national accelerator laboratory

TM-566
1501

# Oscillation Damping Factors for Off-Momentum Orbits in Electron Storage Rings

D.A. Edwards

April 1975

## Introduction

The design orbit of a separated function electron storage ring is usually arranged to pass through the quadrupole centers. The interplay between energy loss due to synchrotron radiation and energy gain from the radiofrequency system then leads to damping of betatron and synchrotron oscillations. However, if the synchronous orbit does not correspond to the design orbit, synchrotron radiation in the quadrupoles becomes a significant factor, and for sufficiently large deviation of the synchronous momentum from that appropriate to the design orbit, either the synchrotron oscillations or the radial betatron oscillations will become anti-damped.*

That this will be the case may be seen as follows. For constant synchronous energy, $E_s$, the equations of motion for a synchrotron oscillation are

$$\frac{d}{dt}\frac{\Delta E}{E_s} = \frac{\omega_s}{2\pi}\frac{eV}{E_s}(\sin\phi - \sin\phi_s) - \frac{\omega_s}{2\pi}\frac{1}{E_s}\left[U(E) - U(E_s)\right]$$

$$\frac{d\phi}{dt} = \frac{h\omega_s}{\gamma_t^2}\frac{\Delta E}{E_s} \qquad (1)$$

*We are considering only situations in which the damping of the vertical betatron oscillations is unperturbed. If there is coupling between the two modes of betatron oscillation, or if the ring does not lie in a plane, the effects on the vertical motion should also be taken into account.

where U(E) is the energy radiated per turn by a particle traversing a closed orbit for energy E, the other quantities have their conventional meanings, and the limit $E \gg mc^2$ has been taken. In the small amplitude approximation, the above relations may be combined to give

$$\frac{d^2}{dt^2}\left(\frac{\Delta E}{E_s}\right) + 2\alpha_s \frac{d}{dt}\left(\frac{\Delta E}{E_s}\right) + \Omega_s^2 \left(\frac{\Delta E}{E_s}\right) = 0$$

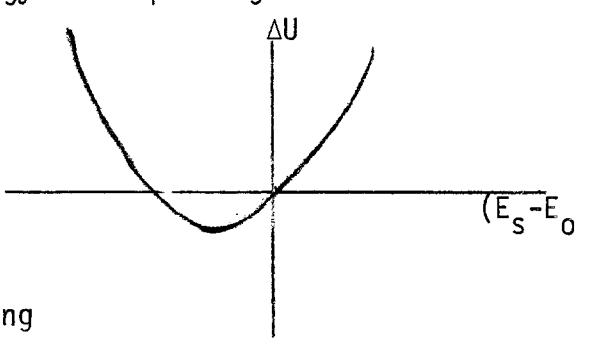
$$\Omega_s^2 \equiv -\omega_s^2 \quad \frac{heV\cos\phi_s}{2\pi E_s \gamma_t^2} \cos\phi_s \qquad (2)$$

$$\alpha_s \equiv \frac{\omega_s}{4\pi}\left(\frac{dU}{dE}\right)_{E_s}$$

Here, $\Omega_s$ is the angular frequency of synchrotron oscillations. Generally, $\alpha_s^2 \ll \Omega_s^2$, and the solution will represent a damped oscillation, i.e.,

$$\frac{\Delta E}{E_s} = Ae^{-\alpha_s t} \cos(\Omega_s t + \delta)$$

The quantity $\Delta U \equiv U(E_s) - U(E_0)$, where $E_0$ is the energy corresponding to the design orbit, has the character sketched at the right. As the synchronous energy is raised above $E_0$, $(dU/dE)_{E_s}$ increases and the synchrotron oscillations become more strongly damped. But according to Robinson's theorem,[1] the increased damping of the synchrotron oscillations occurs at the expense of weakened damping of the horizontal betatron oscillations, and ultimately, for some $E_s > E_0$, the latter will become anti-damped. Similarly, as $E_s$ is lowered below $E_0$,



the synchrotron oscillations will become less strongly damped, and for $E_s$ less than that at the minimum of the $\Delta U$ curve, these oscillations will anti-damp, though the horizontal betatron oscillations will be strongly damped.

