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STOPBANDS OF A FOCUSING SYSTEM

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(A)

In a periodic focusing system, a stopband appears when the phase advance per period of the betatron oscillation is $n\pi$ $(n = 0,1,2,...)$. The width of this stopband is finite so that parameters (field gradient of quadrupoles, for example) must be changed by a certain amount to regain the stable betatron oscillation. Within the stopband, the betatron motion is unstable, that is, the amplitude grows exponentially. The growth factor per period is different for different stopbands and, within a stopband, it takes different values depending on parameters of the system. In a circular machine like the main ring, the beam intensity would decrease more or less exponentially as the beam makes many turns if the focusing system happens to be sitting in one of these stopbands. This is true even for a perfect machine in which its closed orbit coincides with the geometrical axis of the focusing system. Four stopbands nearest the design point are shown in Fig. 1 for the main ring when the injection energy is 7 GeV. Note that, corresponding to a phse advance of $n\pi$, one gets the tune $6n\pi/2\pi = 3n$ since

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there are six periods in the main ring. Uncertainties in the average field gradient (B_F' and B_D') are mostly due to remnant fields (~2.5 G/cm) which are not necessarily uniform for all quadrupoles. In Fig. 1, a small effect of dipoles on ν_y ($\Delta\nu_y \simeq +0.05$ at $\nu_y = 20$) is not included. The growth factors per turn in these stopbands are given in Fig. 2. For example, if the average B_F' is 51 G/cm and the average B_D' is 46.5 G/cm ~47 G/cm, the amplitude grows $(2.4)^n$ times in n turns.

(B)

Courant-Snyder formalism of the betatron oscillation is well known when the motion is stable. Since there is no detailed presentation of the formalism in their classical paper for unstable betatron oscillations, the following sketch may be of some interest to non-specialists.

Consider a linear motion in a periodic system with the period L,

$$\xi'(s) = f(s)\xi(s) + g(s)\eta(s) \qquad (1)$$

$$\eta'(s) = h(s)\xi(s) - f(s)\eta(s) \qquad (2)$$

with $g(s) > 0$. For example, in a synchrotron, $\xi = x$ or y, $\eta = dx/ds$ or dy/ds, $f(s) = 0$, $g(s) = 1$ and $h(s)$ specifies the focusing action of each element in the synchrotron. For a certain application (synchrocyclotrons or cyclotrons with an azimuthally varying magnetic field) it is necessary to take $f(s) \neq 0$ and $g(s) \neq 1$. The transfer matrix from s to s + L can be written formally

$$M(s) \equiv M(s+L|s)$$

$$= \begin{pmatrix} \cos\sigma + \alpha(s)\sin\sigma & \beta(s)\sin\sigma \\ -\gamma(s)\sin\sigma & \cos\sigma - \alpha(s)\sin\sigma \end{pmatrix} \tag{3}$$

where σ is independent of s and $\alpha(s)$, $\beta(s)$, and $\gamma(s)$ are all periodic with the period L. The value of the quadratic form

$$W \equiv \gamma(s)\cdot\xi^2(s) + 2\alpha(s)\xi(s)\eta(s) + \beta(s)\cdot\eta^2(s) \tag{4}$$

is independent of s. Two eigensolutions of this system are

$$\xi_A(s) = \beta^{1/2}(s)e^{i\psi(s)}, \quad \eta_A = \beta^{-1/2}(s)[-\alpha(s)+i]e^{i\psi(s)} \tag{5}$$

$$\xi_B(s) = \beta^{1/2}(s)e^{-i\psi(s)}, \quad \eta_B = \beta^{-1/2}(s)[-\alpha(s)-i]e^{-i\psi(s)} \tag{6}$$

where $\psi(s+L) - \psi(s) = \sigma$. Betatron oscillation parameters satisfy the following relations

