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NORMAL MODE ANALYSIS OF COUPLED BETATRON OSCILLATIONS INCLUDING COOLING

BY

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I. Introduction

A simple and economical way to cool the horizontal betatron oscillation has been suggested.[1] By cooling the verical betatron oscillation and by simultaneously coupling it to the horizontal betatron oscillation, one expects to cool both components of the betatron oscillations simultaneously. The coupling can be achieved either by skewing a quadrupole magnet or by introducing a longitudinal magnetic field, as is present in the electron cooling system. Note that it is important to locate the coupling component in a dispersion-free region inorder to avoid introducing vertical dispersion.

This memo contains a theoretical analysis of the above system. The results of this analysis show that the cooling rates of the vertical and horizontal oscillation are comparable provided that the lattice is tuned close to a coupling resonance ($\nu_x - \nu_z + \ell \cong 0$), and provided that the coupling time is much smaller than the cooling time. This phenomenon is due to the complete mixing of the vertical and horizontal betatron oscillation on resonance.

Two normal modes for the betatron oscillation exist, but they do not coincide with the uncoupled horizontal and vertical modes. Consequently the horizontal emittance fails

to be a constant of the motion, and the vertical emittance fails to decay exponentially at a single rate. Instead there is a beating effect as the energy is exchanged (via the coupling) between the modes. In addition both emittances decay exponentially but not with a single decay rate. There are two decay rates corresponding to the two normal modes.

The purpose of this memo is to obtain expressions for the normal mode cooling rates as a function of lattice parameters (coupling strength, β_x, β_z, etc.) and to obtain the dynamics of the vertical and horizontal emittances.

Two methods are employed to obtain the rates: One is exact (normal modes of a linear lattice), and the other is approximate (resonance approximation). The advantage of the latter method is that the physics is clearer, that the rates have a simpler form and are therefore easier to derive. For this reason, the formalism of the former method is set up, but cooling rates are derived only within the resonance approximation.

The organization of this memo is as follows: Normal mode analysis is reviewed in section II in the form of a one-dimensional oscillator. The notion of a "damped invariant" is introduced for non-Hamiltonian systems and in the linear case is related to normal modes. In section III the

exact transformation matrix to normal modes is derived assuming a linear lattice. The relationship between normal mode cooling rates and the eigenvalues of the exact lattice transfer matrix[2], $M(S+LIS)$, is obtained. In section IV, action-angle coordinates are employed to obtain the resonance approximation of the equations of motion. The equations are then transformed to linear equations, and normal mode cooling rates are obtained as a function of Fourier transforms of lattice parameters. In section V, a numerical comparison between the theory of section III and the resonance approximation is discussed. Section VI contains a brief summary of the conditions for coupling-cooling experiments. Physical values for coupling strengths and cooling rates are substituted into the expressions of section IV to determine the feasability of coupling-cooling experiments.

systems, there are pitfalls that can be avoided provided the one-dimensional oscillator is well understood.

Consider the following equations of motion:

$$\frac{dq}{ds} = p \quad , \qquad 2.1$$

$$\frac{dp}{ds} = -kq - \gamma p \quad , \qquad 2.2$$

where k and γ are independent of s. The normal coordinates, p_1 and p_2, for this system are linear in q and p:

$$p = TX \qquad 2.3$$

where

$$p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \qquad 2.4$$

$$X = \begin{bmatrix} q \\ p \end{bmatrix} \qquad 2.5$$

and T is an 2×2 invertable matrix such that

$$\frac{dP_n}{ds} = \mu_n P_n \qquad 2,6$$

where $\mu_n = i\omega_n + \gamma_n$. The oscillation frequency of P_n is ω_n, and it's damping rate is γ_n. For the above equations of motion, one can easily show that

$$T = \begin{bmatrix} \omega - i\frac{\gamma}{2} & -i \\ \omega + i\frac{\gamma}{2} & i \end{bmatrix} , \qquad 2,7$$

where

$$\omega = \left(k - \frac{\gamma^2}{4}\right)^{1/2} , \qquad (2,8)$$

and

$$\mu_1 = +i\omega - \frac{\gamma}{2} , \qquad (2,9)$$

$$\mu_2 = -i\omega - \frac{\gamma}{2} . \qquad (2,10)$$

The normal mode coordinates are given by

$$P_1 = \left(\omega - \frac{i\gamma}{2}\right) q - ip , \qquad (2,11)$$

$$P_2 = \left(\omega + \frac{i\gamma}{2}\right) q + ip . \qquad (2,12)$$

The dynamics of the normal coordinates is obtained by integrating (2,6);

$$p_n = p_n^0 e^{\mu_n s} \quad , \qquad (2,13)$$

where p_n^0 is a complex valued constant.

For the case of $\gamma = 0$, there exists an invariant, W, given by

$$W = p_1 p_1^* = p_2 p_2^* \quad . \qquad (2,14)$$

If the damping is included, there no longer is an invariant which is a differentiable function of q and p. Nonetheless there is what we term a "damped invariant" (DI). It is given by (2,14) and possesses the property that $dW/ds = -\gamma W$; i.e. dynamically W decreases exponentially with s.

The notion of a damped invariant can be defined for a general dynamical system. A damped invariant W is a differentiable function of q, p and s such that

i) $W(q,p,s)$ is real valued

ii) $\frac{dW}{ds} = \Gamma W$

iii) Γ is a real valued constant

iv) $W(q,p,s+L) = W(q,p,s)$

where L = length of central closed orbit. Constants of the motion satisfie i) – iv) provided $\Gamma = 0$.

Note that condition iv) appears at first to contradict condition ii)

Nevertheless it doesn't. It is a consequence of the fact th W must be a continuous function of S. This point is clarified below.

For the one dimensional problem above, a W satisfying i) – iv) is given by (2.11). Substitution of (2.11) into (2.14) yields:

$$W = p^2 + kq^2 + \gamma pq \quad , \qquad 2.15)$$

where $\Gamma = -\gamma$. In this example, W depends only on q and p, and therefore trivially satisfies condition iv). In a real lattice, however, k depends explicitly on S. Consequently W must depend on all three coordinates q, p and S (as does the Courant-Snyder invariant in the undamped case).

This DI (2.15) is not unique. A family of DI's given by AW^n, where A and n are constants, also satisfies conditions i) – iv), provided one assumes $\Gamma = -n\gamma$. The W in (2.15) is more familiar, however, than the other DI's because it is proportional to the energy (or emittance) in the $\gamma \to 0$ limit.

Consequently, the emittance decays approximately exponentially at the rate γ provided $\gamma \ll k$; i.e.

$$p^2 + kq^2 \cong |p_0|^2 e^{-\gamma s} \quad , \qquad 2.16$$

where p_0 is a constant.

