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COUPLING IMPEDANCE MEASUREMENT

A.G. Ruggiero

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Summary

A technique to measure the coupling impedance between beam and surrounding is explained here. The knowledge of this impedance is important because it specifies the boundary conditions for the solution of the beam field. The impedance is taken to be also varying along the longitudinal coordinate, although the measurement gives only the average value, which, on the other hand, is just the quantity required in the theory.

The MKS system of the measure units is used here.

To the beam in a circular accelerator we can associate two functions, the charge per unit length

$$\lambda = \lambda(\theta-\Omega t) = \sum_{n=-\infty}^{+\infty} \lambda_n e^{i(n\theta-\omega_n t)}$$

and the current

$$I = \beta c\lambda = \sum_{n=-\infty}^{+\infty} I_n e^{i(n\theta-\omega_n t)}$$

where θ is the angle in the longitudinal direction, t is the time, Ω is the angular revolution frequency, βc is the beam velocity and $\omega_n = n\Omega$ with n any integer.

Because λ and I are periodic in θ we also wrote down their Fourier series expansions.

The longitudinal electric field produced on the beam axis is[1]

$$E = \sum_{n=-\infty}^{+\infty} E_n e^{i(n\theta-\omega_n t)} \tag{1}$$

where

$$E_n = -\frac{C_{bw}}{4\pi\varepsilon_o R}\frac{\partial\lambda_n}{\partial\theta} - Z_n' I_n \tag{2}$$

where ε_o is the dielectric constant in the vacuum, R is the average radius of the accelerator, C_{bw} is the remaining capacitance per unit length between beam and surrounding wall after the magnetic cancellation. For circular geometry it is, for instance,

$$C_{bw} = (1-\beta^2) \quad (1+2 \ln \frac{b}{a})$$

with a the beam radius (assumed constant) and b the pipe inner radius. Z_n' is the (complex) impedance per unit length characteristic of the media surrounding the beam at the boundary.

We wish to emphasize, at this point, the difference between the two terms at the right hand side of Eq. (2). The quantity C_{bw} depends exclusively on the geometry of the beam and of the inner size of the tank (or pipe) which surrounds the beam. Z_n' depends more on the electro-magnetic properties of the media, like resistivity, magnetic permeability and dielectric constant although geometry is also important. For example, the effect of the variation of the pipe size along the accelerator is included in Z_n'. When a theoretical model is available, it is possible to calculate Z_n' as it was done in several cases[2]. Nevertheless, it has been clear to many people for a long time, that likely the best idea is to measure directly Z_n'. For this purpose, a paper describing the measurement has also been produced[3]. In this paper, we wish to report the theory of the measurement in more detail.

The main idea is to consider the beam-surrounding combination as a transmission line, where the beam is the inner conductor and what surrounds it constitutes the outer conductor. Thus, it is possible to measure Z_n' by making use of a conductive wire at the place of the beam. We shall consider here only elements of a circular accelerator, like a magnet, a pipe, a discontinuity in the vacuum tank and so on.

A transmission line as well as an element of the accelerator is characterized by a distributed impedance and an ensemble of lumped impedances, all of them contributing to Z_n'.

Other parameters describing the transmission line are

ℓ - the physical length

L - the inductance per unit length (real)

C - the capacitance per unit length (real)

G - the leakage conductance per unit length (real)

Z_R - loading impedance (complex)

Z_{cond} - impedance per unit length of the inner conductor (complex)

In the following, we shall assume G = 0, and we introduce the impedance per unit length

$$X = Z'_{cond} + Z_n' - i\omega_n L \tag{3}$$

and the admittance per unit length

$$Y = - i\omega_n C \ . \tag{4}$$

Denoting the voltage across the line by V and the current along the line by I we have the usual transmission line equations

$$dV/dx = - XI \tag{5}$$

$$dI/dx = - YV \tag{6}$$

where x is the coordinate along the line. We shall consider only Z_n' as a function of x. The equation for the current is

$$d^2I/dx^2 = (XY)I$$

which has the following general solution

$$I = A_1 \, e^{-\int_0^x \gamma(x')dx'} - A_2 \, e^{\int_0^x \gamma(x')dx'} \tag{7}$$

where A_1 and A_2 are two constants.

From (6), then, we have

$$V = Z_o(x)\left[A_1\, e^{-\int_o^x \gamma(x')dx'} + A_2\, e^{\int_o^x \gamma(x')dx'}\right] \tag{8}$$

where $\gamma(x)$ is the propagation function and $Z_o(x)$ is the characteristic impedance function

$$\gamma(x) = \sqrt{XY} \qquad \text{and} \qquad Z_o(x) = \sqrt{X/Y}\ .$$

In the case Z_n' is a constant, then, also γ and Z_o would be constants.

