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STABILITY and DYNAMICAL IVARIANTS for INTRA-BEAM SCATTERING

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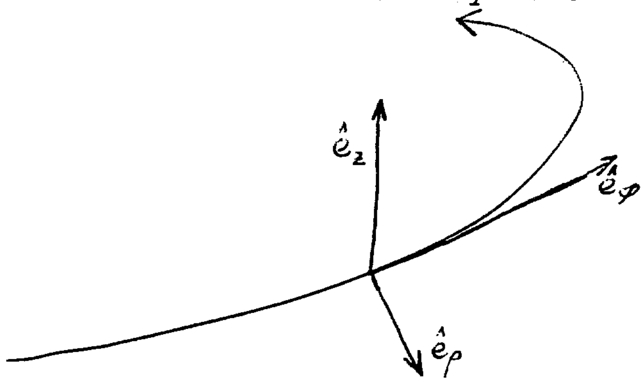
I. Introduction

Intra-beam scattering (IBS) of charged particles within a coasting storage ring beam has been discussed first by Piwinski[1] and more recently by Bjorken in a more general way.[2] The subject has been considered by Derbener and McIntyre as well.[3,4] These approaches all consider the complete collision integral for Coulomb collisions appropriately averaged over all pairs of beam particles. In this note only the invariances of the problem are considered (e.g. the fact that the collisions are point like and elastic). Piwinski's "invariant" is almost trivially derived. Its relationship to (1) total energy conservation (2) azimuthal component of angular momentum conservation and (3) the Courant-Snyder invariants is manifested.

For the general strong focusing lattice it is shown why no analogous invariant (as was conjectured by Piwinski) holds. In this case the Noether theorem symmetry related to the Courant-Snyder invariants is elucidated.

II. Weak Focusing Invariants

Define "weak focusing" to be a lattice with azimuthal (ϕ) symmetry. Then with the coordinate system.



we know that this azimuthal symmetry of the Lagrangian ($\vec{A}$, the potential constant in ϕ) implies, for a single particle:

$$\text{Const.} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \equiv \mathcal{P}_\phi \equiv L_z = \rho \left(P_\phi + e A_\phi \right) \tag{1}$$

The quantity $\langle L_z \rangle$, summing L_z over an entire beam, must also be exactly constant even under IBS (as long as the collisions are "point" collisions and are elastic). The sum of each particles' energy is also invariant. We may evaluate it, expanding each E_i about a central orbit P_o

$$E_i = \sqrt{(P_o + \Delta P_\phi)^2 + P_x^2 + P_z^2 + m^2} \tag{2}$$

$$\approx \tilde{E} + 1/\tilde{E} \left(P_o \Delta P_\phi + 1/2 \left(P_x^2 + P_z^2 \right) + 1/2\, \Delta P_\phi^2 / \gamma^2 \right)$$

where $\tilde{E} = \sqrt{P_o^2 + m^2}$. We may express this also in terms of the

<u>total</u> momentum deviation $\delta \equiv \Delta P_\phi + (P_x{}^2 + P_z{}^2)\ 1/2P_o$ so that the sum invariant $\langle E_i \rangle$ is

$$Const. = \langle E_i \rangle = \widetilde{E} + \frac{P_o^2}{\widetilde{E}} \left(\frac{\langle \Delta P_\phi \rangle}{P_o} + \frac{1}{2}(P_x^2 + P_z^2) + \frac{1}{2} \frac{\langle \Delta P_\phi^2 \rangle}{P_o^2 \gamma^2} \right) \quad (3)$$

Now we wish to express (1) in similar fashion. We expand ρ, P_ϕ and A_ϕ about a central orbit ρ_o, P_o, $Z=0$. For constant gradient magnetic field:

$$B_z = B_o + (\rho - \rho_o) \frac{\partial B_z}{\partial \rho} \quad (4)$$

The azimuthal summetry implies only A_ϕ non-zero:

$$A_\phi = \frac{1}{2} B_o \rho - \frac{1}{2} \rho \rho_o \frac{\partial B_z}{\partial \rho} + \frac{1}{3} \rho^2 \frac{\partial B_z}{\partial \rho} + \frac{1}{6} \frac{\rho_o^3}{\rho} \frac{\partial B_z}{\partial \rho} - \frac{1}{2} z^2 \frac{\partial B_\rho}{\partial z} \quad (5)$$