The accuracy of this expression is checked by Inverting (2.13) to obtain $p(s)$, $q(s)$, and substituting into p^2+kq^2 to obtain the exact s-dependence of the equivalence; i.e.

$$q = \frac{|p_0|}{\omega} e^{-\frac{\gamma s}{2}} \sin(\omega s + \delta) \quad , \qquad 2.17$$

$$p = \frac{|p_0|}{\omega} e^{-\frac{\gamma s}{2}} \left[\omega \cos(\omega s + \delta) - \frac{\gamma}{2} \sin(\omega s + \delta) \right], \qquad 2.18$$

where p_0 and δ are constants. Thus:

$$p^2 + kq^2 = |p_0|^2 e^{-\gamma s} \left[1 + O(\gamma/\omega) \right] . \qquad 2.19$$

The terms $O(\gamma/\omega)$ are neglected in the $\gamma \ll \omega$ limit, thus verifying (2.16).

It is important to clarify condition (iv), because it plays a key role in the derivation in section III. As an example, consider the above system for $\gamma = 0$. Condition (i) for $\rho = 0$ is insufficient because it is only a local condition, and therefore it doesnt gaurentee that W is a global invariant. For example, all functions W that are quadratic in q and p such that $\oint W ds = 0$ are given by

$$W = A(p^2+kq^2) + B\left[(p^2-kq^2)\cos(2\omega s + \delta) + 2\omega p q \sin(2\omega s + \delta) \right] . \qquad 2.20$$

where A, B, and δ are arbitrary constants. In order for W to be a global invariant, it must be a continuously differentiable function of q, p, and s everywhere. Condition (iv) is consequence of W being a continuous function of s. The function W in (2.20) satisfies condition (iv) if either $B=0$ or $2\omega L = 2N\pi$. This implies that $W = p^2 + kq^2$ is the only quadratic global invariant for almost all ω.

Condition (iv) is also important in the case of non-zero damping. If W is given by (2.15), then a second function $\hat{W}$ defined by

$$\hat{W} = e^{\hat{\Gamma} s} W \quad , \qquad (2.21)$$

also satisfies $d\hat{W}/ds = (\hat{\Gamma} - \gamma)\hat{W}$. Nevertheless, it fails to satisfy condition (iv), and therefore is not a DI. The imposition of condition (iv) eliminates the possibility exemplified by (2.21).

In the next section, the formalism for obtaining normal coordinates and DI's is developed for linear, non-Hamiltonian, non-autonomous systems. For these systems, condition (iv) is an essential requirement needed to obtain a normal mode coupling rates.

III. Normal Coordinates for Linear Systems

In this section, the formalism for obtaining the normal coordinates and DI's is developed for the following set of differential equations:

$$\frac{dX}{ds} = F X \quad ; \quad X\Big|_{s=s_0} = X_0 \quad , \qquad 3,1$$

where X is a $2N$-component column vector,

$$X = \begin{bmatrix} q_1 \\ p_1 \\ \vdots \\ q_N \\ p_N \end{bmatrix} \quad , \qquad 3,2$$

F is a $2N \times 2N$ matrix periodic in s ($F(s+L) = F(s)$),

$$F = \begin{bmatrix} F_{11} & \cdots & F_{1N} \\ \vdots & & \vdots \\ F_{N1} & \cdots & F_{NN} \end{bmatrix} \quad , \qquad 3,3$$

and X_0 is a column vector of initial conditions. The matrix F is assumed to include both coupling and damping.

The normal coordinates are assumed to be linear functions of $\{X_i\}$:

$$Y = TX \quad , \qquad 3,4$$

where T is a $2N \times 2N$ matrix periodic in S (i.e. $T(S)$ (see condition iv), such that

$$\frac{d\rho_n}{dS} = \mu_n \rho_n \quad . \qquad 3.5$$

The μ_n are eigenvalues to be defined below. In matrix notation, Eq. (3.5) takes the form

$$\frac{d\rho}{dS} = \mu \rho \quad , \qquad 3.6$$

where μ is an $2N \times 2N$ diagonal matrix with components

$$\mu_{nm} = \delta_{nm} \mu_n \quad . \qquad 3.7$$

Under the above assumptions, an expression for T can be derived. A differential equation satisfied by T is obtained by substituting (3.4) into (3.6) using (3.1):

$$\frac{d\rho}{dS} = \frac{dT}{dS} X + T \frac{dX}{dS} \quad , \qquad 3.8$$

$$= \left(\frac{dT}{dS} + TF\right) X \quad . \qquad 3.9$$

Equating (3.9) to (3.6) yields:

$$\left(\frac{dT}{dS} + TF - \mu T\right) X = 0 \quad . \qquad 3.10$$

This equation is valid for all X, thus

$$\frac{dT}{dS} + TF - \mu T = 0$$

Eq. (3,11) is the differential equation satisfied by T. The general solution is obtained as follows: Define a matrix $\hat{T}$ via

$$T = e^{\mu(s-s_0)}\,\hat{T} \quad , \qquad 3,12$$

then (3,11) becomes

$$\frac{d\hat{T}}{ds} + \hat{T}F = 0 \quad . \qquad 3,13$$

The general solution to (3,13) is a function of the general solution to (3.1). Let $M(s|s_0)$ be the evolution matrix for (3,1); i.e.,

$$X = M(s|s_0)\,X_0 \quad , \qquad 3,14$$

Similarly, if X is replaced by a $N \times N$ invertible matrix B;

$$\frac{dB}{ds} = FB \quad , \quad B\Big|_{s=s_0} = B_0 \qquad 3,15$$

then the general solution to (3,15) is

$$B = M(s|s_0)\,B_0 \quad . \qquad (3,16)$$

Multiplication of (3,15) on the left by $\hat{T}$, multiplication of (3,13) on the right by B, and addition of the resulting equations yields

$$\frac{d}{ds}(\hat{T}B) = 0 \quad . \qquad (3,17)$$

Integration of 3,17 yields

$$\hat{T} = C_0 B^{-1} , \qquad (3,18)$$

where C_0 is a constant matrix.
Substitution of the inverse of (3.16) into (3,18) yields:

$$\hat{T} = T_0 M^{-1}(S|S_0) , \qquad 3,19$$

where $T_0 \equiv C_0 B_0^{-1}$.

