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FIRST ORDER FIXED POINTS FOR A NONLINEAR DELTA FUNCTION PERTURBATION OF BETATRON OSCILLATIONS

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Customarily the transverse motion of the particles in a circular accelerator is described relative to an equilibrium orbit i.e., an orbit which exactly repeats itself each turn. Courant and Snyder[1] demonstrate that for physically realistic magnetic fields such a closed curve must exist. When looking at the effect of non-linear components of the magnetic field it is common to ingore their effect on the equilibrium orbit and to consider small oscillations about it. Such investigations may disclose regions of stability in phase space about stable fixed points and unstable regions separated from them by curves (separatrices) passing through unstable fixed points. Figure 1 is a typical picture arising from a sextupole field component whose azimuthal distribution has a large fourier amplitude for the harmonic nearest to 3ν, where the tune ν is the number betatron oscillations per turn in the absence of the non-linear field components. The coordinates in the phase space plot are the displacement x and the slope $x' = dx/ds$ for a particle at a particular location $s = s_o$ along the equilibrium orbit. The points p_1 to p_3 are stable fixed points; that is, particles with coordinates near these points will remain in the neighborhood turn after turn. If one follows a particle starting near p_1 turn-by-turn he will find it near p_2 and p_3 before

before it returns to p_3 every third turn. For this reason these fixed points are called third order fixed points. The points labled q_1 to q_3 are third order unstable fixed points. A particle at q_1 returns every third turn but neighboring particles drift away. The point labled p_o is the closed orbit position, a first order fixed point which is at zero by definition of the variable x. What has been lost in this treatment which ignores the interaction of the non-linear terms with the linear terms is the fact that p_o has an unstable partner q_o(2). Normally q_o lies far from the region of interest but in the special example discussed below rather modest seeming closed-orbit distortion is sufficient to bring q_o inside q_1, q_2 and q_3. In this circumstance the motion is governed by the first order separatrix and the familiar picture is substantially changed.

Because the existence of the closed orbit is not to be taken for granted the variable to be used has as its origin an arbitrary closed curve like, for instance, the center of the vacuum chamber. The equation for the transverse motion in the radial direction is

$$x'' + K^2(s)x = -\Delta B(x,s)/B\rho \tag{1}$$

where K^2 is the alternating gradient focusing term and ΔB contains the contribution of all terms in the magnetic field except the quadrupole . This is the equation which results if kinematic (x') terms are ignored above the second order but all terms in the field expansion are retained. $\Delta B(0,s)$ is the amount by which the guide field exceeds that needed to hold a particle on the reference trajectory. A periodic solution for eq. (1) may be written

$$x(s) = -\frac{\beta\nu}{2\sin\pi\nu}\int_0^{2\pi}\beta^{3/2}\frac{\Delta B}{B\rho}\cos\nu(\pi+\phi-\psi)\,d\psi \tag{2}$$

where

$$0 \leq \phi(s) = \int_0^s \frac{ds}{\nu\beta} \leq 2\pi \tag{3}$$

and β is the Courant Snyder[1] β belonging to eq. (1) with $\Delta B \equiv 0$. When ΔB is not a function of x eq. (2) gives an immediate expression for the closed orbit. When ΔB is a function of x equation (2) remains valid but may not be easy to use.

To simplify matters a δ function $\left[\int_0^L \delta(s-s_0)\,ds = 1\right]$ distribution of ΔB is used to dispose of the integration:

$$\Delta B(x,s) = \mathcal{B}(x)\,\ell\,\delta(s) \tag{4}$$

choosing s = 0 as the origin of ϕ one can write

$$x(s) = -\beta\,(\mathcal{B}\ell/B\rho)\,\cos\nu(\phi(s)+\pi)/(2\sin\pi\nu). \tag{5}$$

When s is set to zero one gets an equation for x_0, the first order fixed points or closed orbit at s = 0:

$$x_0 = -\beta_0\,\left[\mathcal{B}(x_0)\,\ell/B\rho\right]/2\tan\pi\nu. \tag{6}$$

When $\mathcal{B}$ is independent of x this is the usual result for a closed orbit kink.

