, \tag{53}$$

$$\xi_1^V = \frac{K^2(k)}{4}\frac{k'^4 sn_{10}^2 cn_{10}^2}{dn_{10}^2}, \tag{54}$$

$$\xi_1^H = \frac{K^2(k)}{4}\left\{k'^2(1-2sn_{10}^2) + \frac{dn_{10}^2}{sn_{10}^2 cn_{10}^2}\right\}, \tag{55}$$

where sn_{10}, cn_{10}, dn_{10} have arguments $(2K(k)y_0/h, k')$ or $(K(k)(1-g)w/h, k')$. For $w/h \to \infty$, the case of parallel plates, Eqs. (53), (54) and (55) become

$$\mathcal{E}_1^V = -\mathcal{E}_1^H = \frac{\pi^2}{48},$$

$$\xi_1^V = \frac{\pi^2}{16},$$

$$\xi_1^H = 0, \tag{56}$$

in agreement with Zotter's results[2].

The vertical force on a beam oscillating vertically with amplitude Δ is $\Delta \cdot \partial F(z, z_1)/\partial x_1$ evaluated at the location of the unperturbed beam. The force is composed of the electric part and the magnetic part. From Maxwell's equations, the latter is just β^2 times the former and opposite in sign. Thus, according to the definition given by Eq. (31), the transverse impedances due to the space-charge effects of the images are related to the electric image tuneshifts coefficients by

$$Z_{V,H}^{im}\Big|_{sp\,ch} = -\frac{il}{e\beta I_0}\frac{4e\lambda}{\pi\varepsilon_0 h^2}\frac{1}{\gamma^2}\left(\xi_1^{V,H} - \mathcal{E}_1^{V,H}\right), \tag{57}$$

where $\gamma^2 = (1-\beta^2)^{-1}$. The contribution due to the beam itself is[1]

$$Z_{V,H}^{self}\Big|_{sp\,ch} = \frac{il Z_0}{2\pi\beta^2\gamma^2}\frac{1}{a^2}, \tag{58}$$

which depends on the fields at the very edge of the beam and is therefore independent of the geometry of the beampipe when the beam radius is much smaller than its distance from the wall of the beampipe. Combining Eqs. (57) and (58), we obtain for the transverse impedances due to space-charge

$$Z_{V,H}\Big|_{spch} = \frac{i\ell Z_0}{2\pi\beta^2\gamma^2}\left\{\frac{1}{a^2} - \frac{8}{h^2}\left(\xi_1^{V,H} - \varepsilon_1^{V,H}\right)\right\}. \tag{59}$$

Substituting Eqs. (53), (54) and (55) into Eq. (59), we finally get

$$Z_V\Big|_{spch} = \frac{i\ell Z_0}{2\pi\beta^2\gamma^2}\left\{\frac{1}{a^2} - \frac{2K^2(k)}{h^2}\left[\frac{k'^4 sn_{10}^2 cn_{10}^2}{2dn_{10}^2} + \frac{k'^2(1-2sn_{10}^2)}{3}\right.\right.$$

$$\left.\left. + \frac{dn_{10}^2(3-4sn_{10}^2+4sn_{10}^4)}{6sn_{10}^2 cn_{10}^2}\right]\right\}, \tag{60}$$

$$Z_H\Big|_{spch} = \frac{i\ell Z_0}{2\pi\beta^2\gamma^2}\left\{\frac{1}{a^2} - \frac{2K^2(k)}{h^2}\left[\frac{k'^4 sn_{10}^2 cn_{10}^2}{2dn_{10}^2} + \frac{2k'^2(1-2sn_{10}^2)}{3}\right.\right.$$

$$\left.\left. + \frac{dn_{10}^2(3+4sn_{10}^2-4sn_{10}^4)}{6sn_{10}^2 cn_{10}^2}\right]\right\}, \tag{61}$$

where the arguments for sn_{10}, cn_{10} and dn_{10} are $(K(k)(1-g)w/h,\ k)$.

