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# Potential of a 3-Dimensional Charge Distribution

## (Powerful Calculating Method & Its Applications)

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Abstract

A new expression for the electrical potential for a 3-dimensional non-uniform charge distribution is obtained. The electrical potential is obtained as a limiting case of the general solution to the diffusion equation /1/. It is shown how the new expression can be used to obtain simple expressions for sample charge distribution.

## 1. Introduction

Instead of solving directly the Poisson equation

$$\nabla^2 \phi(\mathbf{x}) = -4\pi \rho(\mathbf{x}), \tag{1-1}$$

we consider the diffusion equation with a stationary sources

$$\nabla^2 \psi(\mathbf{x}, t) - A^2 \frac{\partial \psi(\mathbf{x}, t)}{\partial t} = -4\pi \rho(\mathbf{x}), \tag{1-2}$$

where $A^2$ is a diffusion parameter. If the second term $A^2 \frac{\partial \psi}{\partial t}$ in the left-hand side of (1-2) vanishes for an arbitrary time in the limit $A \to 0$, the potential $\phi(\mathbf{x})$ could be obtained from $\psi(\mathbf{x}, t)$ simply by going to the limit $A \to 0$ :

$$\phi(\mathbf{x}) = \lim_{A \to 0} \psi(\mathbf{x}, t). \tag{1-3}$$

## 2. General Solution

In order to handle the problem for solutions of the diffusion equation, it is necessary to determine the Green's function

$$G(\mathbf{x}, t ; \boldsymbol{\xi}, \tau),$$

which satisfies the equation

$$\left(\nabla^2 - A^2 \frac{\partial}{\partial t}\right) G(\mathbf{x}, t ; \boldsymbol{\xi}, \tau) = -\delta(\mathbf{x} - \boldsymbol{\xi})\, \delta(t - \tau), \tag{2-1}$$

with boundary condition

$$G(\mathbf{x}, t ; \boldsymbol{\xi}, \tau) = 0, \tag{2-2}$$

for $t \leqq \tau$ or $\mathbf{x} = \infty$ .

Accordingly we exploit the fact that the delta function can be written in terms of orthonormal functins :

$$\delta(\mathbf{x}-\boldsymbol{\xi})\delta(t-\tau) = \frac{1}{(2\pi)^4}\iint_{-\infty}^{\infty} e^{i[\mathbf{k}\cdot(\mathbf{x}-\boldsymbol{\xi}) - \omega(t-\tau)]}\, d\mathbf{k}\cdot d\omega \quad (2\text{-}3)$$

We expand the Green's function in similar fashion :

$$G(\mathbf{x},t;\boldsymbol{\xi},\tau) = \frac{1}{(2\pi)^4}\iint_{-\infty}^{\infty} g(\mathbf{k},\omega)\, e^{i[\mathbf{k}\cdot(\mathbf{x}-\boldsymbol{\xi}) - \omega(t-\tau)]}\, d\mathbf{k}\cdot d\omega \quad (2\text{-}4)$$

Then substitution into (2-1) leads to an equation for $g(\mathbf{k},\omega)$:

$$g(\mathbf{k},\omega) = \frac{-1}{i\omega A^2 - k^2} \qquad (\, k^2 = |\mathbf{k}|^2 \,) \quad (2\text{-}5)$$

Furthermore substituting (2-5) into (2-4), we obtain for G :

$$G(\mathbf{x},t;\boldsymbol{\xi},\tau) = \frac{-1}{(2\pi)^4}\int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot(\mathbf{x}-\boldsymbol{\xi})}\, d\mathbf{k} \int_{-\infty}^{\infty} \frac{e^{-i\omega(t-\tau)}}{i\omega A^2 - k^2}\, d\omega \quad (2\text{-}6)$$

We note that in (2-6) the integration over $d\omega\, d\mathbf{k}$ can be performed directly. First consider the integration over $d\omega$