Thus the general solution to (3,11) is

$$T = e^{\mu(S-S_0)} T_0 M^{-1}(S|S_0) . \qquad 3,20$$

The local condition (3,6) is insufficient to determine μ and T_0. However, Application of the continuity condition to (3,
$T(S+L) = T(S)$ uniquely determines the components of the eigenvalue matrix μ and determines T_0 up to multiplication on the left by an arbitrary complex diagonal matrix. From (3,20) one obtains

$$T(S+L) = e^{\mu(S-S_0)} e^{\mu L} T_0 M^{-1}(S+L|S_0) . \qquad 3,21$$

Since

$$M^{-1}(S+L|S_0) = M^{-1}(S|S_0) M^{-1}(S+L|S) , \qquad (3,22)$$

Equating (3,21) to (3,20) yields

$$e^{\mu L} T_0 M^{-1}(S|S_0) M^{-1}(S+L|S) = T_0 M^{-1}(S|S_0) . \qquad 3.23$$

Rearrangement of (3.23) yields

$$e^{\mu L} = T_0 M^{-1}(s|s_0) M(s+L|s) M(s|s_0) T_0^{-1} . \qquad 3.24$$

Since

$$M^{-1}(s|s_0) M(s+L|s) M(s|s_0) = M(s_0+L|s_0) , \qquad 3.25$$

it is clear from 3.24 that T_0 is a matrix which diagonalizes the transfer matrix $M(s_0+L|s_0)$:

$$e^{\mu L} = T_0 M(s_0+L|s_0) T_0^{-1} , \qquad 3.26$$

and $e^{\mu L}$ is a diagonal matrix with components equal to the eigenvalues of the transfer matrix. In the special case of stable, undamped motion, the components of $\mu L/2\pi$ are the tunes of the lattice. The matrix T_0 is unique up to multiplication on the left by an arbitrary complex valued diagonal matrix. This arbitrariness is due to the arbitrariness of the initial phase and amplitude of the ϕ_n coordinates.

The case of F = (constant matrix) is considered here, for use in section IV. The evolution matrix in this case is given by

$$M(s|s_0) = \exp[F(s-s_0)] , \qquad 3.27$$

and

$$M^{-1}(s|s_0) = \exp[-F(s-s_0)] . \qquad 3.28$$

The T_0-matrix satisfies

$$T_0 e^{FL} T_0^{-1} = e^{\mu L} \quad . \qquad 3.29$$

Due to the simple form of $M(s|s_0)$, the matrix T_0 also satisfies

$$T_0 F T_0^{-1} = \mu \, , \qquad 3.30$$

and conversely. Consequently it is sufficient to diagonalize F. Thus

$$T = e^{\mu(s-s_0)} T_0 e^{-F(s-s_0)} T_0^{-1} T_0 \, , \qquad 3.31$$

$$= T_0 \, , \qquad 3.32$$

In this special case, the T-matrix is equal to T_0, and therefore independent of s.

The evolution matrix, $M(s|s_0)$, can be expressed as a function of T and μ. It is obtained by inverting (3.20) to obtain

$$M(s|s_0) = T^{-1} e^{\mu(s-s_0)} T_0 \, , \qquad 3.33$$

and substituting (3.32) into (3.33) to obtain

$$M(s|s_0) = T_0^{-1} e^{\mu(s-s_0)} T_0 \, . \qquad 3.32$$

In principle, the normal mode analysis is completed. More detailed information (normal coordinates, tunes, decay rates etc) for a specific system is obtained by taking the following steps: 1) Obtain the evolution

matrix $M(s|s_0)$ and it's inverse $M^{-1}(s|s_0)$. 2) Find the eigenvalues of the transfer matrix $M(s_0+L|s_0)$. 3) Construct T_0, Eq. (3.26). 4) Construct T, Eq. (3.20). 5) Obtain the normal coordinates, Eq. (3.4).

There are $2N$-damped invariants (not all independent) given by

$$W_n = \rho_n \rho_n^* \quad , \qquad 3.33$$

$$= \sum_\ell \sum_m T_{n\ell} T_{nm}^* X_\ell X_m \quad . \qquad 3.34$$

The W_n satisfy

$$\frac{dW_n}{ds} = \Gamma_n W_n , \qquad 3.35$$

where

$$\Gamma_n = 2\,\mathrm{Re}\,(\mu_n) \quad . \qquad 3.36$$

Even though the above analysis is exact, simple analytical expressions for decay rates and tunes are, in general, difficult to obtain. For this reason, one is motivated to seek a simpler, although approximate, model. The resonance approximation is examined in the next section for the cases of coupling via skewed quads and/or a longitudinal magnetic field. It will be seen that simple expressions for normal coordinates, tunes, and cooling rates can be derived within this model.

a Fourier series in $\exp(i(2\pi s/L))$. Resonant and angle-independent terms are retained. The resulting equations of motion are then transformed to a set of linear equations which are independent of s.

The normal mode analysis of the previous section is applied to these equations. Simple expressions for tunes, cooling rates, and normal modes results for the case of cooling time >> coupling time (defined rigorously below). This is the physical regime of interest as is seen in section V.

The exact equations of motion are assumed to take the form:

$$\frac{dq_n}{ds} = \frac{\partial H}{\partial p_n}, \qquad 4,1$$

$$\frac{dp_n}{ds} = -\frac{\partial H}{\partial q_n} \quad \text{for} \quad n \neq 2 \qquad 4,2$$

$$\frac{dp_2}{ds} = -\frac{\partial H}{\partial q_2} - \gamma(s)\, p_2 \qquad 4,3$$

where

$$H = \frac{p_1^2}{2P_0} + \frac{p_2^2}{2P_0} + P_0\left(k_1\frac{q_1^2}{2} + k_2\frac{q_2^2}{2}\right) +$$

$$\frac{P_0 k}{2}\left[2\sin^2\theta\,(q_2^2 - q_1^2) - 2\sin 2\theta\, q_1 q_2\right] +$$

$$+ G(q_1 p_2 - q_2 p_1) + P_0 F(q_1^2 + q_2^2), \qquad 4.4$$

where $F = G^2/2$.

the 4th term on the R.H.S. of (4.4) is due to shewing quads at angles $\theta(s)$, and the 5th and 6th terms are due to the longitudinal magnetic field.

In action-angle coordinates, the equations of motion take the form:

$$\frac{dJ_n}{ds} = -\frac{\partial H}{\partial \sigma_n} + \frac{\partial J_n}{\partial p_2}(-\gamma p_2), \qquad 4.5$$

$$\frac{d\sigma_n}{ds} = \frac{\partial H}{\partial J_n} + \frac{\partial J_n}{\partial p_2}(-\gamma p_2), \qquad 4.6$$

where

$$J_n = \frac{P_0}{2\beta_n}\left[q_n^2 + \left(\frac{\beta_n p_n}{P_0} + \alpha_n q_n\right)^2\right], \qquad 4.7$$

$$\sigma_n = -\tan^{-1}\left(\frac{p_n\beta_n + \alpha_n q_n P_0}{q_n P_0}\right) - \chi_n(s), \qquad 4.8$$

$$\chi_n = -\frac{2\pi}{L}\nu_n s + \int_0^s \frac{ds'}{\beta_n(s')}, \qquad 4.9$$

$$\nu_n = \frac{1}{2\pi}\int_0^L \frac{ds}{\beta_n(s)}. \qquad 4.10$$

Note that α_n and β_n are the usual lattice functions, and that the J_n are the usual Courant-Snyder invariants or emittances.