The quantities $\varepsilon_1^V - \xi_1^V$ and $\varepsilon_1^H - \xi_1^H$ are plotted in Figure 5 as functions of g for various values of w/h. Their properties are similar to those of the form factors F_V and F_H in the last section. The comments there apply to here also. In Figure 5, we have also plotted for comparison[2]

$$\varepsilon_1^{V,H} - \xi^{V,H} = \frac{1}{2(1-g)^2}$$

for an off-centered beam inside a circular beampipe.

VI. Computer program

A handy computer program "RECT" has been written to compute the form factors F_L, F_V, F_H for the longitudinal, vertical and horizontal transverse impedances due to the resistive walls of a rectangular beampipe with the beam on the horizontal axis of the beampipe but displaced from the center.

Also computed are the electric image tuneshifts ε_1^V, ε_1^H, ξ_1^V, ξ_1^H as well as $\varepsilon_1^V - \xi_1^V$ and $\varepsilon_1^H - \xi_1^H$ which are proportional to the vertical and horizontal transverse impedance due to the space-charge effects of the images.

This program can be readily accessed from a Fermilab terminal by the statement: GET, RECT/UN=94786 with CERNLIB attached. The input is in the namelist form with group name "DATA". An example is $DATA N = 50, WH = 3.50$, where WH is the ratio of the width to the height of the rectangular beampipe. The results are given for each deviation of the beam from the center of the beampipe from g = 0 to 1 - 1/N in steps of 1/N, (g = 0.00, 0.02, 0.04, ···, 0.98 in the above example). The default for N is 100. Many sets of data can be entered at the same time.

References

1. King-Yuen Ng, Fermilab UPC-149

2. B. Zotter, CERN/ISR-TH/74-11

   B. Zotter, Nucl, Inst. and Meth. 129, 377 (1975)

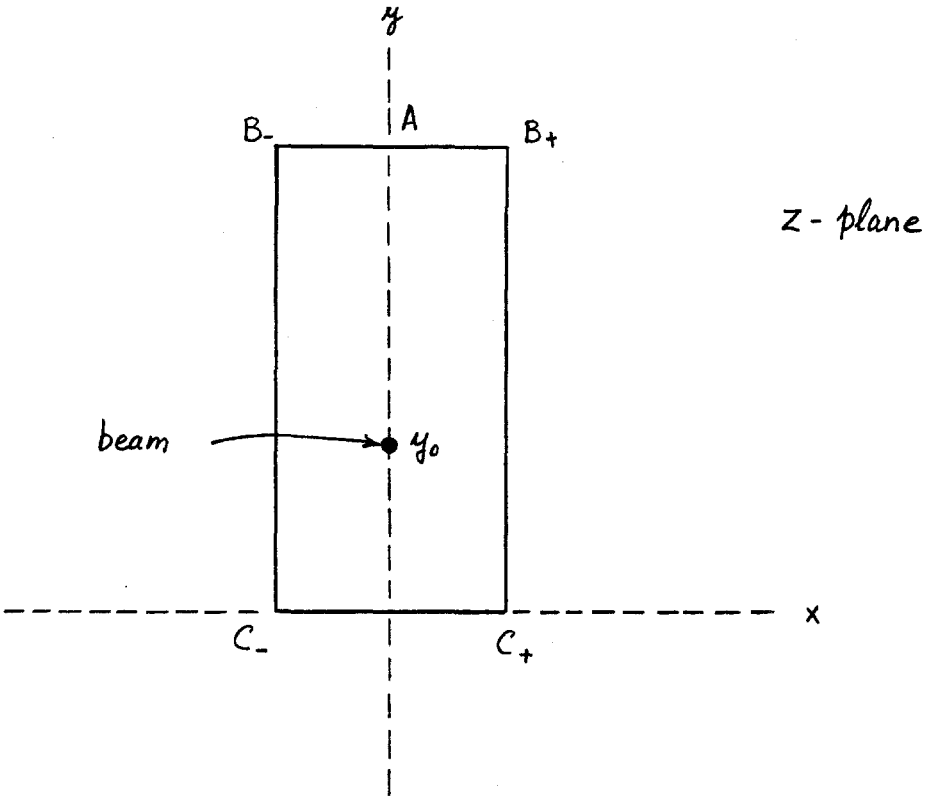

Figure 1 (a)