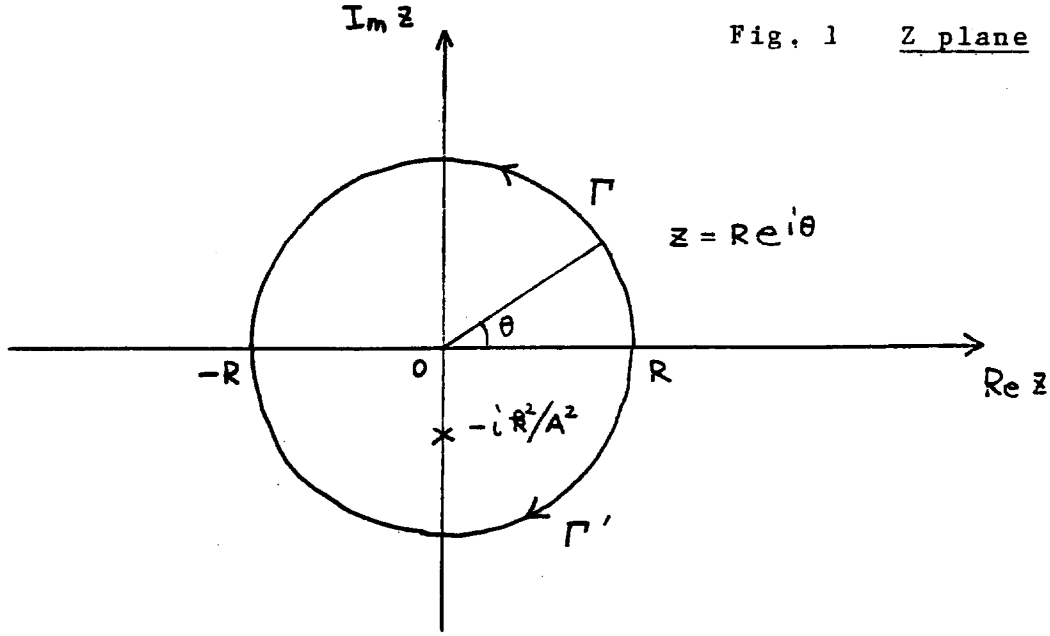
$$I(t-\tau) = \int_{-\infty}^{\infty} \frac{e^{-i\omega(t-\tau)}}{i\omega A^2 - k^2}\, d\omega \quad (2\text{-}7)$$

In order to calculate the integration (2-7), we consider the complex integral

$$\oint_C \frac{e^{-iz(t-\tau)}}{izA^2 - k^2} \, dz \quad , \tag{2-8}$$

along a closed contour cosisting of the real axis from -R to R and a semicircle $\Gamma$ or $\Gamma'$ as in Fig.1.

Fig. 1 Z plane




We note that the pole of the integrand of (2-8) occurs at $z = -ik^2/A^2$ on the imaginary axis of z. Let $z = Re^{i\theta}$. Then

$$e^{-iz(t-\tau)} = e^{R(t-\tau)\sin\theta} \cdot e^{-iR(t-\tau)\cos\theta} \quad . \tag{2-9}$$

When $t < \tau$, from (2-9) the value of the integration along the upper semicircle $\Gamma$ $(0 < \theta < \pi)$ vanishes in the limit $R \to \infty$. Combining this result and the fact that the integrand is an analytical function continuous within and on the smooth closed contour including $\Gamma$, then we have

$$I(t-\tau) = 0 \quad , \qquad t < \tau \ . \tag{2-10}$$

For $t > \tau$, the integration along the lower semicircle $\Gamma'(-\pi < \theta < 0)$ vanishes in the limit $R \to \infty$ from the similar reason to the above case. Here employing calculus of residues, we obtain

$$I(t-\tau) = -2\pi i \sum \text{residues of } \frac{e^{-iz(t-\tau)}}{izA^2 - k^2}, \quad t > \tau, \tag{2-11}$$

in the lower half plane. If we introduce the notation

$$(\mathrm{Res})_z = \text{residue of } \frac{e^{-iz(t-\tau)}}{izA^2 - k^2}, \tag{2-12}$$

the value of $(\mathrm{Res})_z$ is then

$$\begin{aligned}(\mathrm{Res})_{z=-ik^2/A^2} &= \lim_{z \to -ik^2/A^2} \{ z + ik^2/A^2 \} \cdot \frac{e^{-iz(t-\tau)}}{izA^2 - k^2} \\ &= \frac{1}{iA^2} e^{-\frac{k^2}{A^2}(t-\tau)}.\end{aligned} \tag{2-13}$$

From these results, we find

$$I(t-\tau) = \begin{cases} 0 & t < \tau, \\ -\frac{2\pi}{A^2} e^{-\frac{k^2}{A^2}(t-\tau)} & \tau < t. \end{cases} \tag{2-14}$$

Accordingly we obtain

$$G(x,t;\xi,\tau) = \begin{cases} 0 & t < \tau, \\ \frac{1}{(2\pi)^3 A^2} \int_{-\infty}^{\infty} e^{\left[-\frac{k^2}{A^2}(t-\tau) + i\mathbf{k}\cdot(x-\xi)\right]} d\mathbf{k} & t > \tau. \end{cases} \tag{2-15}$$

By rearranging the exponent in the integration, it becomes possible to evaluate each term of the product exactly. The first term may be written

$$\int_{-\infty}^{\infty} e^{-\frac{k_1}{A^2}(t-\tau) + i k_1 (x-\xi_1)} \, dk_1 \quad , \tag{2-16}$$

and by an appropriate change of variable this equals

$$\frac{A\sqrt{\pi}}{(t-\tau)^{1/2}} \cdot e^{-\frac{(x-\xi_1)^2 A^2}{4(t-\tau)}} \quad . \tag{2-17}$$

