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EXPANSION OF MAGNETOSTATIC POTENTIAL ON MEDIAN PLANE

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Let R(s) be the radial position of the closed orbit assumed to lie in the plane y=0. Then, if the radial coordinate r=R(s)+x, and the local curvature of R(s) is $k(s) \equiv 1/\rho(s)$, Laplace's equation for the potential becomes[1]

$$\frac{1}{1+kx} \cdot \frac{\partial}{\partial x} \left[(1+kx)\frac{\partial \Phi}{\partial x}\right] + \frac{\partial^2 \Phi}{\partial y^2} + \frac{1}{1+kx} \cdot \frac{\partial}{\partial s} \left[\frac{1}{1+kx} \cdot \frac{\partial \Phi}{\partial s}\right] = 0 \ . \qquad (1)$$

Assume an expansion of the form:

$$\Phi = \sum_{m=0} \sum_{n=0} \frac{A_{m,n}(s)}{m!n!} x^m y^n \ . \qquad (2)$$

Substitute this potential into Eq. (1), multiply by $(1+kx)^3$, and adjust the general indices to yield $x^m y^n$ as a factor common to all terms. This yields the following recurrence relation:

$$\begin{aligned} &A_{m+2,n}+(3m+1)kA_{m+1,n}+m(3m-1)k^2A_{m,n}+m(m-1)^2k^3A_{m-1,n} \\ &+A_{m,n+2}+3mkA_{m-1,n+2}+3m(m-1)k^2A_{m-2,n+2} \\ &+m(m-1)(m-2)k^3A_{m-3,n+2}-mk'A'_{m-1,n}+A''_{m,n}+mkA''_{m-1,n}=0 \ , \end{aligned} \qquad (3)$$

where the prime designates differentiation with respect to the longitudinal coordinate s. It is to be noted that all coefficients for which the indices m and n are less than zero are zero.

SPECIFIC LOW ORDER TERMS

The lowest order coefficients $A_{m,0}$ and $A_{m,1}$ are completely arbitrary. These terms in fact represent the fields on the reference curve [2] (x=y=0).

$$A_{00} = \int B_s ds \qquad A_{01} = B_y$$

$$A_{10} = B_x \qquad A_{11} = \frac{\partial B_y}{\partial x}$$

$$A_{20} = \frac{\partial B_x}{\partial x} \qquad A_{21} = \frac{\partial^2 B_y}{\partial x^2}$$

$$A_{30} = \frac{\partial^2 B_x}{\partial x^2} \qquad - - -$$

$$- - - \qquad - - - \tag{4}$$

Equation (3) serves to relate all other coefficients to those in Eq. (4). Note that n only appears as a subscript.

<u>n=2</u>

$$(m=0)\quad A_{02} = -A_{20} - kA_{10} - A''_{00} \tag{5}$$

$$(m=1)\quad A_{12} = -A_{30} - kA_{20} + k^2A_{10} + k'A'_{00} - A''_{10} + 2kA''_{00} \tag{6}$$

$$\begin{aligned}(m=2)\quad A_{22} = {} & -A_{40} - kA_{30} + 2k^2A_{20} - 2k^3A_{10} - 6kk'A'_{00} \\ & +4kA''_{10} - 6k^2A''_{00} + 2k'A'_{10} - A''_{20}\end{aligned} \tag{7}$$

$$\begin{aligned}(m=3)\quad A_{32} = {} & -A_{50} - kA_{40} + 3k^2A_{30} - 6k^3A_{20} + 6k^4A_{10} \\ & +24k^3A''_{00} + 36k^2k'A'_{00} - 18k^2A''_{10} - 18kk'A'_{10} \\ & + 6kA''_{20} + 3k'A'_{20} - A''_{30}\end{aligned} \tag{8}$$

$$\begin{aligned}(m=4)\quad A_{42} = {} & -A_{60} - kA_{50} + 4k^2A_{40} - 12k^3A_{30} + 24k^4A_{20} - 24k^5A_{10} \\ & -120k^4A''_{00} - 240k^3k'A'_{00} + 96k^3A''_{10} + 144k^2k'A'_{10} \\ & -36k^2A''_{20} - 18kk'A'_{20} + 8kA''_{30} + 4k'A'_{30} - A''_{40}\end{aligned} \tag{9}$$

n=3

$$(m=0)\quad A_{03} = -A_{21} - k\,A_{11} - A''_{01} \tag{10}$$

$$(m=1)\quad A_{13} = -A_{31} - k\,A_{21} + k^2A_{11} + k'A'_{01} - A''_{11} + 2kA''_{01} \tag{11}$$

$$\begin{aligned}(m=2)\quad A_{23} = {} & -A_{41} - k\,A_{31} + 2k^2A_{21} - 2k^3A_{11} - 6\,kk'A'_{01} \\ & + 4k\,A''_{11} - 6\,k^2\,A''_{01} + 2k'A'_{11} - A''_{21}\end{aligned} \tag{12}$$