$$d\alpha/ds = -h\beta - g\gamma, \tag{7}$$

$$d\beta/ds = 2(f\beta - g\alpha), \tag{8}$$

$$d\gamma/ds = -2(h\alpha + f\gamma), \tag{9}$$

$$d\psi(s)/ds = g/\beta, \tag{10}$$

$$\gamma\beta = 1 + \alpha^2 \tag{11}$$

So far, no restriction has been imposed on the trace of matrix M(s). If $|\mathrm{TrM}| < 2$, all parameters (α, β, γ, ψ, W, σ) are real and the motion is stable. If $|\mathrm{TrM}| = 2$, $\cos\sigma = \pm 1$

so that $\sigma = n\pi$ $(n = 0,1,2,...)$. Courant-Snyder formalism "breaks down altogether" (their own words). The primary interest here is the case when $|TrM| > 2$. There are two possibilities:

(a) $TrM > 2, \quad \cos\sigma > 1$

$\sigma = 2n\pi + i\mu \quad (\mu = \text{real})$

$\cos\sigma = \cosh(\mu), \quad \sin\sigma = i\sinh(\mu)$

(b) $TrM < -2, \quad \cos\sigma < -1$

$\sigma = (2n+1)\pi + i\mu \quad (\mu = \text{real})$

$\cos\sigma = -\cosh(\mu), \quad \sin\sigma = -i\sinh(\mu)$

It is necessary to make a convention

$$\sin\sigma \equiv \text{positive imaginary}$$

so that

$$\mu > 0 \quad \text{if} \quad TrM > 2$$

$$\mu < 0 \quad \text{if} \quad TrM < -2.$$

The opposite convention is equally valid. Whichever convention is used, it should be consistent with the relation

$$\sigma \equiv \psi(s+L) - \psi(s) = \int_s^{s+L} g(s)/\beta(s) \quad ds. \tag{12}$$

Since elements of the matrix M(s) are all real quantities, $\alpha(s)$, $\beta(s)$, and $\gamma(s)$ are pure imaginary,

$$\alpha(s) \equiv i\underline{\alpha}(s), \qquad \beta(s) \equiv i\underline{\beta}(s), \qquad \gamma(s) \equiv i\underline{\gamma}(s).$$

From (11),

$$\underline{\gamma}\,\underline{\beta} = \underline{\alpha}^2 - 1 \quad (\underline{\alpha},\underline{\beta},\underline{\gamma} = \text{real}). \tag{13}$$

It is important to note here that $\underline{\gamma}$ and $\underline{\beta}$ can be positive, negative or zero. (For $|\mathrm{TrM}| < 2$, γ and β are always positive by definition.) Since

$$M_{12} \equiv \beta \sin \sigma = (i\underline{\beta})i|\sinh(\mu)| = -\underline{\beta}|\sinh(\mu)|,$$

sign $(\underline{\beta})$ = -sign (M_{12}). Also, sign $(\underline{\gamma})$ = sign (M_{21}). From (5) and (6),

$$\begin{aligned}
(\xi_A,\eta_A)_{s+L} &= e^{-\mu}(\xi_A,\eta_A)_s && \text{for } \cos\sigma > 1,\ \mu > 0 \\
&= -e^{-\mu}(\xi_A,\eta_A)_s && \text{for } \cos\sigma < -1,\ \mu < 0 \\
&= e^{\mu}(\xi_B,\eta_B)_s && \text{for } \cos\sigma > 1,\ \mu > 0 \\
&= -e^{\mu}(\xi_B,\eta_B)_s && \text{for } \cos\sigma < -1,\ \mu < 0.
\end{aligned}$$

Since general solutions can be expressed as a linear combination of two eigensolutions, their amplitudes grow exponentially. The invariant quadratic form (4) represents two pairs of hyperbolas

$$\underline{\gamma}\xi^2 + 2\underline{\alpha}\xi\eta + \underline{\beta}\eta^2 = C(\equiv W/i); \quad C > 0 \text{ and } C < 0.$$

Asymptotes of these hyperbolas are

$$\xi = \frac{-\underline{\alpha} \pm 1}{\underline{\gamma}}\,\eta \qquad \text{or} \qquad \eta = \frac{-\underline{\alpha} \pm 1}{\underline{\beta}}\,\xi. \tag{14}$$

If $\underline{\beta} = 0$ or $\underline{\gamma} = 0$, one asymptote coincides with η-axis or ξ-axis. If both $\underline{\gamma}$ and $\underline{\beta}$ are zero, ξ- and η-axes become two asymptotes.