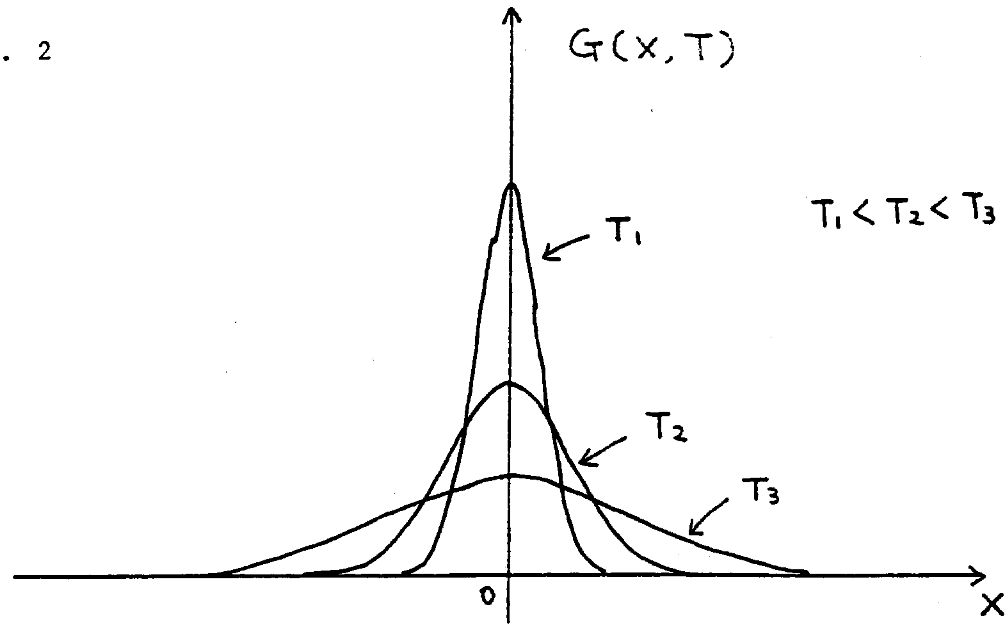
In the similar way, remained integrations over $dk_2 dk_3$ also are performed. Introducing these results into the expression (2-15) for $G(\mathbf{x}, t ; \boldsymbol{\xi}, \tau)$ we obtain

$$G(\mathbf{x}, t ; \boldsymbol{\xi}, \tau) = \begin{cases} 0 & t < \tau , \\ \dfrac{A}{2^3 \pi^{3/2}} \dfrac{e^{-\frac{A^2}{4(t-\tau)} |\mathbf{x}-\boldsymbol{\xi}|^2}}{(t-\tau)^{3/2}} & t > \tau . \end{cases} \tag{2-18}$$

The Green's function (2-18) represents heat diffusion from a point source $\mathbf{x} = \boldsymbol{\xi}$ at the time $t = \tau$. The one dimensional Green's function G(X,T) is plotted in Fig.2 for several values of T ($\equiv t - \tau$). Note that the curve has a sharpe maximum at X ($\equiv x - \xi_1$)=0, and that the width of the curve increases with increasing T. The quantity $2\sqrt{T}/A$ is a measure of this width. At T=0 there is zero width due to the fact that the heat has just been added and is all concentrated at X=0. As T changes from zero, the temperature immediately rises everywhere. The most pro-nounced rise occurs, of course, near X=0, that is, for $X \leq 2\sqrt{T}/A$.

Thus the general solution to the diffusion equation at a point $\mathbf{x}$ , assuming a specified source distribution, is obtained

Fig. 2




as a linear superposition of the effects at $\mathbf{x}$ due to point sources located at different $\boldsymbol{\xi}$'s :

$$\psi(\mathbf{x}, t) = 4\pi \int_0^{\infty} d\tau \int_{-\infty}^{\infty} d\boldsymbol{\xi} \; G(\mathbf{x}, t\,;\, \boldsymbol{\xi}, \tau)\, \rho(\boldsymbol{\xi}) \qquad (2\text{-}19)$$

Now we consider the term $A^2 \frac{\partial \psi}{\partial t}$ in the diffusion equation (1-2). Introducing the notation

$$\eta = \frac{4(t-\tau)}{A^2} , \qquad (2\text{-}20)$$

and substituting (2-18) into (2-19), we write

$$\psi(\mathbf{x}, t) = \frac{1}{\pi^{1/2}} \int_0^{4t/A^2} d\eta \int_{-\infty}^{\infty} d\boldsymbol{\xi} \; \rho(\boldsymbol{\xi}) \; \frac{e^{-\frac{|\mathbf{x}-\boldsymbol{\xi}|^2}{\eta}}}{\eta^{3/2}} . \qquad (2\text{-}21)$$

By using (2-21), we can eliminate

$$A^2 \frac{\partial \psi}{\partial t} = \frac{4}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\frac{|\mathbf{x}-\boldsymbol{\xi}|^2}{4t/A^2}}}{(4t/A^2)^{3/2}} \rho(\boldsymbol{\xi})\, d\boldsymbol{\xi} . \qquad (2\text{-}22)$$