The Hamiltonian in action-angle coordinates is obtained by substituting the inverse of (4.7) (4.8),

$$q_n = \left(\frac{2\beta_n J_n}{p_0}\right)^{1/2} \cos(\sigma_n + \chi_n) \quad , \qquad 4,11$$

$$p_n = -\left(\frac{2p_0 J_n}{\beta_n}\right)^{1/2} \left[\sin(\sigma_n + \chi_n) + \alpha_n \cos(\sigma_n + \chi_n) \right] , \qquad 4,1$$

into (4,4):

$$H = J_1\omega_1 + J_2\omega_2 + 2\beta_1(F - k\sin^2\theta) J_1 \cos^2(\sigma_1 + \chi_1) +$$
$$+ 2\beta_2(F + k\sin^2\theta) J_2 \cos^2(\sigma_2 + \chi_2) +$$
$$+ 2(J_1 J_2)^{1/2} \left\{ \begin{array}{l} \cos(\sigma_1+\chi_1)\cos(\sigma_2+\chi_2)[-k\sin 2\theta(\beta_1\beta_2)^{1/2} + \\ + G\left(\alpha_1\left(\frac{\beta_2}{\beta_1}\right)^{1/2} - \alpha_2\left(\frac{\beta_1}{\beta_2}\right)^{1/2}\right)] + \\ -G\left(\frac{\beta_1}{\beta_2}\right)^{1/2}\cos(\sigma_1+\chi_1)\cos(\sigma_2+\chi_2) + G\left(\frac{\beta_2}{\beta_1}\right)^{1/2}\cos(\sigma_2+\chi_2)\sin(\sigma_1+ \end{array} \right.$$

4,13

where $\omega_n = 2\pi\nu_n / L$. The damping terms in (4.5)(4.6) are given by

$$p_2 \frac{\partial J_2}{\partial p_2} = 2J_2 [\sin^2(\sigma_2+\chi_2) + \alpha_2 \sin(\sigma_2+\chi_2)\cos(\sigma_2+\chi_2)] , \qquad 4,14$$

$$p_2 \frac{\partial \sigma_2}{\partial p_2} = \sin(\sigma_2+\chi_2)\cos(\sigma_2+\chi_2) + \alpha_2 \cos^2(\sigma_2+\chi_2) . \qquad 4,15$$

Expansion of (4,13) – (4,15) in a Fourier series in $\exp(2\pi i s/L)$ and retention of the resonance and angle independent terms, yields the resonance Hamiltonian:

$$H_r = J_1(\omega_1 + \delta\omega_1) + J_2(\omega_2 + \delta\omega_2) + $$

$$-|\Omega_\ell| \cos(\sigma_1 - \sigma_2 + 2\pi s\ell/L + \delta_\ell), \qquad 4,16$$

and the angle independent terms due to damping:

$$\langle \gamma p_2 \frac{\partial J_2}{\partial p_2} \rangle_r = \bar{\gamma} J_2 , \qquad 4,17$$

$$\langle \gamma p_2 \frac{\partial \sigma_2}{\partial p_2} \rangle_r = \omega_\gamma . \qquad 4,18$$

The parameters appearing in (4,16)–(4,18) are Fourier coefficients defined by

$$\delta\omega_1 = \int_0^L \frac{ds}{L} \beta_1 [F - K\sin^2\theta] , \qquad 4,19$$

$$\delta\omega_2 = \int_0^L \frac{ds}{L} \beta_2 [F + K\sin^2\theta] , \qquad 4,20$$

$$|\Omega_\ell| e^{i\delta_\ell} = \int_0^L \frac{ds}{L} e^{-2\pi i s\ell/L + i(\chi_1 - \chi_2)} \left[K\sin 2\theta (\beta_1\beta_2)^{1/2} + G\alpha_2 \left(\frac{\beta_1}{\beta_2}\right)^{1/2} - G\alpha_1 \left(\frac{\beta_2}{\beta_1}\right)^{1/2} + iG\left(\left(\frac{\beta_1}{\beta_2}\right)^{1/2} + \left(\frac{\beta_2}{\beta_1}\right)^{1/2}\right) \right.$$

4,

The s-dependence in H_r (Eq. 4.10) is eliminated via a canonical transformation using a generating function of the second kind:

$$W = I_1 \left(\sigma_1 + 2\pi \frac{\ell s}{L} + \delta_\ell \right) + I_2 \sigma_2 \quad , \qquad 4.24$$

The new action-angle coordinates $\{J_n, \phi_n\}$ are obtained via

$$J_n = \frac{\partial W}{\partial \sigma_n} = I_n \, , \qquad 4.25$$

$$\phi_1 = \frac{\partial W}{\partial I_1} = \sigma_1 + 2\pi \frac{\ell s}{L} + \delta_\ell \, , \qquad 4.26$$

$$\phi_2 = \frac{\partial W}{\partial I_2} = \sigma_2 \quad , \qquad 4.27$$

and $H \rightarrow H_r + \partial W / \partial s$. The final form for the equations of motion in the resonance approximation is

$$\frac{dI_1}{ds} = -\Omega (I_1 I_2)^{1/2} \sin(\phi_1 - \phi_2) \qquad 4.28$$

$$\frac{d\phi_1}{ds} = \Omega_1 - \frac{\Omega}{2} \left(\frac{I_2}{I_1} \right)^{1/2} \cos(\phi_1 - \phi_2) \qquad 4.29$$

$$\frac{dI_2}{ds} = \Omega (I_1 I_2)^{1/2} \sin(\phi_1 - \phi_2) - \gamma I_2 \qquad 4.30$$

$$\frac{d\phi_2}{ds} = \Omega_2 - \frac{\Omega}{2}\left(\frac{I_1}{I_2}\right)^{1/2} \cos(\phi_1 - \phi_2) , \qquad 4.31$$

where

$$\Omega = |\Omega_\ell| , \qquad 4.32$$

$$\Omega_1 = \omega_1 + \delta\omega_1 + 2\pi\ell/L , \qquad 4.33$$

$$\Omega_2 = \omega_2 + \delta\omega_2 - \omega\gamma , \qquad 4.34$$

Linear equations of motion are obtained from (4.29)–(4.34) via the following coordinate transformations:

$$Q_n = (2I_n)^{1/2} \cos\phi_n , \qquad 4.35$$

$$P_n = -(2I_n)^{1/2} \sin\phi_n , \qquad 4.36$$

(Note that these Q's and P's aren't the same as the q's and p's of (4.1) – (4.8)). The equations of motion expressed in these coordinates are

$$\frac{dQ_1}{ds} = \Omega_1 P_1 - \frac{\Omega}{2} P_2 \qquad 4.37$$

$$\frac{dP_1}{ds} = -\Omega_1 Q_1 + \frac{\Omega}{2} Q_2 \qquad 4.38$$

$$\frac{dQ_2}{ds} = \Omega_2 P_2 - \frac{\gamma}{2} Q_2 - \frac{\Omega}{2} P_1 \qquad 4.39$$

$$\frac{dP_2}{ds} = -\Omega_2 Q_2 - \frac{\gamma}{2} P_2 + \frac{\Omega}{2} Q_1 \qquad 4.40$$

In general, there is no simple expression for the solution to 4.41. Nevertheless a simple approximate solution results when $\bar{\gamma} \ll |\Omega|$ (Cooling rate $\ll$ coupling rate).

