



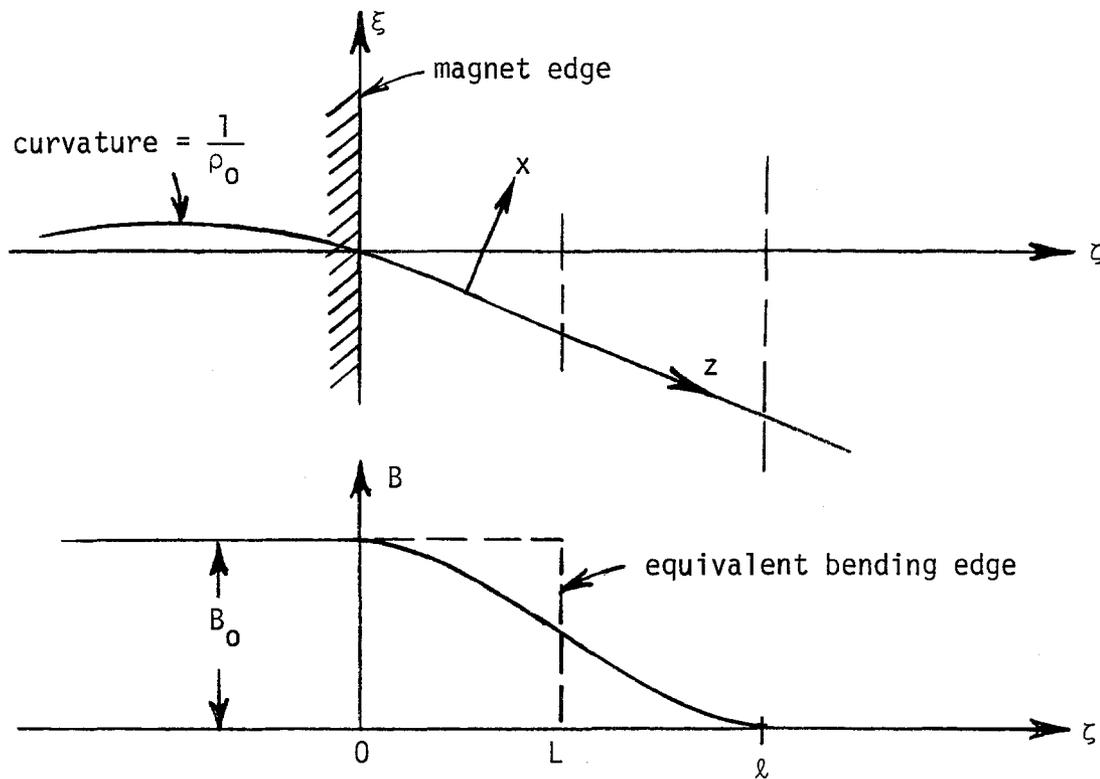
Transfer Matrices Across Soft Magnet Edges

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In this paper we derive the expanded forms for the transfer matrices across an arbitrary two-dimensional fringe-field, i.e. the field is independent of the coordinate along the magnet edge.

The Coordinates



The field on the mid-plane is written as

$$B(\zeta) = B_0 b(\zeta) \quad b(\zeta) = \begin{cases} 1 & \zeta = 0 \\ 0 & \zeta = l \end{cases} \quad (1)$$

where  $\ell$  is a measure of the "thickness" of the fringe-field. We also have the following relations between differentials

$$d\zeta = dz \cos\theta = dx \sin\theta \quad (2)$$

where  $\theta = \theta(\zeta)$  = angle between  $z$  and  $\zeta$ -axes and is given by

$$d\theta = \frac{Bdz}{B_0\rho_0} = \frac{1}{\rho_0} \frac{bd\zeta}{\cos\theta}$$

or, when integrated

$$\sin\theta = \sin\theta_0 + \frac{1}{\rho_0} \int_0^\zeta bd\zeta \quad (3)$$

where  $B_0\rho_0$  is the magnetic rigidity of the particle. The total variation of  $\theta$  in the fringe-field is generally small. In the following we shall simplify computation by assuming  $\theta$  to be constant having the value at the "equivalent bending edge" located at  $\zeta = L = \int_0^\ell bd\zeta$ . The normally used "hard edge" angle  $\bar{\theta}$  is defined by

$$\sin\bar{\theta} = \sin\theta_0 + \frac{1}{\rho_0} \int_0^\ell bd\zeta. \quad (4)$$