Going to the limit $A \to 0$, we write

$$\lim_{A\to 0} A^2 \frac{\partial \psi}{\partial \tau} = \frac{4}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left\{ \lim_{A\to 0} \frac{e^{-\frac{|\mathbf{x}-\boldsymbol{\xi}|^2}{4t/A^2}}}{(4t/A^2)^{3/2}} \right\} \rho(\boldsymbol{\xi})\, d\boldsymbol{\xi} \tag{2-23}$$

The bracketed expression, in the integrand vanishes, due to exponentials falling off faster than any polynomial. In the limit $A \to 0$, the diffusion equation (1-2) therefore is equivalent to the Poisson equation (1-1). This means that we can take

$$\lim_{A\to 0} \psi(\mathbf{x}, t) \tag{2-24}$$

as a solution of the Poisson equation (1-1) :

$$\begin{aligned} \phi(\mathbf{x}) &= \lim_{A\to 0} \frac{1}{\sqrt{\pi}} \int_0^{4\tau/A^2} \frac{dq}{q^{3/2}} \int_{-\infty}^{\infty} d\boldsymbol{\xi}\, \rho(\boldsymbol{\xi})\, e^{-\frac{|\mathbf{x}-\boldsymbol{\xi}|^2}{q}} \\ &= \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{dq}{q^{3/2}} \int_{-\infty}^{\infty} d\boldsymbol{\xi}\, \rho(\boldsymbol{\xi})\, e^{-\frac{|\mathbf{x}-\boldsymbol{\xi}|^2}{q}} . \end{aligned} \tag{2-25}$$

It will be seen in the following examples how this expression (2-25) can be used to obtain simple expressions for $\phi(\mathbf{x})$. Simple in the sense that the usual 3-dimensional integral expression for $\phi(\mathbf{x})$ :

$$\phi(\mathbf{x}) = \int_{-\infty}^{\infty} d\boldsymbol{\xi}\, \frac{\rho(\boldsymbol{\xi})}{|\mathbf{x}-\boldsymbol{\xi}|} ,$$

can be reduced to an one-dimensional integral.

## 3. Examples

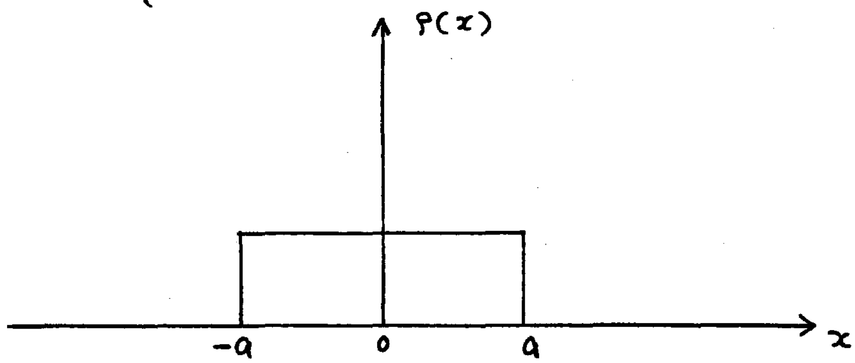
In this section, the several cases of a non-uniform charge distribution are considered. An exact expression for the potential produced by them is derived. We will assume that a charge distribution function $\rho(\mathbf{x})$ is normalized by means of

$$Ne = \int_{-\infty}^{\infty} \rho(\mathbf{x})\, d\mathbf{x} \tag{3-1}$$

where N = number of charged particle, and e = unit charge .

Example (a) Rectangular Distribution.

$$\rho(\mathbf{x}) = \begin{cases} \dfrac{Ne}{8abc} & |x| \leqq a,\ |y| \leqq b,\ |z| \leqq c\,, \\ 0 & |x| > a,\ |y| > b,\ |z| > c\,. \end{cases} \tag{3-a}$$




Substitution of (3-a) into (2-25) yields

$$\phi(\mathbf{x}) = \frac{Ne}{8abc\sqrt{\pi}} \int_0^{\infty} \frac{dq}{q^{3/2}} \int_{-a}^{a} d\xi_1 \int_{-b}^{b} d\xi_2 \int_{-c}^{c} d\xi_3 \, \exp -\frac{1}{q}\left[(x-\xi_1)^2+(y-\xi_2)^2+(z-\xi_3)^2\right] \tag{3-a-1}$$

The integration of the type of

$$I = \int_{-\ell}^{\ell} e^{-\frac{(s-\xi)^2}{q}} \, d\xi \,, \tag{3-a-2}$$

is rewitten in terms of the probability or error function Erf(x)

$$\mathrm{Erf}(x) = \int_0^x e^{-x^2} \, dx \,. \tag{3-a-3}$$

Changing the integration variable

$$\xi' = \frac{s-\xi}{\sqrt{q}} \,, \tag{3-a-4}$$

yields

$$I = \sqrt{q} \int_{\frac{s-\ell}{\sqrt{q}}}^{\frac{s+\ell}{\sqrt{q}}} e^{-\xi'^2} \, d\xi' \,. \tag{3-a-5}$$