The approximate eigenvalues are obtained as perturbations of the $\bar{\gamma}=0$ eigenvalues. Let μ_0 be the solution to (4.41) for $\bar{\gamma}=0$; i.e.,

$$(\mu_0^2 - \Omega_1\Omega_2 + \Omega_{\bar{\Omega}}^2)^2 + \mu_0^2(\Omega_1+\Omega_2)^2 = 0 \quad . \qquad 4.42$$

The solutions to (4.42) are given by

$$\mu_0 = \begin{Bmatrix} \pm i\,\omega_1 \\ \pm i\,\omega_2 \end{Bmatrix} \qquad 4.43$$

where

$$\omega_1 = \frac{1}{2}\left[\Omega_1 + \Omega_2 - (\Delta^2 + \Omega^2)^{1/2}\right] , \qquad 4.44$$

$$\omega_2 = \frac{1}{2}\left[\Omega_1 + \Omega_2 + (\Delta^2 + \Omega^2)^{1/2}\right] , \qquad 4.45 .$$

and

$$\Delta = \Omega_2 - \Omega_1 , \qquad 4.46$$

To first order in $\bar{\gamma}$, the eigenvalues are obtained by assuming

$$\mu = \mu_0 + \delta\mu_0 , \qquad 4.47$$

substituting (4.47) into (4.41), and retaining only terms linear in $\delta\mu_0$ and $\bar{\gamma}$. Solving this equation for $\delta\mu_0$ yields

$$\delta\mu_0 = -\frac{\bar{\gamma}}{4}\left(1 \pm \frac{\Delta}{(\Delta^2+\Omega^2)^{1/2}}\right) , \qquad 4.48$$

where the + sign corresponds to the perturbation of $\pm i\omega_2$, and the − sign corresponds to the perturbation of $\pm i\omega_1$. Consequently, the eigenvalues of F come in pairs

$$\mu = \begin{Bmatrix} -\gamma_1 \pm i\omega_1 \\ -\gamma_2 \pm i\omega_2 \end{Bmatrix} , \qquad 4.49$$

where

$$\gamma_1 = \frac{\bar{\gamma}}{4}\left(1 - \frac{\Delta}{(\Delta^2+\Omega^2)^{1/2}}\right) , \qquad 4.50$$

$$\gamma_2 = \frac{\bar{\gamma}}{4}\left(1 + \frac{\Delta}{(\Delta^2+\Omega^2)^{1/2}}\right) , \qquad 4.51$$

An approximate expression for T_0 can be obtained from these eigenvalues. Unlike the eigenvalues (4.50)/(4.51), however, it is sufficient to obtain T_0 to zeroth order in $\bar{\gamma}$. The damping contribution to T_0 is $\mathcal{O}(\bar{\gamma}/\Omega)$,

and therefore can be neglected (see for example Eq. (2.11) (2.1:
A straight forward, yet tedious, calculation yields T_0 to zeroth order in F:

$$T_0 = \begin{bmatrix} 1 & -i & \lambda & -i\lambda \\ 1 & i & \lambda & i\lambda \\ -\lambda & i\lambda & 1 & -i \\ -\lambda & -i\lambda & 1 & i \end{bmatrix}, \qquad 4.52$$

where

$$\lambda = \frac{\Omega}{\Delta + (\Delta^2 + \Omega^2)^{1/2}}. \qquad 4.53$$

The normal coordinates obtained from T_0 via $P = T_0 X$ are given by

$$P_1 = Q_1 - iP_1 + \lambda(Q_2 - iP_2), \qquad 4.54$$

$$P_2 = P_1^*, \qquad 4.55$$

$$P_3 = Q_2 - iP_2 - \lambda(Q_1 - iP_1), \qquad 4.56$$

$$P_4 = P_3^*, \qquad 4.57$$

with associated eigenvalues ($dP_n/dS = \mu_n P_n$):

$$\mu_1 = -\gamma_1 + i\omega_1, \qquad 4.58$$

$$\mu_2 = -\gamma_1 - i\omega_1, \qquad 4.59$$

$$\mu_3 = -\gamma_2 + i\omega_2, \qquad 4.60$$

$$\mu_4 = -\gamma_2 - i\omega_2. \qquad 4.61$$

The two damped invariants ($dW_n/dS = -\Gamma_n W_n$) are

$$W_1 = \tfrac{1}{2} P_1 P_1^* \quad , \qquad 4.6$$

$$= \tfrac{1}{2}(Q_1^2 + P_1^2) + \tfrac{1}{2}\lambda^2 (Q_2^2 + P_2^2) + \lambda (Q_1 Q_2 + P_1 P_2) , \qquad 4.6$$

$$= I_1 + \lambda^2 I_2 + 2\lambda (I_1 I_2)^{1/2} \cos(\phi_1 - \phi_2) , \qquad 4.64$$

with $\Gamma_1 = -2\gamma_1$, and

$$W_2 = \tfrac{1}{2} P_2 P_2^* \qquad 4.65$$

$$= \tfrac{1}{2}(Q_2^2 + P_2^2) + \tfrac{1}{2}\lambda^2 (Q_1^2 + P_1^2) - \lambda (Q_1 Q_2 + P_1 P_2) , \qquad 4.66$$

$$= I_2 + \lambda^2 I_1 - 2\lambda (I_1 I_2)^{1/2} \cos(\phi_1 - \phi_2) , \qquad 4.6$$

with $\Gamma_2 = -2\gamma_2$. Note that in the for off resonance limit ($\Delta \gg |\Omega|$), $W_1 \to I_1$, $W_2 \to I_2$, due to $\lambda \to 0$.