Consider, however, some other cases. If $\mathcal{B}\ell/B\rho = Qx$ we have only x = 0 as a root so that a pure quadrupole term has no effect on the first order fixed points. If one has a pure sextupole term

$$B\ell/B\rho = S\ x^2/2 \tag{7}$$

there results a quadratic equation for x_o with roots

$$x_{o,s/u} = 0,\ -2\ \tan\ \pi\nu/\beta_o S. \tag{8}$$

The stable fixed point is the p_o of figure 1. Here, however, is the unstable partner $x_{o,u} = q_o$ which could be in the region of interest for ν sufficiently close to an integer. Of course the contribution of the sextupole to the integer resonance is familiar and one ordinarily expects to stay well away from integral ν.

A little more interesting is the example

$$B\ell/B\rho = D + Sx^2/2. \tag{9}$$

The first order fixed points are

$$x_{o,s/u} = -2\ \tan\pi\nu\ /\beta S\ +/-\ \sqrt{(2\ \tan\pi\nu\ /\beta S)^2 - 2D/S}\ . \tag{10}$$

Here $x_{o,s}$ is the displaced equilibrium orbit which goes to $x_{o,s} = 0$ as $D = 0$. One sees that if

$$DS = (DS)_c = 2(\tan\pi\nu\ /\beta)^2 \tag{11}$$

the stable and unstable fixed points coalesce leaving no stable region. For greater DS values x_o is complex. Therefore, within the region in which the magnetic field is represented by equation (9) there is no equilibrium orbit when $DS \geq (DS)_c$. The value of the closed orbit distortion when $DS = (DS)_c$ is

$$x_c = -\beta\, D_c/\tan \pi\nu = \sqrt{2D/S} \tag{12}$$

which is just twice the distortion produced by D_c alone. Any attempt to employ this type of resonance for, say, extraction or injection must take account of aperture required for this orbit distortion.

Although it was easy to obtain these results because of the δ-function distribution of ΔB the same sort of thing can happen with perfectly smooth distributions. In a 1959 accelerator conference paper Laslett and Symon[3] analyze at considerable length the fixed points for an equation like

$$x'' + \nu^2 x = D \cos s/R + S\, x^2 \cos 3s/R \tag{13}$$

where R is the accelerator radius. The analysis required was more sophisticated than for the example used above, but the character of the fixed points was qualitatively similar. The presence of first order fixed points in the δ-function case has also been noted by others.[4,5]

Qualitatively what occurs is that the closed orbit deformation produced by the dipole and sextupole jointly give a sufficient tune shift from the quadrupole term $x_o S$ to bring ν to the nearest integer. The fact that integral tune is involved is clear from the first order character of the fixed points and more explicitly from eq.(11) which shows $(DS)_c \to 0$ as $\nu \to n$. The quantity

$$DS/(DS)_c = \frac{1}{2}\, DS\, (\beta/\tan \pi\nu)^2 \tag{14}$$

is a good measure of the strength of a perturbation DS which one might wish to keep in mind in considering how big sextupoles can safely be in the presence of closed-orbit distortion. The separatrix with one unstable fixed point has generally the "fish" form familiar in synchrotron oscillation studies. The details of the phase plane for eq. (1) with δ-function ΔB depends sensitively on ν because the rich azimuthal harmonic content of the δ-function can lead to fixed points of various orders even for a pure sextupole. The character of the motion for the ΔB of eq. (9) has been explored by particle tracing for a tune just above $6\frac{2}{3}$. For $S>0$ and $D = 0$ the situation is somewhat like that in Figure 1. When D takes on positive values the picture evolves to the general character shown in Figure 2 where one of the third order unstable points has been supplanted by the first order unstable fixed point.

One can see that other multipoles can play a part in an effect of this kind, but the dipole is necessary to produce zero stable area of the first order separatrix for finite sized perturbation.

