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DETERMINATION OF BOOSTER MAGNET SEXTUPOLE STRENGTH

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PURPOSE

To determine the pole face contour sextupole parameter of the gradient magnets that will cause the betatron oscillation frequencies to be independent of momentum.

FORMULATION

Cole[2] gives for the radial and vertical particle motion in a magnetic confining field

$$\frac{d}{ds}\left(X\,\frac{dx}{ds}\right) = \frac{1}{\rho}\left(1+\frac{x}{\rho}\right)X + \frac{e}{p}\left[\left(1+\frac{x}{\rho}\right)B_z - B_s\,\frac{dz}{ds}\right] \tag{1}$$

$$\frac{d}{ds}\left(X\,\frac{dz}{ds}\right) = \frac{e}{p}\left[B_s\,\frac{dx}{ds} - \left(1+\frac{x}{\rho}\right)B_x\right], \tag{2}$$

$$X = \left[\left(1+\frac{x}{\rho}\right)^2 + \left(\frac{dx}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2\right]^{-\frac{1}{2}}. \tag{3}$$

An adequate approximation gives on the median plane

$$\frac{d^2x}{ds^2} = \frac{1}{\rho}\left(1+\frac{x}{\rho}\right) + \frac{e}{p}\left(1+\frac{x}{\rho}\right)^2 B_z \tag{4}$$

and

$$\frac{d^2z}{ds^2} = -\frac{e}{p}\left(1+\frac{x}{\rho}\right)^2 B_x \tag{5}$$

The magnetic fields near the median plane may be derived from the scalar potential

$$\Psi = -B_o z\left\{1 + \left(\frac{1}{B}\cdot\frac{\partial B_z}{\partial x}\right)_0 x + \frac{1}{2}\left(\frac{1}{B_z}\cdot\frac{\partial^2 B_z}{\partial x^2}\right)_0 x^2 - \frac{\frac{1}{\rho}\left(\frac{1}{B_z}\cdot\frac{\partial B_z}{\partial x}\right)_0 + \left(\frac{1}{B_z}\cdot\frac{\partial^2 B_z}{\partial x^2}\right)_0}{6}\,z^2\right\} \tag{6}$$

Thus, letting

$$k_1(s) = \left(\frac{1}{B_z}\cdot\frac{\partial B_z}{\partial x}\right)_o \quad ; \quad k_2(s) = \left(\frac{1}{B_z}\cdot\frac{\partial^2 B_z}{\partial x^2}\right)_o \tag{7}$$

one has

$$B_z = -\frac{\partial\psi}{\partial z} = B_o\left\{1 + k_1 x + \frac{1}{2}k_2x^2 - \frac{1}{2}\left(\frac{1}{\rho}k_1 + k_2\right)z^2\right\} \tag{8}$$

$$B_x = -\frac{\partial\psi}{\partial x} = B_o z\left(k_1 + k_2 x\right) \tag{9}$$

For the betatron motion, Eqs. (4) and (7) give

$$\frac{d^2x}{ds^2} = \frac{1}{\rho} + \frac{x}{\rho^2} + \frac{eB_o}{p}\left\{1 + \left(\frac{2}{\rho} + k_1\right)x + \left(\frac{1}{\rho^2} + \frac{2k_1}{\rho} + \frac{1}{2}k_2\right)x^2 - \frac{1}{2}\left(\frac{k_1}{\rho} + k_2\right)z^2\right\}. \tag{10}$$

Similarly, Eqs. (5) and (9) give

$$\frac{d^2z}{ds^2} = -\frac{eB_o}{p}\left\{k_1 + \left(\frac{2k_1}{\rho} + k_2\right)x\right\}z\,. \tag{11}$$

Let B_o be determined such that, at the momentum $p = p_o$

$$\frac{1}{\rho} + \frac{eB_o}{p_o} = 0. \tag{12}$$

Equations (10) and (11) become (drop z^2 term)

$$\frac{d^2x}{ds^2} + \frac{1}{\rho^2}\left\{\left(2 + \rho k_1\right)\frac{p_o}{p} - 1 + \left(1 + 2\rho k_1 + \frac{1}{2}\rho^2 k_2\right)\frac{p_o}{p}\cdot\frac{x}{\rho}\right\}x = \frac{1}{\rho}\left(1 - \frac{p_o}{p}\right) \tag{13}$$

and

$$\frac{d^2z}{ds^2} - \frac{1}{\rho^2}\cdot\frac{p_o}{p}\left\{\rho k_1 + \left(2k_1 + \rho k_2\right) x\right\} z = 0. \tag{14}$$

For $p = p_o$ the periodic solution of Eq. (13) is

$$X_o = 0. \tag{15}$$

For $p = p_o + \Delta p$, the periodic solution of Eq. (13) is

$$X = x_p(s)\ \frac{\Delta p}{p}\ , \tag{16}$$

where $x_p(s)$ is the dispersion function.