Thus there are two limiting values of $E_s$, above and below $E_o$, at which anti-damping sets in. It is the purpose of this memorandum to estimate the critical values of $E_s$, to identify the manner in which the various lattice parameters influence the stable range in energy, and to comment on the variation of equilibrium beam size within this range.

## Damping Factors

In general, the notation and definitions of Sands[2] will be followed. The partition numbers, $J_i$, are related to the damping factors, $\alpha_i$, by

$$\alpha_i = J_i \alpha_o \tag{3}$$

where $\alpha_o$ is the reciprocal of a characteristic time, which is just twice the time that a particle of energy $E_s$ would radiate an energy $E_s$:

$$\alpha_o = \frac{(\omega_s/2\pi)}{2E_s/U(E_s)} \tag{4}$$

The subscript i stands for the three oscillation modes - synchrotron oscillations and horizontal and vertical betatron oscillations. Robinson's theorem requires that

$$J_E + J_H + J_V = 4 \tag{5}$$

Under our assumptions, the vertical betatron oscillations do not participate in the interchange of damping, and $J_V = 1$ independent of momentum. For the synchrotron and horizontal betatron oscillations respectively, we have

$$J_E = 2 + D$$
$$J_H = 1 - D \tag{6}$$

with

$$D \equiv \frac{2 \oint (\eta K/\rho_s) dz}{\oint (1/\rho_s^2) dz} \tag{7}$$

The definition (7) for D differs from that used by Sands in that we assume "straight" magnets, rather than wedge magnets. The integrals in (7) are to be evaluated along the synchronous orbit, so D is a function of momentum. We will include terms up to the second order in $(E_s - E_o)/E_o$; path length effects are of higher order, so the coordinate z may refer to the design orbit. The radius of curvature $\rho_s$, on the synchronous orbit is always a positive quantity regardless of the sense of the curvature. As in Sands' article, $\eta$ denotes the momentum dispersion function and $K \equiv eB'/p_s$.

On the design orbit, D = 0, so $J_E = 2$ and $J_H = 1$. These are the partition numbers normally ascribed to a separated function ring. When the synchronous orbit differs from the design orbit, the radius of curvature in a quadrupole becomes

$$\frac{1}{\rho_s} = \frac{eB}{p_s} = \frac{eB'\eta}{p_s}\left(\frac{\Delta E}{E_o}\right) = \frac{K_o \eta (\Delta E/E_o)}{1+(\Delta E/E_o)} \tag{8}$$

where $K_o \equiv eB'/p_o$. In a dipole, $\rho_s = \rho_o(1+\Delta E/E_o)$. The numerator of (7) then will become non-zero for $E_s \neq E_o$, while the denominator will receive contributions from the quadrupoles as well as the dipoles. Thus

$$D = 2\,\frac{\Delta E}{E_o}\,\frac{\oint \eta^2 K_o^2 dz}{\sum_i (\ell_{Bi}/\rho_{oi}^2) + (\Delta E/E_o)^2 \oint \eta^2 K_o^2 dz} \tag{9}$$

where the first term in the denominator represents a summation over the various dipoles of length $\ell_{Bi}$ in the ring. Since the dispersion function may have a first order energy dependence, it does not appear with a "0" subscript in the numerator. In the order to which this calculation is carried out, it is irrelevant whether or not $\eta$ is so subscripted in the denominator.