The original interest in the above material was aroused by the possibility of its application to injection into the Booster. Although the results of that investigation are negative the foregoing retains interest as a simple way of looking at first order fixed point behavior in a special but practical case. The fact that the original x, x' variables are used throughout has heuristic value.

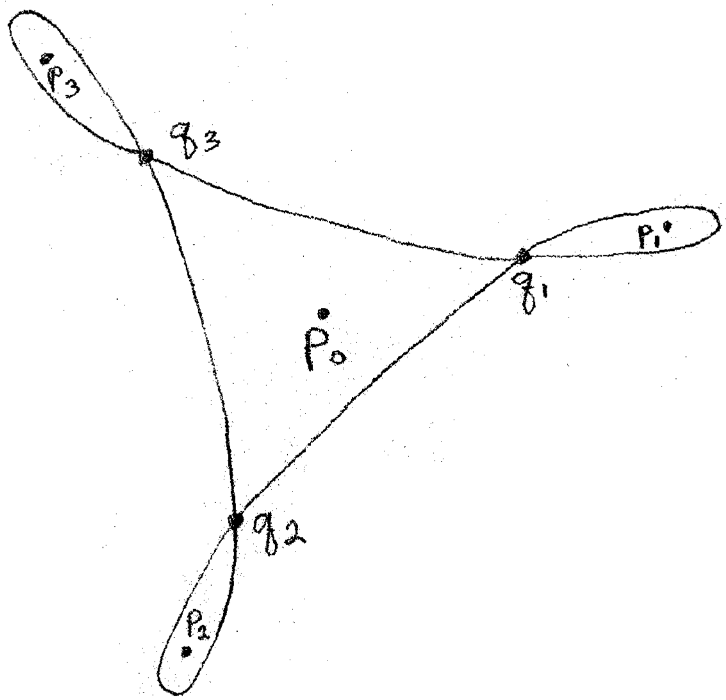

Figure 1

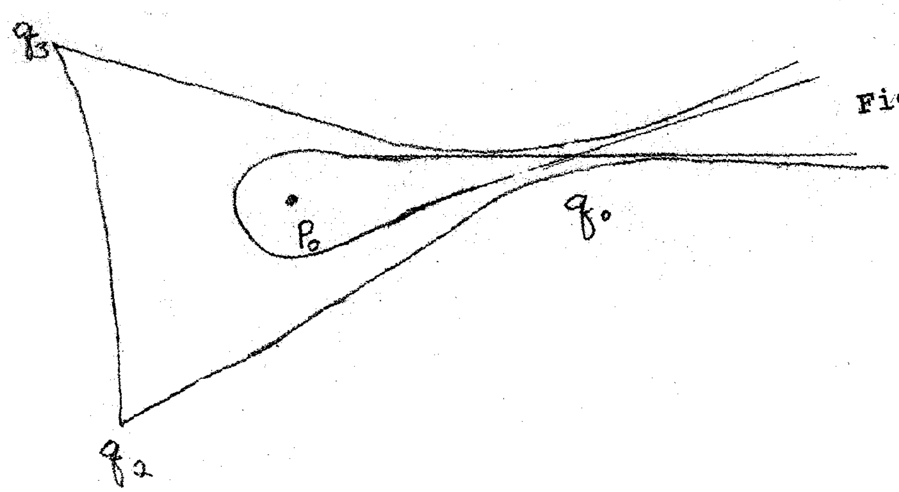

Figure 2

REFERENCES

1. E.D. Courant and H.S. Snyder, Annals of Physics, V3, #1, P1 (1958)

2. P.A. Sturrock, Annals of Physics, V3, #1, P113 (1958)

3. L.G. Laslett and K.R. Symon, Proc. III Int. Conf. on High-Energy Accel. (CERN 1959)

4. E.A. Crosbie, T.K. Khoe, and R.J. Lari, IEEE Trans. on Nucl. Sci., NS-18, P1077 (1971)

5. T.K. Khoe and R.J. Lari, IEEE Trans. on Nucl. Sci., NS-20, P392 (1973)