$$\begin{aligned}(m=3)\quad A_{33} = {} & -A_{51} - k\,A_{41} + 3k^2A_{31} - 6k^3A_{21} + 6k^4A_{11} \\ & + 24k^3A''_{01} + 36k^2k'A'_{01} - 18k^2A''_{11} - 18kk'A'_{11} \\ & + 6kA''_{21} + 3k'A'_{21} - A''_{31}\end{aligned} \tag{13}$$

$$\begin{aligned}(m=4)\quad A_{43} = {} & -A_{61} - kA_{51} + 4k^2A_{41} - 12k^3A_{31} + 24k^4A_{21} - 24k^5A_{11} \\ & - 120\,k^4A''_{01} - 240k^3k'A'_{01} + 96k^3A''_{11} + 144k^2k'A'_{11} \\ & - 36\,k^2A''_{21} - 18kk'A'_{21} + 8kA''_{31} + 4k'A'_{31} - A''_{41}\end{aligned} \tag{14}$$

n=4

$$\begin{aligned}(m=0)\quad A_{04} = {} & A_{40} + 2kA_{30} - k^2A_{20} + k^3A_{10} + 5kk'A'_{00} \\ & - 2kA''_{10} + 4k^2A''_{00} + 2A''_{20} + k''A_{10} + A''''_{00}\end{aligned} \tag{15}$$

n=5

$$\begin{aligned}(m=0)\quad A_{05} = {} & A_{41} + 2kA_{31} - k^2A_{21} + k^3A_{11} + 5kk'A'_{01} \\ & -2kA''_{11} + 4k^2A''_{01} + 2A''_{21} + k''A_{11} + A''''_{01}\end{aligned} \tag{16}$$

VECTOR POTENTIAL

In hamiltonian analysis, it is necessary to have an expression for the vector potential. Using L operator analysis[3], since i_y is the only constant vector in the present coordinate system, let

$$L = i_y x \nabla \tag{17}$$

with this operator a general decomposition of any vector such as the vector potential may be written as

$$A = LU + \nabla x LV + \nabla W. \tag{18}$$

The flux density becomes

$$B = \nabla x A = \nabla x LU - L\nabla^2 V \tag{19}$$

and

$$\nabla x B = -L\nabla^2 U - \nabla x L\nabla^2 V = 0, \tag{20}$$

which may be satisfied rather generally with

$$\nabla^2 U = 0 \tag{21}$$

$$\nabla^2 V = 0 \tag{22}$$

Then

$$B = \nabla x LU = i_y \nabla^2 U - \nabla\frac{\partial U}{\partial y} = - \nabla\frac{\partial U}{\partial y} = \nabla\Phi \tag{23}$$

Hence

$$U = -\int_o^y \Phi dy' + F(x,s) = -\Sigma \frac{A_{mn}}{m!(n+1)!} x^m y^{n+1} + F(x,s). \tag{24}$$

To obtain the function, F(x,s) notice[4] that

$$\nabla^2 U = - \nabla^2_{x,s} \int_o^y \Phi dy' - \frac{\partial^2}{\partial y^2} \int_o^y \Phi dy' + \nabla^2 F = 0 \tag{25}$$

or, since $\nabla^2\Phi = 0$,

$$\nabla^2 F = - \int_o^y \frac{\partial^2\Phi}{\partial y'^2} dy' + \frac{\partial\Phi}{\partial y} = -\frac{\partial\Phi}{\partial y} + \left(\frac{\partial\Phi}{\partial y}\right)_{y=0} + \frac{\partial\Phi}{\partial y} = B_y(x,o,s) \tag{26}$$

Thus

$$\frac{1}{1+kx}\frac{\partial}{\partial x}\left[(1+kx)\frac{\partial F}{\partial x}\right] + \frac{1}{1+kx}\frac{\partial}{\partial s}\left(\frac{1}{1+kx}\frac{\partial F}{\partial s}\right) = \Sigma \frac{A_{m1}(s)}{m!} x^m$$

or

$$\frac{\partial}{\partial x}\left[(1+kx)\frac{\partial F}{\partial x}\right] + \frac{\partial}{\partial s}\left[\frac{1}{1+kx}\frac{\partial F}{\partial s}\right] = \Sigma\frac{(A_{m1}+mkA_{m-1,1})}{m!}x^m. \quad (27)$$

If one sets a particular solution

$$F = \Sigma\frac{C_m(s)}{m!}x^m \quad (28)$$

Then

$$\begin{aligned} &C_{m+2}+(3m+1)kC_{m+1}+m(3m-1)k^2C_m+m(m-1)^2k^3C_{m-1} \\ &\quad -mk'C'_{m-1}+C''_m + mkC''_{m-1} = \\ &\quad A_{m1}+3mkA_{m-1,1}+3m(m-1)k^2A_{m-2,1}+m(m-1)(m-2)K^3A_{m-3,1} \end{aligned} \quad (29)$$

This recursion relation gives for the first few terms

$$C_o = C_1 = 0 \quad (30)$$

$$C_2 = A_{01} \quad (31)$$

$$C_3 = A_{11}-kA_{01} \quad (32)$$

$$C_4 = A_{21}-kA_{11}+3k^2A_{01}-A''_{01} \quad (33)$$

CHOICE OF GAUGE

Since $\nabla^2 V = 0$, the vector potential may be written as

$$A = LU + \nabla(W - \frac{\partial V}{\partial y}) \quad (34)$$

or

$$A = LU + \nabla T, \quad (35)$$

where, since W and $\partial V/\partial y$ are not needed separately they have been combined into T.