To estimate the integral over the quadrupoles, let us ignore the variation of $\eta$ within a quad, so that

$$\oint \eta^2 K_o^2\, dz \Longrightarrow \sum_{\text{quads}} \eta_i^2\, K_{oi}^2 \ell_{Qi} \tag{10}$$

where $\ell_{Qi}$ is the length of the $i^{th}$ quadrupole. Now assume that the dominant contribution to (10) arises in N simple FODO cells, each half cell containing a bend angle of $\pi/N$. Also assume that the value of $\eta$ at each quad is that appropriate to the cell itself - that is $\eta$ is matched at the ends of interaction region or other insertions. Then it is easy to show that (9) becomes

$$D = 2\,(\Delta E/E_o)\,\frac{F}{1+(\Delta E/E)^2 F} \tag{11}$$

with

$$F = 4 \frac{f_B}{f_Q} \frac{1}{\sin^2(\mu_o/2)} \left[(1+ \Delta E/E_o)^2 + \frac{1}{4} \sin^2 (\mu_o/2)\right] \tag{12}$$

In (12), $\mu_o$ is the betatron oscillation phase advance through a normal cell for a particle of energy $E_o$, and $f_B$, $f_Q$ are the fractions of the cell length occupied by dipoles and quads respectively. It is clear that the energy dependence of $\eta$ as reflected in the $\Delta E/E_o$ term in (12) will be unimportant in this case where we have presumed perfect matching.

As an example, take $\mu_o = \pi/3$ and $\ell_B/\ell_Q = 10$. Then F = 170, having dropped the energy dependence of $\eta$, and

$$D = \frac{340\ (\Delta E/E_o)}{1+170\ (\Delta E/E_o)^2} \tag{13}$$

Therefore, for $\Delta E/E_o \approx 0.3\%$, D = 1 and the radial betatron oscillations would no longer damp. And for $\Delta E/E_o \approx -0.6\%$, D = -2 and the synchrotron oscillations would be on the verge of anti-damping. Within this region, the second order term in $\Delta E/E_o$ is unimportant. Interestingly enough, for $\Delta E/E_o = +0.6\%$, D = 2; the partition numbers will be those normally characteristic of a combined function lattice.

The range in energy throughout which both modes of oscillation are stable can be increased by using a higher phase advance per cell or longer quadrupoles. On the other hand, if the normal cell portion of the lattice contains a free oscillation in $\eta$, as might arise from a mismatch at the ends of the insertions, then the stable range will be reduced, since the integral contains $\eta^2$ and so the free oscillation contribution will not average to zero.

## Equilibrium Beam Size

The mean square energy spread $\sigma_E^2$ is given by

$$\left(\frac{\sigma_E}{E_s}\right)^2 = \frac{C_q \gamma_s^2}{J_E} \frac{\oint 1/\rho_s^3 \, dz}{\oint 1/\rho_s^2 \, dz} \tag{14}$$

$$C_q = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} = 3.84 \times 10^{-13} \text{ m.}$$

Now that the energy range of interest has been shown to be small (~1%) and the terms strongly dependent on energy identified, the integrals in (14) need be evaluated only for the design orbit. For the simple lattice of the preceding section,

$$\left(\frac{\sigma_E}{E_s}\right)^2 = \frac{C_q \gamma_s^2}{J_E \rho_o} \tag{15}$$

and so the mean square physical width due to energy oscillations, $\sigma_{H,E}^2$, is

$$\sigma_{H,E}^2 = \eta^2 \left(\frac{\sigma_E}{E_s}\right)^2 = \frac{C_q \gamma_s^2 \eta^2}{J_E \rho_o} \tag{16}$$

The mean square width of the distribution of horizontal betatron oscillation amplitudes is

$$\left(\frac{\sigma_{H,\beta}^2}{\beta}\right) = \frac{C_q \gamma_s^2}{J_H} \frac{\oint (1/\rho_s^3) H dz}{\oint 1/\rho_s^2 dz} \tag{17}$$

with

$$H = \gamma\eta^2 + 2\alpha\eta\eta' + \beta\eta'^2 \tag{18}$$

In (18), $\alpha$, $\beta$, $\gamma$ are the usual Courant-Snyder parameters. Again, we evaluate the integrals in (17) along the design orbit. Actually, Sands gives general estimates for them, but since we have been using a particular lattice, we might as well continue to do so. It may be of some interest to compare the results with Sands' formulae.