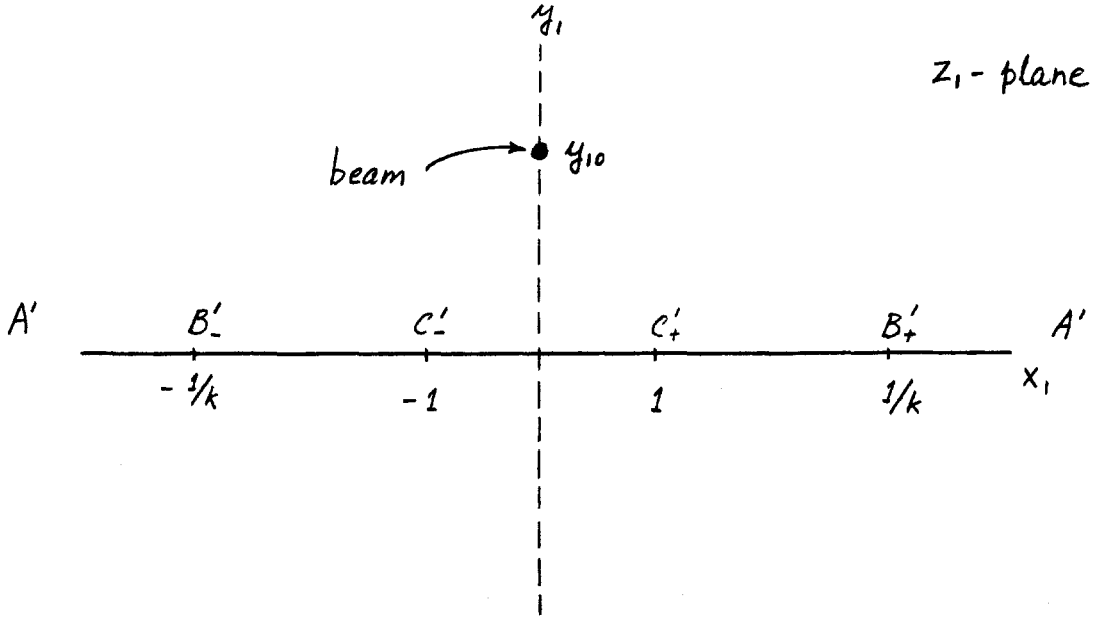

Figure 1 (b)