The dynamics of the omitt amies (I_1 and I_2) for arbitrary λ is obtained from the evolution matrix $M(S|S_0)$. Using (3.32) and the observation that

$$T_0^{-1} = \frac{\tilde{T}^*}{2(1+\lambda^2)} \qquad (4.68)$$

($\tilde{T}$ is the transpose of T), the evolution matrix is given by

$$(1+\lambda^2)M(s|s_0) =$$

$$\begin{bmatrix} C_1 + \lambda^2 C_2 & S_1 + \lambda^2 S_2 & \lambda(C_1 - C_2) & \lambda(S_1 - S_2) \\ -(S_1 + \lambda^2 S_2) & C_1 + \lambda^2 C_2 & -\lambda(S_1 - S_2) & \lambda(C_1 - C_2) \\ \lambda(C_1 - C_2) & \lambda(S_1 - S_2) & C_2 + \lambda^2 C_1 & S_2 + \lambda^2 S_1 \\ -\lambda(S_1 - S_2) & \lambda(C_1 - C_2) & -(S_2 + \lambda^2 S_1) & C_2 + \lambda^2 C_1 \end{bmatrix} \quad 4,$$

where

$$C_n = e^{-\gamma_n(s-s_0)} \cos \omega_n (s - s_0) \quad , \qquad 4,70$$

$$S_n = e^{-\gamma_n(s-s_0)} \sin \omega_n (s - s_0) \quad . \qquad 4,71$$

Since

$$I_1 = \tfrac{1}{2}(X_1^2 + X_2^2) \quad , \qquad 4,72$$

the dynamics of I_1 is given by

$$I_1 = \tfrac{1}{2} \sum_\ell \sum_m (M_{1\ell} M_{1m} + M_{2\ell} M_{2m}) X^0_\ell X^0_m \quad . \qquad 4,73$$

Eq (4,73) describes the evolution of the emittance for one particle. The value of the emittance averaged over the initial conditions of several particles, however, is given by

$$\langle I_1 \rangle = \tfrac{1}{2} \sum_\ell [(M_{1\ell})^2 + (M_{2\ell})^2] \langle X^0_\ell X^0_\ell \rangle \quad . \qquad 4,74$$

$\langle x_\ell^0 x_m^0 \rangle = \delta_{\ell m} \langle x_\ell^0 x_\ell^0 \rangle$. Substitution of the matrix elements $M_{\ell m}$ from (4.69) into (4.74) yields

$$\langle I_1 \rangle = e^{-2\gamma_1 s}(I_1^0 + \lambda^2 I_2^0) + \lambda^2 e^{-2\gamma_2 s}(I_2^0 + \lambda^2 I_1^0) + \qquad 4.75$$

$$+ 2\lambda^2 e^{-(\gamma_1+\gamma_2)s} \cos(\omega_1-\omega_2)s\,(I_1^0 - I_2^0) ,$$

and

$$\langle I_2 \rangle = e^{-2\gamma_2 s}(I_2^0 + \lambda^2 I_1^0) + \lambda^2 e^{-2\gamma_1 s}(I_1^0 + \lambda^2 I_2^0) +$$

$$- 2\lambda^2 e^{-(\gamma_1+\gamma_2)s} \cos(\omega_1-\omega_2)s\,(I_1^0 - I_2^0), \qquad 4.76$$

where I_1^0 and I_2^0 are the values of the averaged emittances at $s = s_0 = 0$.

There are two main features to the emittance dynamics: 1) Both emittances decay exponentially with s but not at a single rate, and 2) There is a beating effect, due to the coupling, with beat frequency $|\omega_1 - \omega_2|$.

The detailed dynamics in (4.75) and (4.76) can be understood by considering the equations in two limits, far off resonance ($|\Delta| \gg \Omega$) and on resonance ($\Delta = 0$). Far off resonance ($\Delta > 0$) one obtains

$$\langle I_1 \rangle \cong I_1^0 e^{-2\gamma_1 s} , \qquad 4.77$$

$$\langle I_2 \rangle \cong I_2^0 e^{-2\gamma_2 s} , \qquad 4.78$$

Both emittances decay exponentially at a single rate. The rate for $\langle I_1 \rangle$, however, is much smaller than the rate for $\langle I_2 \rangle$.

In the on resonance case ($\Delta = 0$),

$$\langle I_1 \rangle = \frac{e^{-\frac{\gamma s}{2}}}{2} \left[I_1^0 + I_2^0 + (I_1^0 - I_2^0) \cos \Omega s \right], \qquad 4.81$$

$$\langle I_2 \rangle = \frac{e^{-\frac{\gamma s}{2}}}{2} \left[I_1^0 + I_2^0 - (I_1^0 - I_2^0) \cos \Omega s \right], \qquad 4.82$$

Both $\langle I_1 \rangle$ and $\langle I_2 \rangle$ decay at the same rate ($= \gamma/2$), and both show a beating effect, provided $I_1^0 \neq I_2^0$. Note that the beat frequency Ω, is called the "Rabi frequency" in NMR.

) This completes the normal mode analysis of the equations of motion in the resonant approximation. The main results of this section are the damping rates of the normal modes (4.50) (4.51), the tunes of the lattice (4.44) (4.45), and the dynamics of the emittances (4.75) and (4.76). These results are valid provided one is close to an ℓth order resonance ($|\Delta| \sim \Omega$). To obtain the exact damping rates for a non-resonance situation, the formalism of section III is required. This, however, has not been done by the author. Nevertheless, the equations obtained in this section can be used to make zeroth order estimates of the dynamics even in the far off resonance case (see Eqs (4.77) - (4.80)).

V Numerical Comparisson

In this section, a numerical comparison is made between the resonance approximation (section IV) and calculations utilizing the theory in section III. The two theories are applied to a model lattice for which Numerical values for tunes and cooling rates are obtained. A brief comparison is made and good agreement is indicated.

The model lattice under consideration is described by the following parameters: Two quads located symmetrically in a lattice in a region of zero dispersion with $\beta_1 = \beta_2 = 30\,m$, $\alpha_1 = \alpha_2 = 10.0$) are shewed at $\theta = 0.5$ milliradians. A betatron cooling section (with cooling rate $1.0\ sec^{-1}$) is located at the end of one of the quads. Finally, the lattice is assumed to be tuned close to a second order resonance; $\nu_1 \cong 3.55$, $\nu_2 = 5.55$, $\ell = -2$.

Due to the coupling, the tunes of the lattice can't be calculated in the usual way (i.e. $2\pi\nu = \int ds/\beta$). Nevertheless, the tunes mod-one can be obtained from eigenvalues of the transfer matrix; i.e. the tunes ν_i (mod-one) are given by

$$2\pi\nu_i = |\,\mathrm{Im}\,(\log(\lambda_i))\,| \,, \qquad 5.1$$

where $\{\lambda_i\}$ are the eigenvalues of $M(S_0 + L_1 S_0)$. Likewise the cooling rates γ_i are given by

$$\gamma_i = |\mathrm{Re}\,(\log(\lambda_i))\,| \,, \qquad 5.2$$

Numerical values for the tunes and cooling rates (converted to units of sec^{-1} assuming a lattice radius of $\rho = 50$) as a function of $\nu_0 - \nu_1 + \ell$ are plotted in figs. 1 and 2. Good agreement (25%) between the exact calculation (section III) and the resonance approximation (section IV) is indicated. Better agreement occurs, provided smaller values of α_1 and α_2 are assumed ($\alpha_1 = \alpha_2 = 1.0$). In this case, the resulting curves coincide (not shown) to within 5%.