The contribution from a half-cell to the integral in the numerator of (20) is

$$\int_0^{\ell_B} \frac{1}{\rho_0^3} H dz = \frac{\ell_B}{\rho_0^3} \left\{ \frac{H(\ell_B)-H(0)}{2} - \left(\frac{\ell_B^2}{12} \frac{d^2H}{dz^2}\right)_{\ell_B/2} - \left(\frac{\ell_B^4}{480} \frac{d^4H}{dz^4}\right) \right\} \tag{19}$$

The sum of the second and third terms in the brackets becomes larger as $\mu$ increases, but for typical parameters, they amount to only 7% of the first term at $\mu = 135^o$. We will neglect them below. In the first term, $H(\ell_B)$ and $H(0)$ mean the values of H at the two ends of the bending region. But outside of the bending region, $\eta$ propagates as a free oscillation and H is an invariant. So we can evaluate the H's at the quad midpoints, where $H = \eta^2/\beta$. Then (19) becomes

$$\int_0^{\ell_B} \frac{1}{\rho_0^3} H dz = \frac{\ell_B}{2\rho_0^3} \left[ \frac{\eta^2_{max}}{\beta_{max}} + \frac{\eta^2_{min}}{\beta_{min}} \right] \tag{20}$$

and for (17) we have

$$\left(\frac{\sigma^2_{H,\beta}}{\beta}\right) = \frac{C_q\gamma_s^{\ 2}}{J_H f_B}\ \frac{1}{\nu_o^3}\left[\left(\frac{(\mu/2)}{\sin\ (\mu/2)}\right)^3 \frac{1+3\ \cos^2(\mu/2)}{4\ \cos\ (\mu/2)}\right] \tag{21}$$

$$\nu_o \equiv N\mu/2\pi$$

The quantity in brackets is 1.08 for $\mu = 60^o$; for small phase advance, a reasonable approximation is to treat it as unity. At $\mu = 135^o$, however, it has grown to 1.9. Here again, free oscillations in $\eta$ do not average to zero in evaluating $\sigma_{H,\beta}$, and the presence of such oscillations can make expressions such as (21) of little value.

The beam width from both horizontal betatron oscillations and synchrotron oscillations is characterized by $\sigma_H^{\ 2} = \sigma_{H,\beta}^{\ 2} + \sigma_{H,E}^{\ 2}$. At a point where $\eta = \eta_{max}$ and $\beta = \beta_{max}$

$$\sigma_H^2 = \frac{C_q\gamma_s^{\ 2}}{\rho_o}\ \eta_{max}^2\left[\frac{1}{J_H}\ \frac{(1+r)}{2} + \frac{1}{J_E}\right] \tag{22}$$

$$r \equiv \left(\frac{\eta_{min}}{\eta_{max}}\right)^2 \frac{\beta_{max}}{\beta_{min}}$$

For small $\mu$, $r \approx 1$ (at $\mu=\pi/2$, $r=1.33$). If we set $r=1$ in (22), then an approximate form for $\sigma_H^{\ 2}$ is

$$\sigma_H^{\ 2} = \frac{C_q\gamma_s^{\ 2}}{\rho_o}\ \eta_{max}^2\ \frac{3}{J_H J_E} \tag{23}$$

A factor of 2 growth in $\sigma_H^{\ 2}$ will result from D = +0.618 or D = -1.618. For the example of the preceding section, where $D \approx 340\ (\Delta E/E_o)$, the corresponding energy range would be $-0.48\% < (\Delta E/E_o) < 0.18\%$.

## Conclusion

Though the damping factors are sensitive functions of the synchronous energy, they do not appear to be excessively so. A region about 1% wide in synchronous energy within which both synchrotron oscillations and horizontal betatron oscillations exhibit damping is readily obtained for typical lattice parameters. As indicated in equation (12), this energy width is not manifestly dependent on the size of the storage ring. The corresponding radiofrequency band will shrink roughly as $1/\nu^2$ as one considers larger rings, but this is not cause for immediate alarm.

## References

[1] K. Robinson, Phys. Rev. 111, 373 (1958).

[2] M. Sands, "The Physics of Electron Storage Rings. An Introduction," in Proceedings of the International School of Physics "Enrico Fermi," Course XLVI, edited by B. Touschek (Academic Press, 1971).