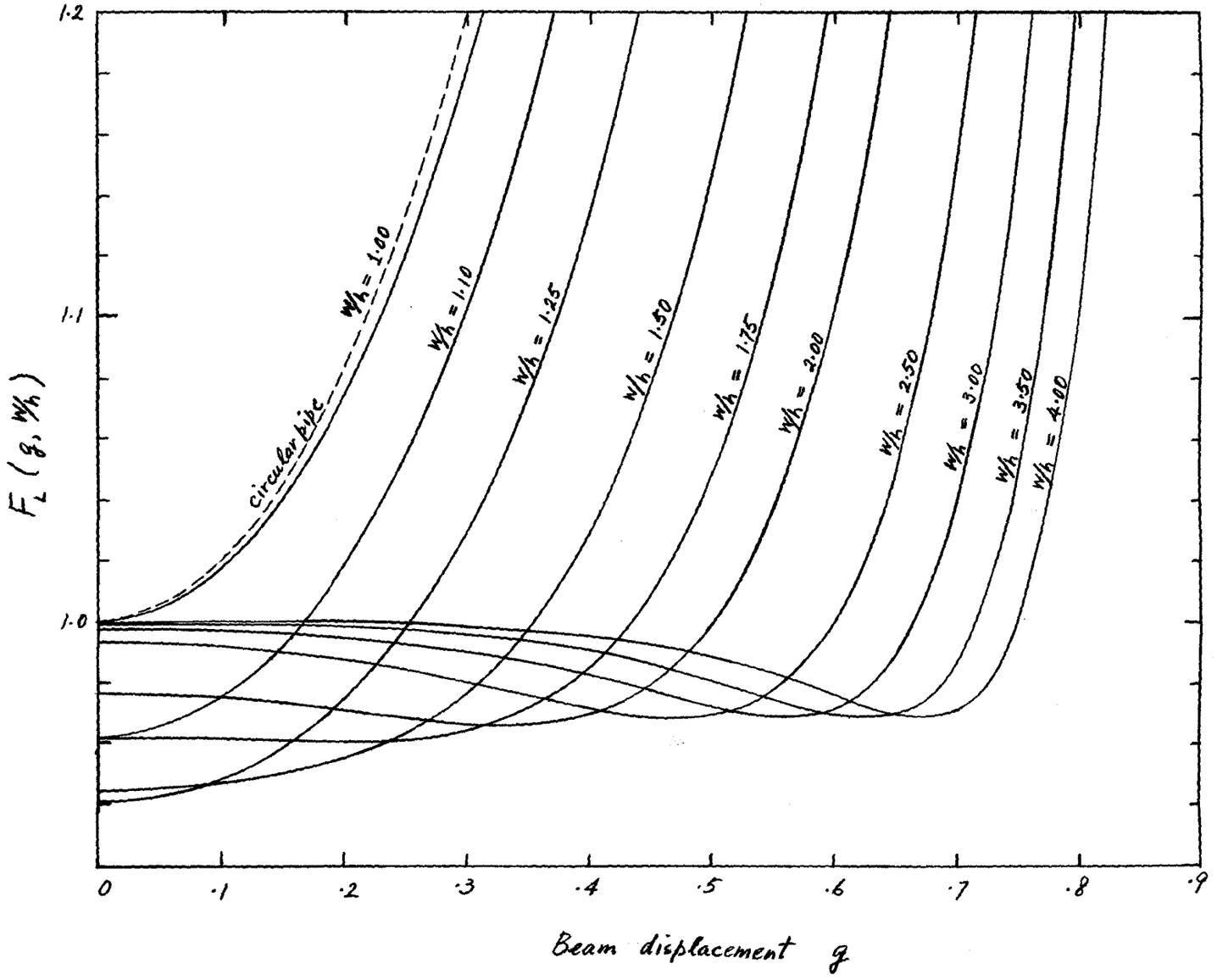
1·2
1·1
1·0
$F_L(g, W/h)$
circular pipe
W/h = 1·00
W/h = 1·10
W/h = 1·25
W/h = 1·50
W/h = 1·75
W/h = 2·00
W/h = 2·50
W/h = 3·00
W/h = 3·50
W/h = 4·00
0
·1
·2
·3
·4
·5
·6
·7
·8
·9
Beam displacement g