For $|s| \leqq \ell$ , (3-a-5) becomes

$$I = \sqrt{q} \left\{ \mathrm{Erf}\left(\frac{\ell+s}{\sqrt{q}}\right) + \mathrm{Erf}\left(\frac{\ell-s}{\sqrt{q}}\right) \right\} . \tag{3-a-6}$$

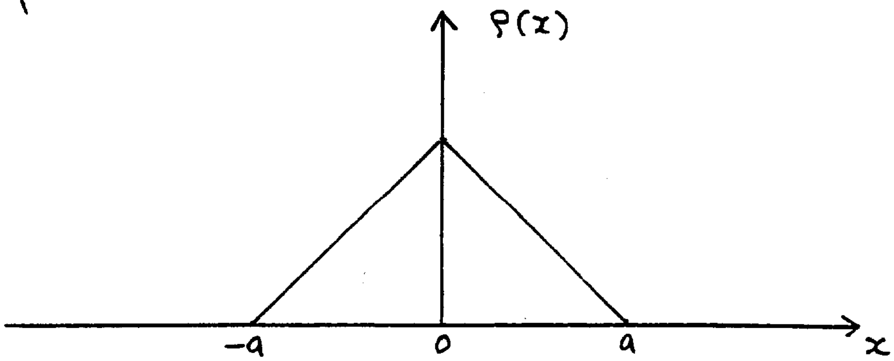
Accordingly we can express the potential $\phi(\mathbf{x})$ in the form

$$\phi(\mathbf{x}) = \frac{Ne}{8abc\sqrt{\pi}} \int_0^\infty dq \left\{ \mathrm{Erf}\left(\frac{a+x}{\sqrt{q}}\right) + \mathrm{Erf}\left(\frac{a-x}{\sqrt{q}}\right) \right\} \left\{ \mathrm{Erf}\left(\frac{b+y}{\sqrt{q}}\right) + \mathrm{Erf}\left(\frac{b-y}{\sqrt{q}}\right) \right\} \left\{ \mathrm{Erf}\left(\frac{c+z}{\sqrt{q}}\right) + \mathrm{Erf}\left(\frac{c-z}{\sqrt{q}}\right) \right\} , \tag{3-a-7}$$

for $|x| \leqq a$ and $|y| \leqq b$ and $|z| \leqq c$.

Example (b) Triangular Distribution.

$$\rho(x) = \begin{cases} \dfrac{Ne}{a^2b^2c^2}\,(a-|x|)(b-|y|)(c-|z|) & |x|\leqq a,\ |y|\leqq b,\ |z|\leqq c\,, \\ 0 & |x|>a,\ |y|>b,\ |z|>c\,. \end{cases} \tag{3-b}$$




Substitution of (3-b) into (2-25), yields

$$\phi(x) = \frac{Ne}{a^2b^2c^2\sqrt{\pi}}\int_0^\infty \frac{dq}{q^{3/2}}\int_{-a}^{a}d\xi_1\int_{-b}^{b}d\xi_2\int_{-c}^{c}d\xi_3\ (a-|\xi_1|)(b-|\xi_2|)(c-|\xi_3|)\times$$
$$\exp -\left[(x-\xi_1)^2+(y-\xi_2)^2+(z-\xi_3)^2\right]/q\,. \tag{3-b-1}$$

As in the previous example, an expression for

$$I = \int_{-l}^{l} |\xi|\, e^{-\frac{(s-\xi)^2}{q}}\, d\xi\,, \tag{3-b-2}$$

is obtained in terms of the error function. After tedious mathematical manipulation including change of the integration variable, we obtain

$$I = \sqrt{q}\left\{ -s\left[2\,\mathrm{Erf}\left(\frac{s}{\sqrt{q}}\right) - \mathrm{Erf}\left(\frac{s+l}{\sqrt{q}}\right) + \mathrm{Erf}\left(\frac{|s-l|}{\sqrt{q}}\right)\right] + \frac{\sqrt{q}}{2}\left[2e^{-\frac{s^2}{q}} - e^{-\frac{(s+l)^2}{q}} - e^{-\frac{(s-l)^2}{q}}\right]\right\}, \tag{3-b-3}$$

for $|s| \leqq \ell$ . Combining (3-c-3) and the results of (a), we write

$$\int_{-\ell}^{\ell} (\ell - |\xi|) \exp\left[-\frac{(s-\xi)^2}{q}\right] d\xi$$
$$= \sqrt{q}\left\{ \ell\left[\mathrm{Erf}\left(\frac{\ell+s}{\sqrt{q}}\right) + \mathrm{Erf}\left(\frac{\ell-s}{\sqrt{q}}\right)\right] - s\left[2\,\mathrm{Erf}\left(\frac{s}{\sqrt{q}}\right) - \mathrm{Erf}\left(\frac{s+\ell}{\sqrt{q}}\right) + \mathrm{Erf}\left(\frac{|s-\ell|}{\sqrt{q}}\right)\right] + \frac{\sqrt{q}}{2}\left[2e^{-\frac{s^2}{q}} - e^{-\frac{(s+\ell)^2}{q}} - e^{-\frac{(s-\ell)^2}{q}}\right]\right\} \quad \text{(3-b-4)}$$

for $|s| \leqq \ell$ .