The angle  $\theta$  we use here is given by

$$\begin{aligned} \sin\theta &= \sin\theta_0 + \frac{1}{\rho_0} \int_0^L bd\zeta \\ &= \sin\bar{\theta} - \frac{1}{\rho_0} \int_L^\ell bd\zeta \end{aligned} \quad (5)$$

or, since the difference between  $\theta$  and  $\bar{\theta}$  is generally small, it is given approximately by

$$\theta = \bar{\theta} - \frac{1}{\rho_0 \cos\bar{\theta}} \int_L^\ell bd\zeta. \quad (6)$$

### Iterative Solutions of the Linear Orbit Equations

The linear orbit equations are

$$x'' - mx = 0, \quad y'' - ny = 0 \quad (\text{prime} = \frac{d}{dz}) \quad (7)$$

where

$$m = -\frac{1}{B_0 \rho_0} \frac{\partial B}{\partial x} - \frac{1}{\rho_0^2} = -\frac{\sin \theta}{\rho_0} \dot{b} - \frac{1}{\rho_0^2} b^2 \quad (8)$$

and

$$n = \frac{1}{B_0 \rho_0} \frac{\partial B}{\partial x} = \frac{\sin \theta}{\rho_0} \dot{b} \quad (\text{dot} = \frac{d}{d\xi}). \quad (9)$$

To solve the equation  $x'' = mx$  by iteration we have:

1. First order

Putting  $x = x(\zeta = 0) = x_0$  in the right-hand-side and integrating we get

$$\begin{cases} x'' = mx_0 \\ x' = x_0' + x_0 \int mdz \\ x = x_0 + x_0' z + x_0 \iint mdz^2 \end{cases} \quad (10)$$

where we used the short hand notation

$$\iint mdz^2 \equiv \int_0^z dz_1 \int_0^{z_1} dz_2 m(z_2) \quad (11)$$

etc.

2. Second order

Substituting the first order  $x$  from Eq. (10) in the right-hand-side and integrating we get

$$\begin{cases} x'' = x_0 m + x_0' m z + x_0 m \iint mdz^2 \\ x' = x_0' + x_0 \int mdz + x_0' \int m z dz + x_0 \int mdz \iint mdz^2 \\ x = x_0 + x_0' z + x_0 \iint mdz^2 + x_0' \iint m z dz^2 \\ \quad \quad \quad + x_0 \iint mdz^2 \iint mdz^2 \end{cases} \quad (12)$$

The regularity is now clear and we can write for the x-transfer matrix

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \quad (13)$$

$$\begin{cases} M_{11} = 1 + \int \int m dz^2 + \int \int m dz^2 \int \int m dz^2 + \dots \\ M_{12} = z + \int \int m z dz^2 + \int \int m dz^2 \int \int m z dz^2 + \dots \\ M_{21} = 0 + \int m dz + \int m dz \int \int m dz^2 + \dots \\ M_{22} = 1 + \int m z dz + \int m dz \int \int m z dz^2 + \dots \end{cases}$$

where the terms in the elements of M are vertically lined up according to the generic order (power of m). When applied to the fringe-field at a magnet edge, however, they should be ordered by the power of the "softness" parameter and the terms should be realigned as