Figure 2

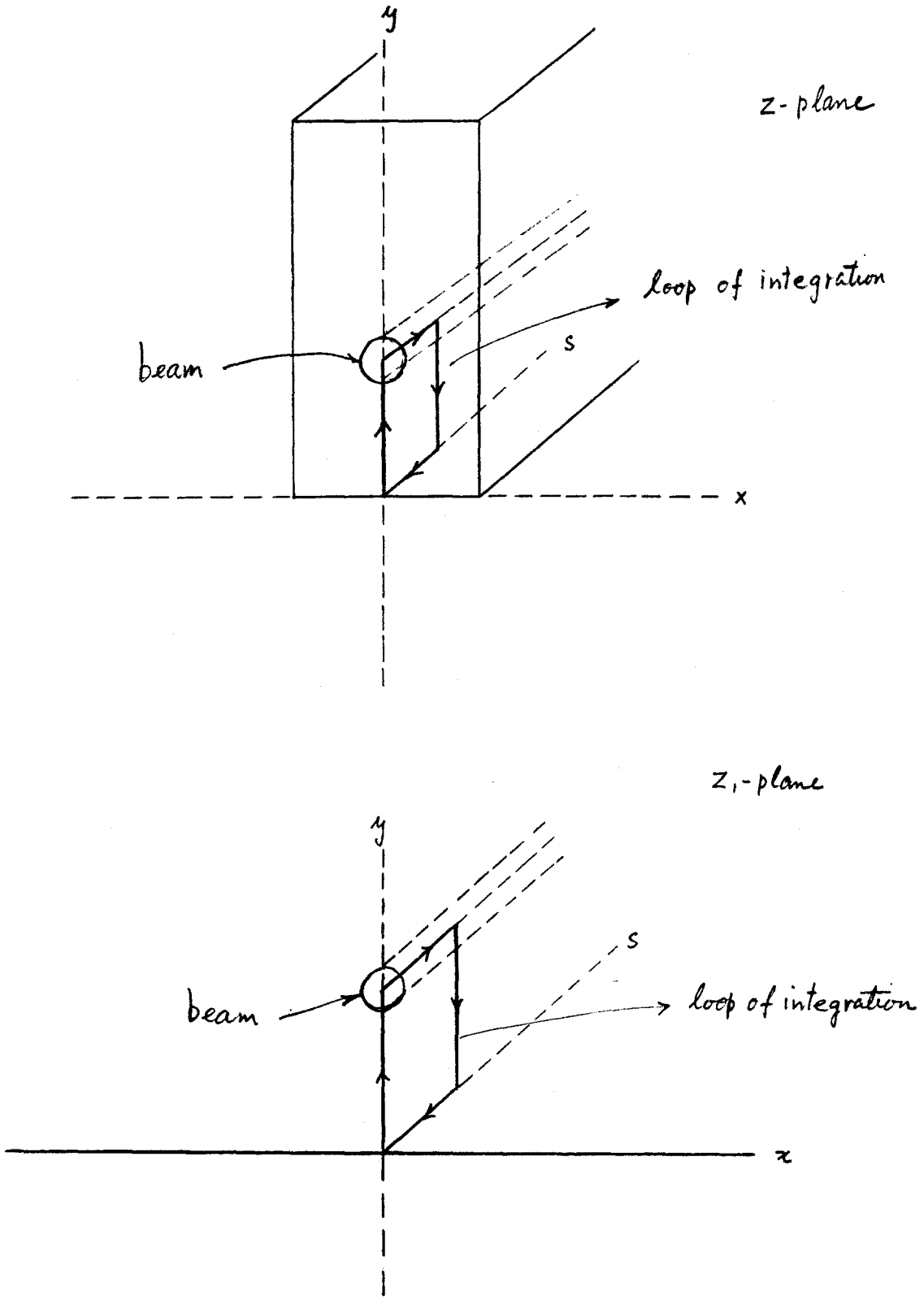
y
z-plane
loop of integration
beam
s
x
z₁-plane
y
s
beam
loop of integration
x