Since $\beta(s)$ can be zero for some values of s, formal expressions like (5), (6) and (12) must be defined more carefully. For example, it is necessary to have eigensolutions finite and continuous everywhere in the system. Because of the periodicity, $\beta(s+L) = \beta(s)$, there are even number of zeros of $\beta(s)$; $s = s_1, s_2, \ldots, s_{2N}$.

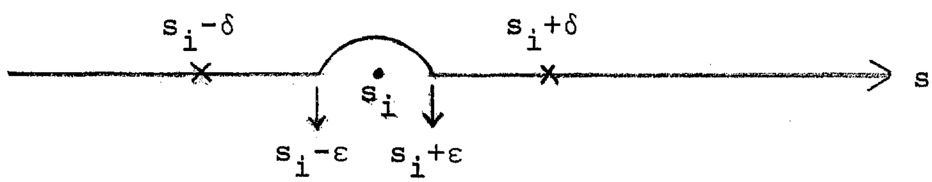
(a) $\beta(s)$ changes from positive to negative imaginary.

For $s_i - \delta < s < s_i + \delta$, $\delta << 1$, one can write

$$\beta(s) = -2ig(s)\ (s-s_i) \tag{15}$$

since $g(s) > 0$ and $\beta'(s) = -2g(s)\alpha(s)$ where $\alpha(s_i) = +i$.

By going above the pole at $s = s_i$, one can show that $\beta^{1/2}(s)$ changes by factor $(-i)$:



with $z \equiv s-s_i$

$s_i-\delta \le s \le s_i-\varepsilon$: $\quad s-s_i = |z|e^{i\pi}$

$$\beta^{1/2}(s) = i\sqrt{2g}\,|z|^{1/2}e^{-i\pi/4}$$

$s_i + \varepsilon \le s \le s_i + \delta$: $\quad s-s_i = |z|$

$$\beta^{1/2}(s) = \sqrt{2g}\,|z|^{1/2}e^{-i\pi/4}.$$

Similarly, the change in $e^{i\psi(s)}$ is $(+i)$:

$$\psi(s) = \psi(s_i-\delta) + \int_{s_i-\delta}^{s} g/\beta\ ds$$

$s \le s_i-\varepsilon$, $\quad \psi(s) = \psi(s_i-\delta) + (i/2)\ln(|z|/\delta)$

$$e^{i\psi(s)} = e^{i\psi(s_i-\delta)}|z|^{-1/2}\delta^{1/2}$$

$$s \geq s_i + \varepsilon, \qquad \psi(s) = \psi(s_i-\delta) + (i/2)\ell n(|z|/\delta) + \pi/2$$

$$e^{i\psi(s)} = e^{i\psi(s_i-\delta)}|z|^{-1/2}\delta^{1/2}(i).$$

Consequently, $\xi_A = \beta^{1/2}(s)e^{i\psi(s)}$ is continuous at $s = s_i$. Note that the phase $\psi(s)$ makes an increase of $\pi/2$. Other functions are also continuous:

$$\eta_A = \beta^{-1/2}(-\alpha + i)e^{i\psi}$$

$$= \beta^{-1/2}\frac{\gamma\beta}{-(\alpha+i)}e^{i\psi} = (i/2)\gamma\beta^{1/2}e^{i\psi}$$

$$\xi_B = \beta^{1/2}e^{-i\psi} = \text{const.}|z| \qquad (s = s_i-\varepsilon)$$

$$= -\text{const.}|z| \qquad (x = s_i+\varepsilon)$$

$$\eta_B = -\beta^{-1/2}(\alpha+i)e^{-i\psi}$$

$$= (-2i)\beta^{-1/2}e^{-i\psi} = \text{const. for both } s = s_i-\varepsilon \text{ and } s = s_i+\varepsilon.$$