$$\begin{cases} M_{11} = 1 + \int \int m dz^2 + \int \int m dz^2 \int \int m dz^2 + \dots \\ M_{12} = 0 + z + \int \int m z dz^2 + \dots \\ M_{21} = \int m dz + \int m dz \int \int m dz^2 + \int m dz \int \int m dz^2 \int \int m dz^2 + \dots \\ M_{22} = 1 + \int m z dz + \int m dz \int \int m z dz^2 + \dots \end{cases} \quad (14)$$

i.e. when  $\ell \rightarrow 0$  (hard edge) M becomes

$$M = \begin{pmatrix} 1 & 0 \\ \int m dz & 1 \end{pmatrix}.$$

### Transfer Matrices

The calculation of the elements of the transfer matrices across the fringe-field from  $\zeta = 0$  to  $\zeta = \ell$  is straightforward. We shall give only a few examples below.

$$\left. \begin{aligned}
 \int_{\text{fringe field}} mdz &= -\frac{\tan\theta}{\rho_0} \int_0^{\ell} b dz - \frac{1}{\rho_0^2 \cos\theta} \int_0^{\ell} b^2 dz \\
 &= -\frac{\tan\theta}{\rho_0} \left[ \int_0^{\ell} b dz + \frac{\ell}{\rho_0 \sin\theta} \left( \frac{1}{\ell} \int_0^{\ell} b^2 dz \right) \right] \\
 \iint_{\text{fringe field}} mdz^2 &= -\frac{\tan\theta}{\rho_0 \cos\theta} \int_0^{\ell} dz \int_0^{\zeta} b dz - \frac{1}{\rho_0^2 \cos^2\theta} \int_0^{\ell} dz \int_0^{\zeta} b^2 dz \\
 &= -\frac{\ell}{\rho_0} \frac{\tan\theta}{\cos\theta} \left[ \frac{1}{\ell} \int_0^{\ell} dz \int_0^{\zeta} b dz + \frac{\ell}{\rho_0 \sin\theta} \left( \frac{1}{\ell^2} \int_0^{\ell} dz \int_0^{\zeta} b^2 dz \right) \right] \\
 \int_{\text{fringe field}} mzdz &= -\frac{\tan\theta}{\rho_0 \cos\theta} \int_0^{\ell} b_{\zeta} dz - \frac{1}{\rho_0^2 \cos^2\theta} \int_0^{\ell} b^2_{\zeta} dz \\
 &= -\frac{\ell}{\rho_0} \frac{\tan\theta}{\cos\theta} \left[ \frac{1}{\ell} \int_0^{\ell} b_{\zeta} dz + \frac{\ell}{\rho_0 \sin\theta} \left( \frac{1}{\ell^2} \int_0^{\ell} b^2_{\zeta} dz \right) \right] \\
 \iint_{\text{fringe field}} mzdz^2 &= -\frac{\tan\theta}{\rho_0 \cos^2\theta} \int_0^{\ell} dz \int_0^{\zeta} b_{\zeta} dz - \frac{1}{\rho_0^2 \cos^3\theta} \int_0^{\ell} dz \int_0^{\zeta} b^2_{\zeta} dz \\
 &= -\frac{\rho_0}{\tan\theta} \left( \frac{\ell}{\rho_0} \frac{\tan\theta}{\cos\theta} \right)^2 \left[ \frac{1}{\ell^2} \int_0^{\ell} dz \int_0^{\zeta} b_{\zeta} dz + \frac{\ell}{\rho_0 \sin\theta} \left( \frac{1}{\ell^3} \int_0^{\ell} dz \int_0^{\zeta} b^2_{\zeta} dz \right) \right].
 \end{aligned} \right\} \tag{15}$$

We shall now simply exhibit the total result.