VI Discussion and Applications

The resonance approximation (section IV) is an accurate model (see section V) for the betatron coupling plus cooling system. Utilization of this model yields simple expressions for normal coordinates, cooling rates, and tunes. The advantage of simple expressions is that the feasability of coupling-cooling experiments can be quickly determined. In this section, specific feasability conditions are delineated and are clarified with numerical calculations.

Two conditions must be met inorder to cool the horizontal betatron oscillation at a rate comparable to the vertical cooling rate: 1) The coupling component (skewed quad or solenoid) should be located in a region of zero horizontal dispersion, to avoid introducing vertical dispersion. 2) The lattice must be tuned sufficiently close to a coupling resonance ($|\Delta| \sim \Omega$).

The second condition is met provided the tune resolution of the lattice is adequate to observe the resonance. More specifically,

$$|\nu_2 - \nu_1 + \ell| \cong L|\Omega_\ell|/2\pi \quad , \qquad 6.1$$

where $|\Omega_\ell|$ is given by (4.21). For sake of illustration, physical values for $|\Omega_\ell|$ are calculated in two special cases:

1) one quad rotated by θ, and 2) longitudinal magnetic field originating from a solenoid.

For the skewed Quad:

$$\frac{L}{2\pi}|\Omega_{el}| \cong K\frac{\theta}{\pi}(\beta_1\beta_2)^{1/2} l_0 \quad , \qquad 5.2$$

where l_0 = length of the quad, and the β-functions are evaluated at the quad. Note that (5.2) is identical to the stop-band width as calculated by Courant & Snyder.[1] Assuming $K = 1.0\ m^{-2}$, $\rho = 50\ m$, $\beta_x = \beta_z = 20\ m$, $l_0 = 0.5\ m$, $\theta = 1.0$ mrad, one finds the width of the resonance to be:

$$\frac{L}{2\pi}|\Omega_{el}| \cong 0.003 \quad . \qquad 5.3$$

The resolution attainable in the experimental cooling ring at Fermilab is 0.001. Therefore this resonance should be observable.

For the solenoid,

$$\frac{L}{2\pi}|\Omega_{el}| \cong \frac{G l_0}{2\pi(\beta_1\beta_2)^{1/2}}\left[(\alpha_2\beta_1 - \alpha_1\beta_2)^2 + (\beta_1 + \beta_2)^2\right]^{1/2} \quad . \qquad 5.4$$

Assuming $B_{solenoid} = 0.01\ T$, $B\rho = 2.14\ Tm$ then $G = B_{sol.}/B\rho \cong 0.005\ m^{-1}$, $l_0 = 5\ m$, $\beta_1 \cong \beta_2$, $\alpha_1 = \alpha_2 = 0$, then

$$\frac{L}{2\pi}|\Omega_{el}| \cong 0.008 \quad . \qquad 5.5$$

This resonance should also be observeable in the experimental cooling ring.

Finally the assumption of $T_{cooling} >> T_{coupling}$ ($\bar{\gamma} << |\Omega$ must be checked. Assuming a cooling time of 1.0 sec., the condition of $\bar{\gamma} << |\Omega_e|$ is satisfied by both the skewed quad ($\bar{\gamma}/\Omega \cong 3\times10^{-4}$) and the solenoid ($\bar{\gamma}/\Omega \cong 1\times10^{-4}$). This calculation justifies the approximations in section IV.

References

1. Fred Mills (Private communication, 1981).
2. E.D. Courant and H.S. Snyder, Annals of Physics 3, 1 (1958).

Fig 1

<u>Lattice Tunes (mod-one)</u>. The solid curves are plots of $|\mathrm{Im}(\log(\lambda_i))|/2\pi$ as a function of $\nu_s - \nu_1 + \ell$, where $\{\lambda_i\}$ are the eigenvalues of the transfer matrix for the lattice described in section II. The dashed curves are plots of the tunes (mod-one) obtained within the resonance approximation. The tunes are plotted mod-one, because the computer yields $-\pi \leq \mathrm{Im}(\log\lambda) \leq \pi$. Note also that the tunes mod-one are plotted relative to 0.55. Good agreement (25%) is indicated between the exact calculations and the resonance approximation. Note that on resonance ($\nu_s - \nu_1 + \ell = 0$), the difference in the resonance approximation tunes is equal to $\hat{\Omega} = \Omega L/2\pi$.

Fig 2

<u>Normal Mode Cooling Rates</u>. The solid curves are plots of $|\mathrm{Re}(\log(\lambda_i))|$ as a function of $\nu_s - \nu_1 + \ell$. The dashed curves are plots of the normal mode cooling rates obtained within the resonance approximation. Note that the rates have been converted to ($\sec^{-1}$). Good agreement (25%) is indicated between the exact rates and the resonance approximation rates. Inspection of either curve reveals that the width of the resonance is given approximately by $\hat{\Omega} = \Omega L/2\pi$.