Figure 3

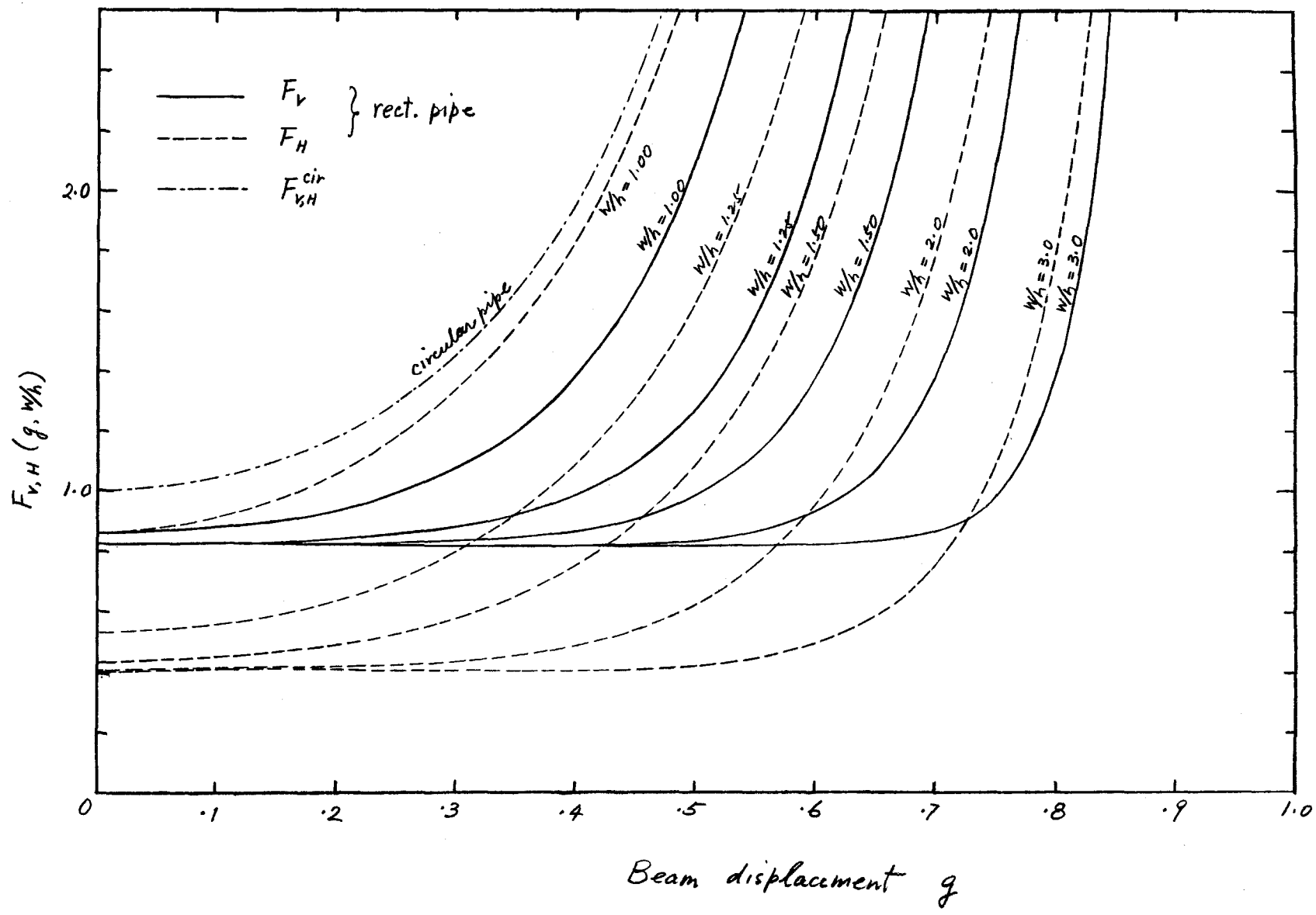
F_V
F_H
$F_{V,H}^{cir}$
} rect. pipe
circular pipe
w/h = 1.00
w/h = 1.00
w/h = 1.25
w/h = 1.25
w/h = 1.50
w/h = 1.50
w/h = 2.0
w/h = 2.0
w/h = 3.0
w/h = 3.0
2.0
1.0
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
$F_{V,H}(g, w/h)$
Beam displacement g