Then $\vec{\nabla} \times \vec{A}$ is consistant with (4) and $\vec{\nabla} \times \vec{B} = 0$. Let $\rho = \rho_o + X_\beta + \alpha \rho_o\ \delta/\bar{p}$, and expand (1) to terms quadratic in deviations:

$$L_z = \rho_o (P_o + \delta) + \frac{1}{2} \alpha \rho_o P_o \frac{\delta^2}{P_o^2} - \frac{\rho_o P_o}{2} \left\{ \left(\frac{x_\beta^2}{\beta_x^2} + x'^2 \right) + \left(\frac{z^2}{\beta_z^2} + z'^2 \right) \right\} \quad (6)$$

where we have used:

$$P_o = -\rho_o e B_o$$

$$1/\beta_x^2 \equiv -\frac{e}{\rho_o P_o} \left(B_o + \rho_o \frac{\partial B_z}{\partial \rho} \right)$$

$$1/\beta_z^2 \equiv \frac{1}{P_o} \frac{\partial B_z}{\partial \rho}$$

$$\alpha \equiv -\beta_x^2 / \rho_o^2$$

$$z' \equiv P_z / P_o \qquad x' \equiv P_x / P_o$$

As expected (6) is made up of identifiable invariants: δ and δ^2, while the last bracket contains the Courant-Snyder invariants divided by β_x and β_z respectively.

III. Weak Focusing IBS

We know that $\langle\dot{E}\rangle = \langle\dot{L}_z\rangle = 0$, however, this does not lead to two independent useful invariants under IBS. Although we can choose P_o such that $\langle\delta\rangle = 0$ it will not be true in general that $\langle\dot{\delta}\rangle = 0$ (even to the quadratic accuracy considered). A general diffusion (e.g. from IBS) will have constraints:

$$0 = \frac{\langle\dot{L}_z\rangle}{\rho_o P_o} = \frac{\langle\dot{\delta}\rangle}{P_o} + \frac{\alpha}{2}\frac{\langle\dot{\delta^2}\rangle}{P_o^2} - \frac{1}{2}\langle\dot{\varepsilon}_x\rangle/\beta_x - \frac{1}{2}\langle\dot{\varepsilon}_z\rangle/\beta_z$$

$$0 = \frac{\langle\dot{E}\rangle}{P_o^2/E} = \frac{\langle\dot{\delta}\rangle}{P_o} + \frac{1}{2\gamma^2}\frac{\langle\dot{\delta^2}\rangle}{P_o^2} \qquad (7)$$

The difference, eliminating $\langle\dot{\delta}\rangle$, gives just Piwinski's "invariant":

$$\left(1/\gamma^2 - \alpha\right)\frac{\langle\delta^2\rangle}{P_o^2} + \langle\varepsilon_x\rangle/\beta_x + \langle\varepsilon_z\rangle/\beta_z \qquad (8)$$

IV. Discussion

For weak focusing the compaction α is ≥ 1 so that (8) always implies unbounded phase space growth. The beam heads for a state of equipartition equivalent to that of gas in a box. From (7) it is clear that the energy for the growing fluctuations comes from average beam deceleration.

Usually we constrain the beam externally such that it does not decelerate. For instance the filter notches in a stochastic cooler or the fixed momentum of the electron beam in an electron cooler will "pull" the coasting beam. This additional influence does not invalidate (8) since, in the derivation from (7), one need only trade $\langle\dot{E}\rangle = 0$ and $\langle\dot{L}_Z\rangle = 0$ to obtain similar expressions with $\langle\dot{\delta}\rangle = 0$.

We see that Piwinski's "invariant" is a very special case, a result of energy and angular momentum conservation. Piwinski speculated that there was a similar invariant for the general lattice obtained from (8) by merely including β_x, β_Z, and α in the average < > about the circumference. If true this would result, typically, in values $\langle\alpha\rangle \ll 1$ thus, implying stability under IBS.

V. General, Strong Focusing Lattice

Bjorken has derived the rates of change of emittances for the general lattice case. However, he does not explicitly settle the issue of whether or not a general "invariant" exists, connecting momentum spreads in the three planes. A further constraint, additional to (3), is necessary to eliminate the p_o change during IBS.

We know that the only general constraint on the particle motion is the Courant-Snyder invariant. To complete the argument (against any further independent invariant!) consider the original Hamiltonian construction of the C-S invariant (paraphrasing their article - using its notation).[5] Courant and Snyder transform the usual Hamiltonian for motion in a static $\vec{B}(\vec{X})$ field to a new Hamiltonian G describing the motion with respect to the equilibrium orbit (for momentum p particles) in terms of the parameter s (path length along the equilibrium orbit). G is itself P_s the <u>generalized</u> momentum conjugate to this path length, which <u>retains</u> the property of explicit s (time) independence which H had. G <u>is</u> the Courant-Snyder invariant. It is clear that its invariance is a consequence (in the Noether Theorem sense) of symmetry in <u>time</u>. But this is entirely equivalent to the usual total <u>energy</u> invariance of each particle between collisions: no additional constraint is implied.

References

1. A. Pininski, Proc. 9th Int. Accel. Conf. (1974) p 405.

2. J. Bjorken, Unpublished September 1981.

3. Ya. Derbener, Fermilab $\bar{p}$ Note 176 (Translation-September 1981).

4. P. McIntyre, Unpublished 1981.

5. E. Courant and H. Snyder, Annals Phys. 3 (1958) p 45-48.