Thus the potential (3-b-1) is described by

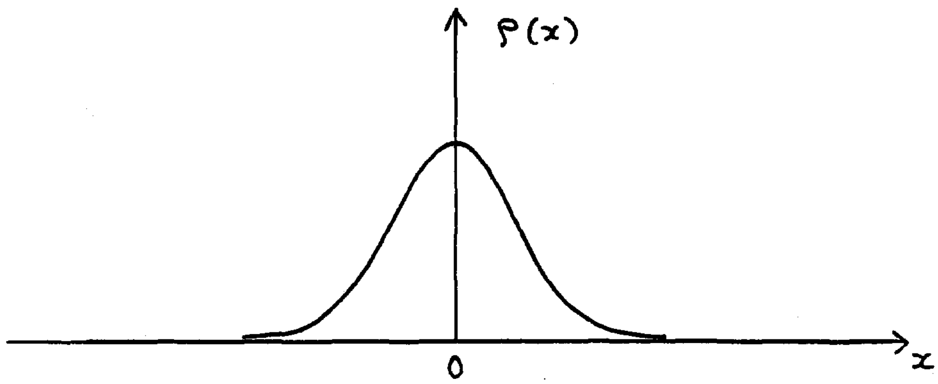
$$\phi(x) = \frac{Ne}{a^2b^2c^2}\int_0^\infty dq \prod_{i=1}^{3}\left\{ \ell_i\left[\mathrm{Erf}\left(\frac{\ell_i+s_i}{\sqrt{q}}\right) + \cdots\cdots \right.\right\} \quad \text{(3-b-5)}$$

( $\ell_i$ = a,b,c, and $s_i$ = x,y,z ).

Example (c) Gaussian Distribution /2/.

$$\rho(x) = \frac{Ne}{(2\pi)^{3/2}abc}\exp\left[-\frac{x^2}{2a^2} - \frac{y^2}{2b^2} - \frac{z^2}{2c^2}\right], \quad \text{(3-c)}$$

where a,b,c are standard deviations.




Substituting (3-c) into (2-25), we have

$$\phi(x) = \frac{Ne}{(2\pi)^{3/2} abc\sqrt{\pi}} \int_0^\infty \frac{dq}{q^{3/2}} \int_{-\infty}^{\infty} d\xi \exp\left[-\frac{|x-\xi|^2}{q} - \frac{\xi_1^2}{2a^2} - \frac{\xi_2^2}{2b^2} - \frac{\xi_3^2}{2c^2}\right] \quad (3\text{-c-1})$$

Integrals of the form

$$I = \int_{-\infty}^{\infty} e^{-\frac{(s-\xi)^2}{q} - \frac{\xi^2}{2\ell^2}} d\xi \quad (3\text{-c-2})$$

can be easily obtained from the integration formula

$$\int_{-\infty}^{\infty} e^{-\alpha^2\xi^2+\beta\xi} d\xi = \frac{e^{\left(\frac{\beta}{2\alpha}\right)^2}}{\alpha}\sqrt{\pi} \qquad (\beta > 0) \quad (3\text{-c-3})$$

For example (3-c-2) becomes

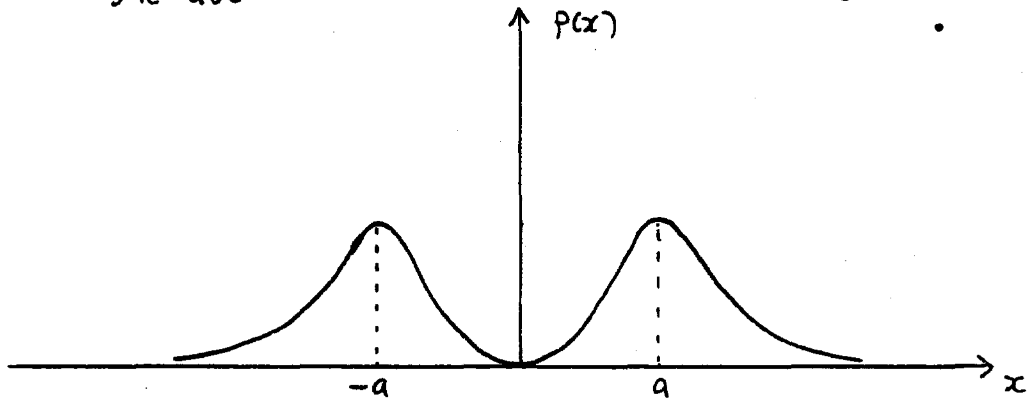
$$I = \ell\sqrt{2\pi}\, q^{\frac{1}{2}} \frac{e^{-\frac{s^2}{2\ell^2+q}}}{\sqrt{2\ell^2+q}} \quad (3\text{-c-4})$$