Figure 4

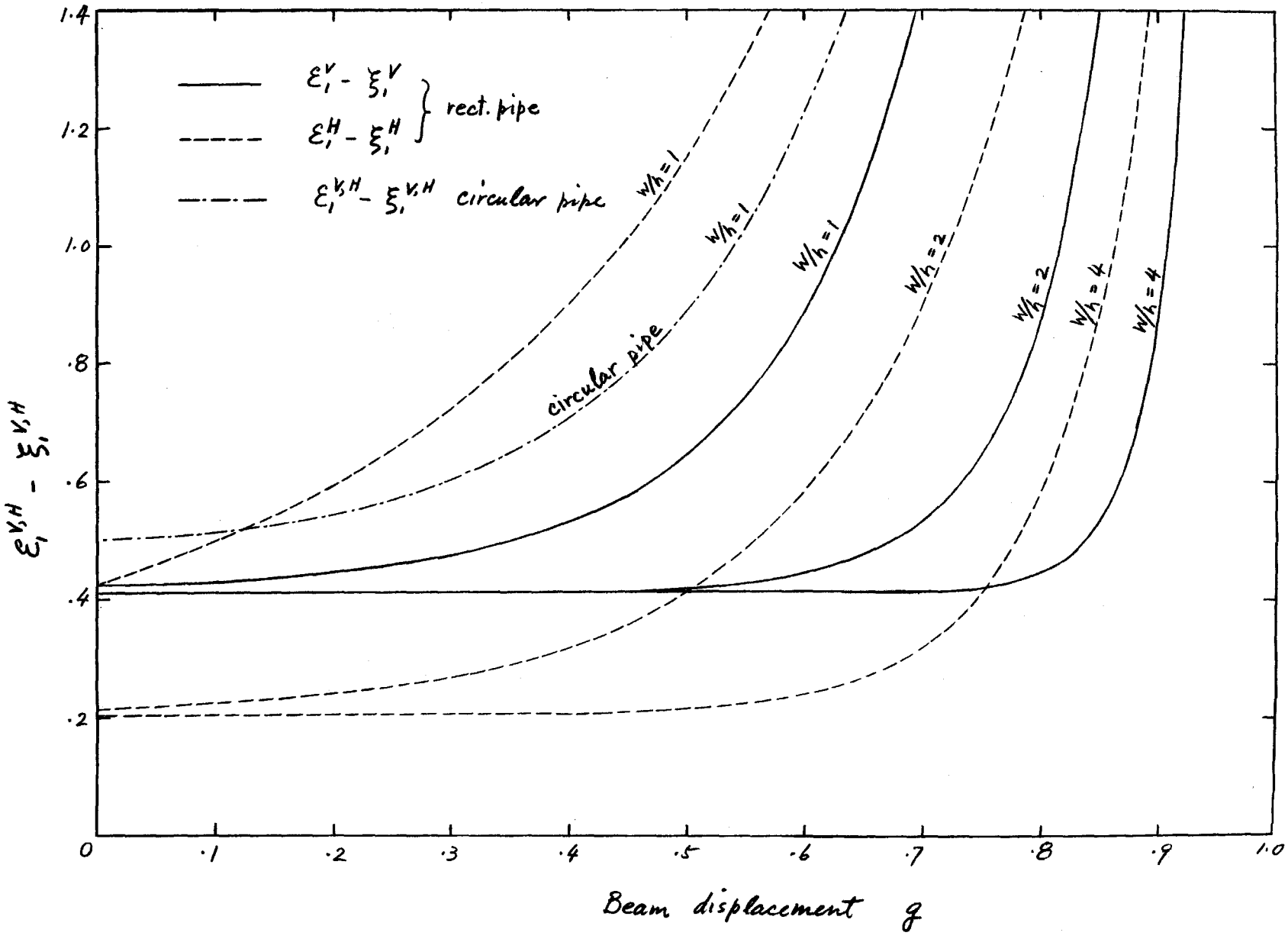
1.4
1.2
1.0
.8
.6
.4
.2
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
$\mathcal{E}_1^V - \xi_1^V$
$\mathcal{E}_1^H - \xi_1^H$
rect. pipe
$\mathcal{E}_1^{V,H} - \xi_1^{V,H}$ circular pipe
w/h = 1
w/h = 1
w/h = 1
w/h = 2
w/h = 2
w/h = 4
w/h = 4
circular pipe
$\mathcal{E}_1^{V,H} - \xi_1^{V,H}$
Beam displacement g

Figure 5