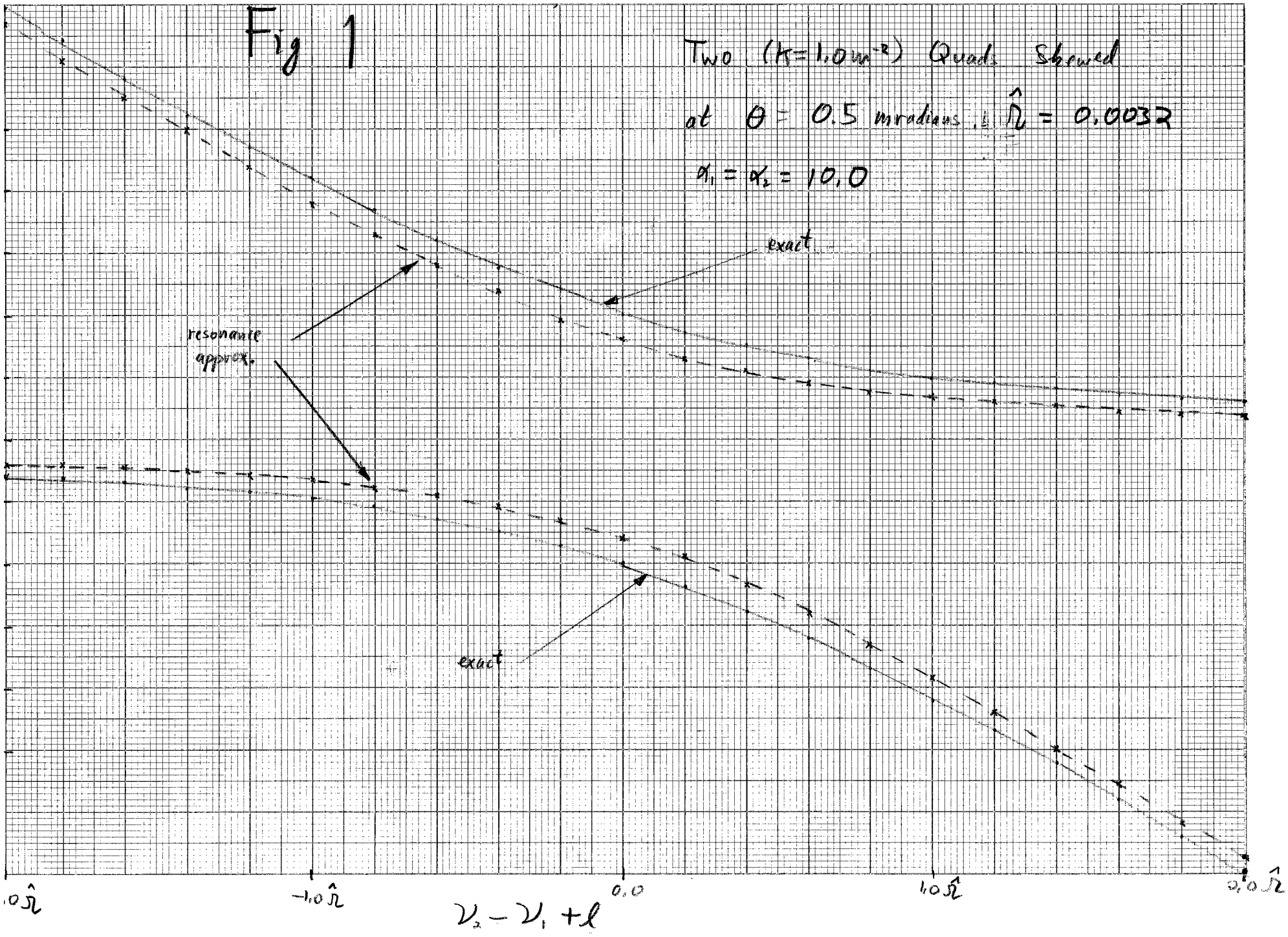
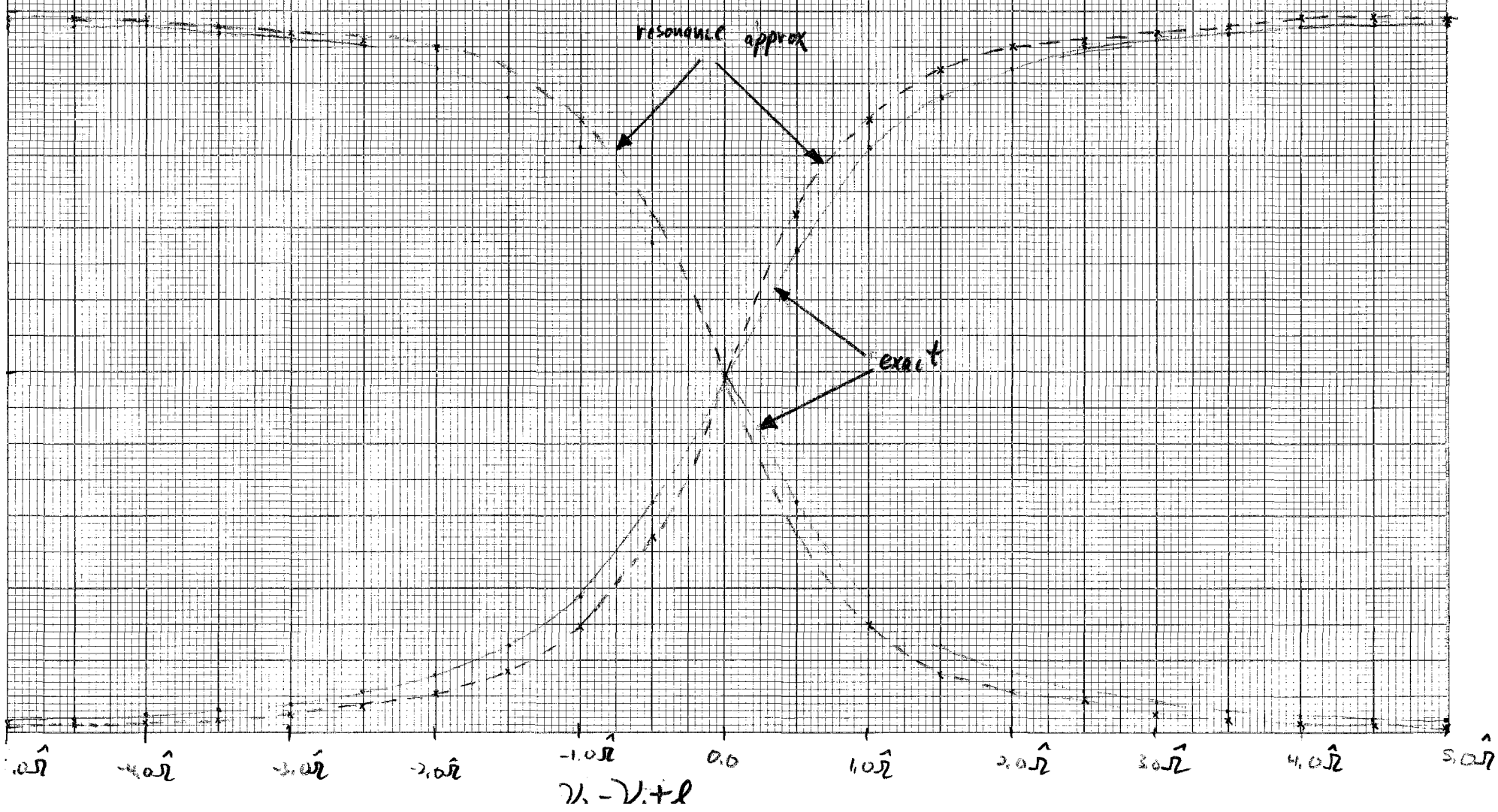


Fig 1
Two (k=1.0 m⁻²) Quads Skewed
at θ = 0.5 mradians, Ω̂ = 0.0032
α₁ = α₂ = 10.0
exact
resonance approx.
exact
.0 Ω̂
-1.0 Ω̂
0.0
1.0 Ω̂
2.0 Ω̂
ν₂ − ν₁ + ℓ

Fig 2

Two ($k = 1.0 m^{-2}$) Quads Showed

at $\theta = 0.5$ inradians, $\frac{\hat{\Omega}}{T} = 0.0032$

$\alpha_1 = \alpha_2 = 10.0$