Inserting (3-c-4) into (3-c-1), we have the exact expression for the potential of the Gaussian distribution

$$\phi(x) = \frac{Ne}{\sqrt{\pi}} \int_0^\infty \frac{\exp\left[-\frac{x^2}{2a^2+q} - \frac{y^2}{2b^2+q} - \frac{z^2}{2c^2+q}\right]}{\sqrt{(2a^2+q)(2b^2+q)(2c^2+q)}} dq \quad (3\text{-c-5})$$

Example (d) Halo Distribution.

$$\rho(\mathbf{x}) = \frac{8\,Ne}{3\pi^{3/2}abc}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\exp\left[-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right] \qquad (3\text{-d})$$




Substituting (3-d) into (2-25), we have

$$\phi(\mathbf{x}) = \frac{8\,Ne}{3\pi^{3/2}abc\sqrt{\pi}}\int_0^\infty \frac{dq}{q^{3/2}}\int_{-\infty}^{\infty}\left(\frac{\xi_1^2}{a^2}+\frac{\xi_2^2}{b^2}+\frac{\xi_3^2}{c^2}\right)\exp\left[-\frac{|\mathbf{x}-\boldsymbol{\xi}|^2}{q}-\frac{\xi_1^2}{a^2}-\frac{\xi_2^2}{b^2}-\frac{\xi_3^2}{c^2}\right]d\boldsymbol{\xi} \qquad (3\text{-d-1})$$

The space integration over $d\boldsymbol{\xi}$ is divided into three parts:

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left(\frac{\xi_1^2}{a^2}+\frac{\xi_2^2}{b^2}+\frac{\xi_3^2}{c^2}\right)\exp\left[-\frac{|\mathbf{x}-\boldsymbol{\xi}|^2}{q}-\frac{\xi_1^2}{a^2}-\frac{\xi_2^2}{b^2}-\frac{\xi_3^2}{c^2}\right]d\boldsymbol{\xi} \\
&= \frac{1}{a^2}\int_{-\infty}^{\infty}\xi_1^2 e^{-\frac{(x-\xi_1)^2}{q}-\frac{\xi_1^2}{a^2}}d\xi_1\int_{-\infty}^{\infty}e^{-\frac{(y-\xi_2)^2}{q}-\frac{\xi_2^2}{b^2}}d\xi_2\int_{-\infty}^{\infty}e^{-\frac{(z-\xi_3)^2}{q}-\frac{\xi_3^2}{c^2}}d\xi_3 \\
&+ \frac{1}{b^2}\int_{-\infty}^{\infty}e^{-\frac{(x-\xi_1)^2}{q}-\frac{\xi_1^2}{a^2}}d\xi_1\int_{-\infty}^{\infty}\xi_2^2 e^{-\frac{(y-\xi_2)^2}{q}-\frac{\xi_2^2}{b^2}}d\xi_2\int_{-\infty}^{\infty}e^{-\frac{(z-\xi_3)^2}{q}-\frac{\xi_3^2}{c^2}}d\xi_3 \\
&+ \frac{1}{c^2}\int_{-\infty}^{\infty}e^{-\frac{(x-\xi_1)^2}{q}-\frac{\xi_1^2}{a^2}}d\xi_1\int_{-\infty}^{\infty}e^{-\frac{(y-\xi_2)^2}{q}-\frac{\xi_2^2}{b^2}}d\xi_2\int_{-\infty}^{\infty}\xi_3^2 e^{-\frac{(z-\xi_3)^2}{q}-\frac{\xi_3^2}{c^2}}d\xi_3
\end{aligned} \qquad (3\text{-d-2})$$

Integrals of the form

$$I_A = \int_{-\infty}^{\infty}\xi^2\, e^{-\frac{(s-\xi)^2}{q}-\frac{\xi^2}{\ell^2}}\,d\xi \qquad (3\text{-d-3})$$

appearing in (3-d-2) are calculated as follows:

Changing the integration variable

$$\xi' = \xi - \frac{s\ell^2}{q+\ell^2} , \tag{3-d-4}$$

we have

$$I_A = e^{-\frac{s^2}{q+\ell^2}} \int_{-\infty}^{\infty} (\xi' + s\gamma)^2 e^{-\frac{\xi'^2}{q\gamma}} d\xi' , \tag{3-d-5}$$

with

$$\gamma = \frac{\ell^2}{q+\ell^2} . \tag{3-d-6}$$

Using the integration formula

$$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}} , \quad \int_{-\infty}^{\infty} x e^{-\alpha x^2} dx = 0 , \quad \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \frac{\sqrt{\pi}}{\sqrt{\alpha}} ,$$

we write each term appearing in (3-d-2) in the form

$$\int_{-\infty}^{\infty} \xi'^2 e^{-\frac{\xi'^2}{q\gamma}} d\xi' = \frac{\sqrt{\pi}\ell^3}{2} \left( \frac{q}{q+\ell^2} \right)^{3/2} , \tag{3-d-7}$$

$$\int_{-\infty}^{\infty} \xi' e^{-\frac{\xi'^2}{q\gamma}} d\xi' = 0 , \tag{3-d-8}$$

$$\int_{-\infty}^{\infty} e^{-\frac{\xi'^2}{q\gamma}} d\xi' = \sqrt{\pi}\, \ell \left( \frac{q}{q+\ell^2} \right)^{1/2} . \tag{3-d-9}$$

Substituting these into (3-d-5), we obtain

$$I_A = e^{-\frac{s^2}{q+\ell^2}} \left\{ \frac{\sqrt{\pi}}{2} \ell^3 \left( \frac{q}{q+\ell^2} \right)^{3/2} + \sqrt{\pi}\, \ell^5 \frac{q^{\frac{1}{2}}}{(q+\ell^2)^{5/2}} s^2 \right\}. \tag{3-d-10}$$

Also integrals of the form

$$I_B = \int_{-\infty}^{\infty} e^{-\frac{(s-\xi)^2}{q} - \frac{\xi^2}{\ell^2}} d\xi , \tag{3-d-11}$$

have been obtained in the previous section; i.e.

$$I_B = \frac{\ell \sqrt{\pi}\, q^{\frac{1}{2}} e^{-\frac{s^2}{\ell^2+q}}}{\sqrt{\ell^2 + q}} . \tag{3-d-12}$$

Lastly, using (3-d-10) and (3-d-12), we can obtain (3-d-1) as an integration over dq alone,

$$\phi(x) = \frac{8Ne}{3\sqrt{\pi}} \int_0^{\infty} dq \, \frac{\exp\left[ -\frac{x^2}{a^2+q} - \frac{y^2}{b^2+q} - \frac{z^2}{c^2+q} \right]}{\sqrt{(a^2+q)(b^2+q)(c^2+q)}} \times$$

$$\left\{ \frac{q}{2} \left( \frac{1}{a^2+q} + \frac{1}{b^2+q} + \frac{1}{c^2+q} \right) + \left[ \left( \frac{ax}{a^2+q} \right)^2 + \left( \frac{by}{b^2+q} \right)^2 + \left( \frac{cz}{c^2+q} \right)^2 \right] \right\}. \tag{3-d-13}$$

## 4. Conclusion

The potential of an ellipsoidal charge distribution (3-dimensional Gaussian for example) can be obtained by applying the traditional method /3/. However the such traditional method seems not to be applicable to cases where charge distributions don't have spatial symmetry. The advantage of the method presented here is that it is applicable even to unsymmetrical cases, provided integrals of the form

$$\int_{-\infty}^{\infty} \rho(\xi_1, \xi_2, \xi_3)\, e^{-\frac{(s_i-\xi_i)^2}{q}}\, d\xi_i \, ,$$

where $s_i$ is x,y,z, and $\xi_i$ is $\xi_1$, $\xi_2$, $\xi_3$, can be performed. We emphasize this point.

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## References and Footnotes

/1/S. Kheifet, " Potential of a Three-Dimensional Gaussian Bunch ", PETRA 119, (1976)

The present idea for solving the Poisson equation is found in the above short report, in which the result of its application to a three-dimensional Gaussian distribution also is given without proofs.

/2/E. Keil, " Beam-Beam Interaction in P-P Storage Rings ", in Proceedings of the First Course of the International School of Particle Accelerators of the Ettore Majorana Centre for Scientific Culture, Erice 1976, CERN 77-13, p315 (1977)

The potential of a 3-dimensinal Gaussian charge distribution is given in the form

$$\phi(x) = -\frac{Ne}{\sqrt{\pi}}\int_0^\infty \frac{1-\exp\left[-\frac{x^2}{2a^2+q}-\frac{y^2}{2b^2+q}-\frac{z^2}{2c^2+q}\right]}{\sqrt{(2a^2+q)(2b^2+q)(2c^2+q)}}\,dq\,,$$

which is different from our present result by the first term of the integration. The existance of this constant potential may mean that the derivation of the above potential has been made by another method /3/.

/3/J.A.Stratton, Electromagnetic Theory, sec.3.27 (McGraw-Hill, New York, 1941)

F.Mills called my attention to the applicability of the traditional method (for example, in the above text) to cases of ellipsoidal distributions.