<u>Summary</u>

$\beta(s)$: positive imaginary to negative imaginary

$\alpha = +i$

By going <u>above</u> the pole,

$$\beta^{1/2} \rightarrow -i\beta^{1/2}, \qquad \psi \rightarrow \psi + \pi/2, \qquad e^{i\psi} \rightarrow ie^{i\psi}.$$

Near $s = s_i$,

$$\xi_A(s) = \text{const.}, \qquad \eta_A(s) = \text{const. } \gamma(s)$$

$$\xi_B(s) = \text{const. } (s-s_i), \quad \eta_B(s) = \text{const.}$$

(b) $\beta(s)$ changes from negative to positive imaginary.

$\alpha = -i$

By going <u>below</u> the pole,

$$\beta^{1/2} \to i\beta^{1/2}, \quad \psi \to \psi + \pi/2, \quad e^{i\psi} \to ie^{i\psi}.$$

Near $s = s_i$,

$$\xi_A(s) = \text{const.}(s-s_i), \; \eta_A(s) = \text{const.}$$

$$\xi_B(s) = \text{const.}, \qquad \eta_B(s) = \text{const.}\ \gamma(s).$$

Thus a pair of poles contribute π to the phase ψ. If $N = 2n$ (even number of pairs), $\sigma = 2n\pi + i\mu$ and for $N = 2n + 1$ (odd number of pairs), $\sigma = (2n+1)\pi + i\mu$:

$$\sigma \equiv \psi(s+L) - \psi(s)$$

$$= P \int_s^{s+L} (g/\beta)ds + (\pi/2)\cdot 2N$$

$$= N\pi - iP \int_s^{s+L} (g/\underline{\beta})ds \equiv N\pi + i\mu.$$

Since $\beta^{1/2}(s)$ is periodic with the period L,

$$\xi_A(s+L) \equiv \beta^{1/2}(s+L)e^{i\psi(s+L)}$$

$$= \beta^{1/2}(s)e^{i\psi(s)}e^{iN\pi}e^{-\mu}$$

$$= \pm\xi_A(s)e^{-\mu}.$$

A general solution (ξ,η) with the initial conditions

$$\xi(s=0) = 0, \qquad \eta(s=0) = 1, \qquad \psi(s=0) = 0$$

can be written as

$$\xi(s) = \beta^{1/2}(s=0)\beta^{1/2}(s) \sin \psi(s)$$

$$\eta(s) = \beta^{1/2}(s=0)\beta^{-1/2}(s)[\cos \psi(s) - \alpha(s) \sin \psi(s)].$$