Consider now the radial betatron oscillations about $x = X$. Let

$$x = X + u. \tag{17}$$

Then, neglecting u^2 and $\left(\Delta p/p\right)^2$,

$$\frac{d^2u}{ds^2} + \frac{1}{\rho^2}\left\{1 + \rho k_1 + \left[\frac{2}{\rho}\left(1 + 2\rho k_1 + \frac{1}{2}\rho^2 k_2\right) x_p - 2 - \rho k_1\right]\frac{\Delta p}{p}\right\} u = 0. \tag{18}$$

Similarly, for the vertical oscillations

$$\frac{d^2z}{ds^2} - \frac{1}{\rho^2}\left\{\rho k_1 + \left[\left(2k_1 + \rho k_2\right) x_p - \rho k_1\right]\frac{\Delta p}{p}\right\} z = 0. \tag{19}$$

From Eq. (18) the radial tune shift becomes

$$\Delta\nu_x = \frac{1}{4\pi\rho^2}\cdot\frac{\Delta p}{p}\int^C\left\{\frac{2}{\rho}\left(1 + 2\ \rho k_1 + \frac{1}{2}\ \rho^2 k_2\right)x_p - 2 - \rho k_1\right\}\beta_x ds, \tag{20}$$

and from Eq. (19) the vertical tune shift becomes

$$\Delta\nu_z = -\frac{1}{4\pi\rho^2}\cdot\frac{\Delta p}{p}\int^C \left\{\left(2k_1 + \rho k_2\right) x_p - \rho k_1\right\} \beta_z ds \tag{21}$$

Another source of tune variation comes from edge focusing. This is given by

$$\Delta\nu_x = \frac{1}{4\pi\rho^2}\frac{\Delta p}{p}\Sigma_i\ \alpha_i\left(1 + \rho k_{1i}\right) x_p\left(s_i\right)\cdot\ \beta_x\left(s_i\right) \tag{22}$$

and

$$\Delta\nu_z = -\frac{1}{4\pi\rho^2}\frac{\Delta p}{p}\Sigma_i\ \alpha_i\ \rho k_{1i}\ x_p\left(s_i\right)\cdot\ \beta_z\left(s_i\right) \tag{23}$$

Numerical Calculations

The magnet parameters k_{1F} and k_{1D} and the orbit functions were determined from the geometrical layout and the condition that $\nu_x = 6.7$ and $\nu_z = 6.8$ via the SYNCH (Garren) code. Using the previous analysis it is then possible to determine k_{2F} and k_{2D} such that the sum of the expressions for $\Delta\nu_x$, Eqs. (20) and (22), and the sum of the expressions for $\Delta\nu_z$, Eqs. (21) and (23), are both zero. The numerical results obtained by utilizing the orbit functions are given in Table I. for three magnet desgins.

Table I. Magnet Parameters

	F	D	F	D	F	D
Turns	48	56	48	64	40	56
B_o(kG)	8.8995	7.5679	8.3512	8.1162	8.1504	8.3170
ℓ (m)	2.8896	2.8896	2.8896	2.8896	2.8896	2.8896
ρ (m)	40.8469	48.0341	43.5287	44.7891	44.6011	43.7077
k_1 (m^{-1})	2.2147	-2.7719	2.3600	-2.5833	2.4180	-2.5204
$\frac{1}{\rho}\int X_p \beta_x ds$ (m^2)	6.1874	1.3971	5.7385	1.4757	5.5761	1.5037
$\int \beta_x ds$ (m^2)	83.9790	32.3197	83.9461	32.3196	83.9325	32.3195
$\frac{1}{\rho}\int X_p \beta_z ds$ (m^2)	1.5011	2.1545	1.3921	2.2757	1.3528	2.3189
$\int \beta_z ds$ (m^2)	20.8800	51.6164	20.8762	51.6405	20.8749	51.6485
k_2 (m^{-2})	.6079	-1.2560	.6702	-1.1926	.6951	-1.1716

References

1 Y. Kobayashi and H. Sasaki, The Influence of Momentum Error on the Operating Point and Pole Face Shaping, INS-TH-21 University of Tokyo (1957).

2 F. T. Cole, MURA-TN-259 (1961).