Since T may be any arbitrary function without altering the magnetic flux density, a gauge is selected whenever T is

specified. Having found U through the L-operator scheme, the simplest gauge is T = 0. This gives

$$A_x = \frac{1}{1+kx}\frac{\partial U}{\partial s} = -\frac{1}{1+kx}\left\{\Sigma\frac{A'_{mn}}{m!(n+1)!}x^m y^{n+1} - \Sigma\frac{C'_m}{m!}x^m\right\} \tag{36}$$

$$A_y = 0$$

$$A_s = -\frac{\partial U}{\partial x} = \Sigma\frac{A_{m,n}}{(m-1)!(n+1)!}x^{m-1}y^{n+1} - \Sigma\frac{C_m}{(m-1)!}x^{m-1} \tag{37}$$

Another useful gauge is found by setting $xA_x + yA_y$ equal to zero. Let T be given by

$$T = \Sigma\frac{D_{mn}(s)}{m!n!}x^m y^n . \tag{38}$$

Then set $xA_x + y A_y = 0$ and obtain

$$-\frac{x}{1+kx}\left\{\Sigma\frac{A'_{mn}}{m!(n+1)!}x^m y^{n+1} - \Sigma\frac{C'_m}{m!}x^m\right\}$$

$$+ x\Sigma\frac{D_{mn}}{(m-1)!n!}x^{m-1}y^n + y\Sigma\frac{D_{mn}}{m!(n-1)!}x^m y^{n-1} = 0$$

or, after adjusting indices

$$\sum_{m,n}\frac{x^m y^n}{m!n!}\left\{(m+n)D_{mn} + m(m+n-1)kD_{m-1,n} - mA'_{m-1,n-1}\right\} = -\sum_m\frac{mC'_{m-1}}{m!}x^m \tag{40}$$

Thus the recursion relation becomes for n=0:

$$D_{mo} + (m-1)D_{m-1,o} = -C'_{m-1} \tag{41}$$

or

$$D_{00} = D_{10} = D_{20} = 0 \tag{42}$$

$$D_{30} = -C'_2 \tag{43}$$

$$D_{40} = -C'_3 + 3kC'_2 \tag{44}$$

$$D_{50} = -C'_4 + 4kC'_3 - 12k^2C'_2 \tag{45}$$

For n = 1, 2, 3, etc. the recursion relation becomes

$$(m+n)D_{mn} + m(m+n-1)kD_{m-1,n} = mA'_{m-1,n-1} \tag{46}$$

or for n = 1

$$D_{01} = 0 \tag{47}$$

$$D_{11} = \frac{1}{2} A'_{00} \tag{48}$$

$$D_{21} = \frac{2}{3} A'_{10} - \frac{2}{3}k\ A'_{00} \tag{49}$$

$$D_{31} = \frac{3}{4} A'_{20} - \frac{3k}{2} A'_{10} + \frac{3k^2}{2} A'_{00} \tag{50}$$

$$D_{41} = \frac{4}{5} A'_{30} - \frac{12}{5}kA'_{20} + \frac{24}{5}k^2A'_{10} - \frac{24}{5}k^3A'_{00} \tag{51}$$

for n = 2

$$D_{02} = 0 \tag{52}$$

$$D_{12} = \frac{1}{3} A'_{01} \tag{53}$$

$$D_{22} = \frac{1}{2} A'_{11} - \frac{1}{2}kA'_{01} \tag{54}$$

$$D_{32} = \frac{3}{5} A'_{21} - \frac{6}{5}k\ A'_{11} + \frac{6}{5}k^2A'_{01} \tag{55}$$

$$D_{42} = \frac{2}{3} A'_{31} - 2kA'_{21} + 4k^2A'_{11} - 4k^3A'_{01} \tag{56}$$

and, for n = 3

$$D_{03} = 0 \tag{57}$$

$$D_{13} = \frac{1}{4} A'_{02} \tag{58}$$

$$D_{23} = \frac{2}{5} A'_{12} - \frac{2}{5}\ kA'_{02} \tag{59}$$

$$D_{33} = \frac{1}{2} A'_{22} - kA'_{12} + k^2A'_{02} \tag{60}$$

$$D_{43} = \frac{4}{7} A'_{32} - \frac{12}{7}\ kA'_{22} + \frac{24}{7}\ k^2A'_{12} - \frac{24}{7}\ k^3A'_{02} \tag{61}$$

Then, in this gauge

$$A_x = \frac{1}{1+kx} \frac{\partial U}{\partial s} + \frac{\partial T}{\partial x} \tag{62}$$

$$A_y = \frac{\partial T}{\partial y} \tag{63}$$

$$A_s = - \frac{\partial U}{\partial x} + \frac{1}{1+kx} \frac{\partial T}{\partial s} \tag{64}$$

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4. E. D. Courant, private communication.