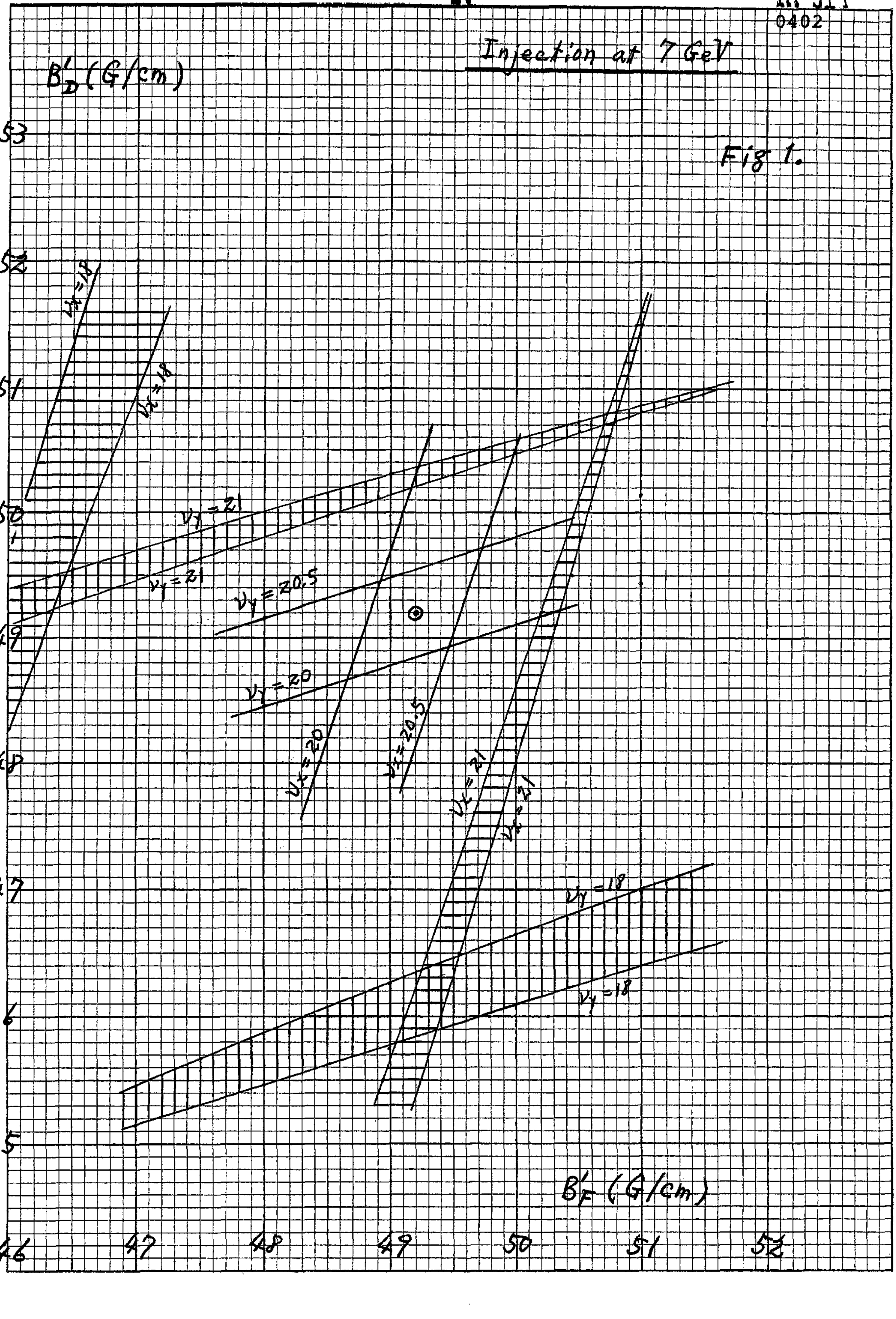
At $s = L$,

$$\xi(s=L) = \beta(s=0) \sin \sigma = -\underline{\beta}(s=0) \sinh |\mu|$$

so that

$$\text{sign } (\underline{\beta})_{s=0} = -\text{sign } (\xi)_{s=L}$$

thereby defining the sign of $\underline{\beta}$ for all s.