1. Horizontal transfer matrix M

$$\left\{ \begin{aligned}
 M_{11} &= 1 - \epsilon(a_1 + a_{11}\lambda) + \epsilon^2(a_2 + a_{21}\lambda + a_{22}\lambda^2) + \dots \\
 M_{12} &= \frac{\rho_0}{\tan\theta} \epsilon [1 - \epsilon(b_1 + b_{11}\lambda) + \dots] \\
 M_{21} &= -\frac{\tan\theta}{\rho_0} [(c_1 + c_{11}\lambda) - \epsilon(c_2 + c_{21}\lambda + c_{22}\lambda^2) \\
 &\quad + \epsilon^2(c_3 + c_{31}\lambda + c_{32}\lambda^2 + c_{33}\lambda^3) + \dots] \\
 M_{22} &= 1 - \epsilon(d_1 + d_{11}\lambda) + \epsilon^2(d_2 + d_{21}\lambda + d_{22}\lambda^2) + \dots
 \end{aligned} \right. \tag{16}$$

where

$$\varepsilon \equiv \frac{\ell}{\rho_0} \frac{\tan\theta}{\cos\theta}, \quad \lambda \equiv \frac{\ell}{\rho_0} \frac{1}{\sin\theta},$$

and  $\frac{\ell}{\rho_0}$  may be considered as the "softness" parameter. The numerical coefficients are given by:

$$\begin{aligned} \left\{ \begin{array}{l} a_1 = \frac{1}{\ell} \iint \dot{b} \\ c_1 = \int \dot{b} = -1 \end{array} \right. & \quad \left\{ \begin{array}{l} b_1 = \frac{1}{\ell^2} \iint \dot{b}\zeta \\ d_1 = \frac{1}{\ell} \int \dot{b}\zeta \end{array} \right. \\ \\ \left\{ \begin{array}{l} a_{11} = \frac{1}{\ell^2} \iint b^2 \\ c_{11} = \frac{1}{\ell} \iint b^2 \end{array} \right. & \quad \left\{ \begin{array}{l} b_{11} = \frac{1}{\ell^3} \iint b^2\zeta \\ d_{11} = \frac{1}{\ell^2} \int b^2\zeta \end{array} \right. \\ \\ \left\{ \begin{array}{l} a_2 = \frac{1}{\ell^2} \iint \dot{b} \iint \dot{b} \\ c_2 = \frac{1}{\ell} \int \dot{b} \iint \dot{b} \end{array} \right. & \quad \left\{ \begin{array}{l} b_2 = \frac{1}{\ell^3} \iint \dot{b} \iint \dot{b}\zeta \\ d_2 = \frac{1}{\ell^2} \int \dot{b} \iint \dot{b}\zeta \end{array} \right. \tag{17} \\ \\ \left\{ \begin{array}{l} a_{21} = \frac{1}{\ell^3} \left[ \iint \dot{b} \iint b^2 + \iint b^2 \iint \dot{b} \right] \\ c_{21} = \frac{1}{\ell^2} \left[ \int \dot{b} \iint b^2 + \int b^2 \iint \dot{b} \right] \end{array} \right. & \quad \left\{ \begin{array}{l} b_{21} = \frac{1}{\ell^4} \left[ \iint \dot{b} \iint b^2\zeta + \iint b^2 \iint \dot{b}\zeta \right] \\ d_{21} = \frac{1}{\ell^3} \left[ \int \dot{b} \iint b^2\zeta + \int b^2 \iint \dot{b}\zeta \right] \end{array} \right. \\ \\ \left\{ \begin{array}{l} a_{22} = \frac{1}{\ell^4} \iint b^2 \iint b^2 \\ c_{22} = \frac{1}{\ell^3} \int b^2 \iint b^2 \end{array} \right. & \quad \left\{ \begin{array}{l} b_{22} = \frac{1}{\ell^5} \iint b^2 \iint b^2\zeta \\ d_{22} = \frac{1}{\ell^4} \int b^2 \iint b^2\zeta \end{array} \right. \end{aligned}$$