If the line is terminated at the far end with the loading impedance Z_R, we can calculate A_1 and A_2 imposing that at $x = \ell$ it is $I = I_R$ and $V = Z_R I_R$.

We finally have

$$V = \frac{Z_o I_R}{2Z_{ou}}\left\{(Z_{ou}+Z_R)\, e^{\int_x^\ell \gamma(x')dx'} + (Z_R - Z_{ou})\, e^{-\int_x^\ell \gamma(x')dx'}\right\} \tag{9}$$

and

$$I = \frac{I_R}{2Z_{ou}}\left\{(Z_{ou}+Z_R)\, e^{\int_x^\ell \gamma(x')dx'} - (Z_R - Z_{ou})\, e^{-\int_x^\ell \gamma(x')dx'}\right\} \tag{10}$$

where

$$Z_{ou} = Z_o(\ell)\ .$$

The input impedance Z is defined as

$$Z = \frac{V(o)}{I(o)}$$

or, from (9) and (10)

$$Z = Z_{oi} \frac{Z_R + Z_{ou} \tanh \alpha}{Z_{ou} + Z_R \tanh \alpha} \tag{11}$$

where

$$Z_{oi} = Z_o(0) \qquad \text{and} \qquad \alpha = \int_0^{\ell} \gamma(x')dx'.$$

Two measurements of Z can be performed: one with the far end shorted (Z_R=o) which would give Z_{short}, and another with the far and open ($Z_R=\infty$) which would give Z_{open}. We have, from (11),

$$Z_{short} = Z_{oi} \tanh \alpha$$

$$Z_{open} = Z_{oi}/\tanh \alpha$$

or, conversely,

$$Z_{oi} = \sqrt{Z_{short} \cdot Z_{open}} \tag{12}$$

$$\tanh \alpha = \sqrt{Z_{short} \cdot Z_{open}} \tag{13}$$

From the last of these equations we have

$$\alpha = \log|A| + i(\delta+2\pi m) \tag{14}$$

where $|A|$ and δ are the magnitude and the angle of the complex quantity

$$A = \left(\frac{1 + \sqrt{Z_{short}/Z_{open}}}{1 - \sqrt{Z_{short} \; Z_{open}}} \right)^{1/2} \tag{15}$$

m is any integer number. To determine its value one has to guess a priori the magnitude of the propagation velocity.

Since one is interested only in the average impedance per unit length, one can actually replace the function $Z_n'(x)$ with its average $\overline{Z}_n'$ over the line length. From the definition of α, then, we have

$$\log|A| + i(\delta+2\pi m) = \ell\sqrt{-i\omega_n\ C(Z'_{cond}+\overline{Z}_n'-i\omega_n L)} \tag{16}$$

which can be solved for $\overline{Z}_n'$ assuming we know C, L and Z'_{cond}.

If the inner conductor is made of a cylinder of radius a of a ferromagnetic material, Z'_{cond} is easily calculated

$$Z'_{cond} = \frac{(1-i)\rho}{2\pi a\delta} \tag{17}$$

where ρ is the resistivity of the material in $\Omega\cdot m$ and δ is the skin depth which for copper is

$$\delta = \frac{6.64}{\sqrt{\omega_n/2\pi}} \quad cm$$

and ω_n is expressed in cycles per second. Eq. (17) applies only when the conductor radius a is larger (but not necessarily much larger) than the skin depth. For copper, this is the case when $\frac{\omega_n}{2\pi} \gtrsim 10$ kHz.

Finally, C and L can be calculated from the shape of the inner and outer conductors of the line. For complicated geometries, when the calculation is difficult, they can be measured by using conducting-paper techniques[4]. Observe that C and L are mutually related by

$$CL = c^2$$

thus the knowledge of one gives immediately the other.

References

1. See, for example,

   A.M. Sessler and V.G. Vaccaro, CERN Report 67-2, Feb. 1967, or A.G. Ruggiero, NAL Int. Rep. FN-219, Dec. 1970
2. See, for example,

   A.M. Sessler and V.G. Vaccaro, CERN Report 68-1, Jan. 1968, or A.G. Ruggiero, NAL Int. Rep. RN-220, Jan. 1972
3. A. Faltens and others, "An Analog Method For Measuring The Longitudinal Coupling Impedance Of A Relativistic Particle Beam With Its Environment", Proceedings of the 8th Intl. Conference on High-Energy Accelerators, CERN, 1971, page 338
4. R.E. Matick, "Transmission Lines For Digital And Communication Networks", McGraw-Hill Book Co., page 342