Injection at 7 GeV
Fig 1.
B'_D (G/cm)
B'_F (G/cm)
νx = 18
νx = 18
νy = 21
νy = 21
νy = 20.5
νy = 20
νx = 20
νx = 20.5
νx = 21
νx = 21
νy = 18
νy = 18
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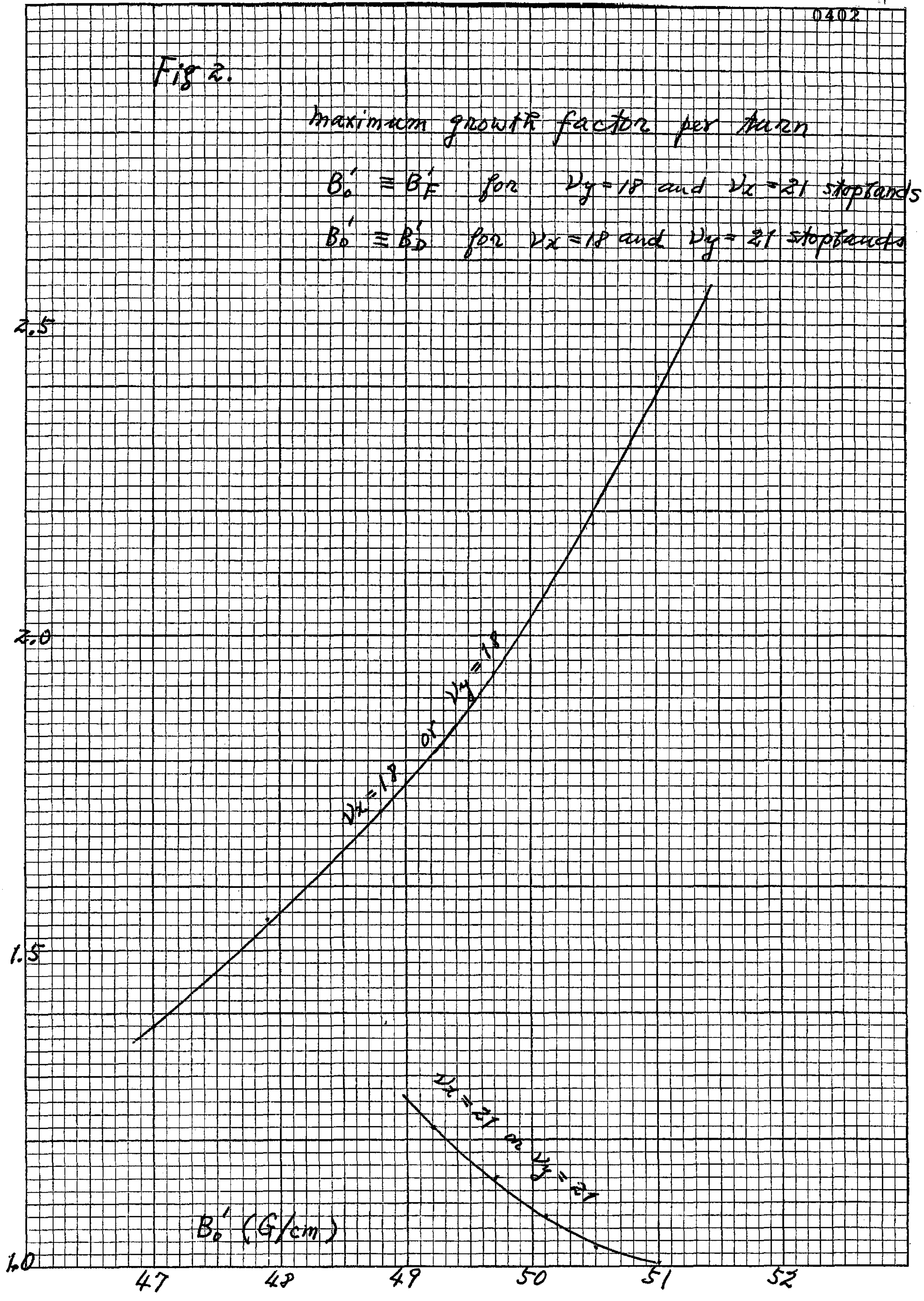
Fig 2.
maximum growth factor per turn
$B_0' \equiv B_F'$ for $\nu_y = 18$ and $\nu_x = 21$ stopbands
$B_0' \equiv B_D'$ for $\nu_x = 18$ and $\nu_y = 21$ stopbands
2.5
2.0
1.5
1.0
$\nu_x = 18$ or $\nu_y = 18$
$\nu_x = 21$ or $\nu_y = 21$
B_0' (G/cm)
47
48
49
50
51
52