$$\left\{ \begin{array}{l} c_3 = \frac{1}{l^2} \int \dot{b} \iint \dot{b} \iint \dot{b} \\ c_{31} = \frac{1}{l^3} \left[ \int \dot{b} \iint \dot{b} \iint b^2 + \int \dot{b} \iint b^2 \iint \dot{b} + \int b^2 \iint \dot{b} \iint \dot{b} \right] \\ c_{32} = \frac{1}{l^4} \left[ \int b^2 \iint b^2 \iint \dot{b} + \int b^2 \iint \dot{b} \iint b^2 + \int \dot{b} \iint b^2 \iint b^2 \right] \\ c_{33} = \frac{1}{l^5} \int b^2 \iint b^2 \iint b^2 \end{array} \right. \quad \text{etc.}$$

where we have omitted all  $d\zeta$  and where the limits of integration are as exemplified by the coefficients explicitly exhibited in Eqs. (15).

The simplest is the linear fringe-field given by

$$b = 1 - \frac{\zeta}{l} \quad \text{and} \quad \dot{b} = -\frac{1}{l}. \quad (18)$$

For this we have, to  $\epsilon^2$  terms

$$\left\{ \begin{array}{llll} a_1 = -\frac{1}{2}, & b_1 = -\frac{1}{6}, & c_1 = -1, & d_1 = -\frac{1}{2} \\ a_{11} = \frac{1}{4}, & b_{11} = \frac{1}{20}, & c_{11} = \frac{1}{3}, & d_{11} = \frac{1}{12} \\ a_2 = \frac{1}{24}, & b_2 = \frac{1}{120}, & c_2 = \frac{1}{6}, & d_2 = \frac{1}{24} \\ a_{21} = -\frac{13}{360}, & b_{21} = -\frac{13}{2520}, & c_{21} = -\frac{7}{60}, & d_{21} = -\frac{7}{360} \\ a_{22} = \frac{1}{160}, & b_{22} = \frac{1}{1440}, & c_{22} = \frac{1}{84}, & d_{22} = -\frac{1}{672} \\ c_3 = -\frac{1}{120}, & c_{31} = \frac{11}{1260}, & c_{32} = -\frac{211}{90720}, & c_{33} = \frac{1}{7392}. \end{array} \right. \quad (19)$$

## 2. Vertical transfer matrix N

Without the centripetal term  $\frac{1}{\rho^2}$  in  $n$  the vertical transfer matrix is considerably simpler and is given by

$$N = \begin{pmatrix} 1 + a_1 \epsilon + a_2 \epsilon^2 + \dots & \frac{\rho_0}{\tan \theta} \epsilon (1 + b_1 \epsilon + \dots) \\ \frac{\tan \theta}{\rho_0} (c_1 + c_2 \epsilon + c_3 \epsilon^2 + \dots) & 1 + d_1 \epsilon + d_2 \epsilon^2 + \dots \end{pmatrix} \quad (20)$$

where the coefficients are as given in Eqs. (17) and (19).

Conventionally one continues the magnet interior matrix to the "equivalent bending edge", multiplies it by an "equivalent edge matrix", then continues with the drift (field-free) matrix from the equivalent edge onward. In this case the "equivalent edge matrices" are then, for a dipole edge

$$\text{horizontal} \quad \begin{pmatrix} 1 & -\frac{\rho-L}{\cos\theta} \\ 0 & 1 \end{pmatrix} M \begin{pmatrix} \cos\frac{L}{\rho_0 \cos\theta} & -\rho_0 \sin\frac{L}{\rho_0 \cos\theta} \\ \frac{1}{\rho_0} \sin\frac{L}{\rho_0 \cos\theta} & \cos\frac{L}{\rho_0 \cos\theta} \end{pmatrix}$$

$$\text{vertical} \quad \begin{pmatrix} 1 & -\frac{\rho-L}{\cos\theta} \\ 0 & 1 \end{pmatrix} N \begin{pmatrix} 1 & -\frac{L}{\cos\theta} \\ 0 & 1 \end{pmatrix}$$

The case of the three-dimensional fringe-field, i.e.  $B = B(\xi, \zeta) = B_0 b(\xi, \zeta)$  will be treated in a separate report.