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INSTABILITY OF A WITNESS BUNCH IN A PLASMA BUBBLE

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Abstract

The stability of a trailing witness bunch, accelerated by a plasma wake accelerator (PWA) in a blow-out regime, is discussed. The instability growth rate as well as the energy spread, required for BNS damping, are obtained. A relationship between the PWA power efficiency and the BNS energy spread is derived.

THE LU EQUATION

When the drive bunch is intense enough, it produces an electron-free plasma bubble, with an average radius, *R*, exceeding the characteristic plasma length, $R \gg k_p^{-1}$. In this case, the shape of the bubble $r_b(z)$ can be described by the Lu equation [@Lu]:

$$r_b r_b'' + 2r_b'^2 + 1 = \frac{4\lambda(z)}{r_b^2};$$

$$\lambda(z) = \frac{N_d}{(2\pi)^{3/2}\sigma n_e}\exp\left(-\frac{z^2}{2\sigma^2}\right); \qquad (1)$$

$$\int \lambda(z)\,dz = \frac{N_d}{2\pi n_e}.$$

Here $\lambda(z)$ is the beam line density normalized in agreement with Ref [@Lu], N_d is the number of electrons in the drive bunch, n_e is the plasma density and z is the distance from the drive bunch; zero boundary conditions sufficiently ahead of the bunch are assumed for r_b and r_b'. After the substitution $r_b \to \sigma r_b$; $z \to \sigma z$, the equation is made to depend on a single parameter *q*:

$$r_b r_b'' + 2r_b'^2 + 1 = \frac{4\lambda(z)}{r_b^2};$$

$$\lambda(z) = \frac{q}{(2\pi)^{3/2}}\exp\left(-\frac{z^2}{2}\right); \qquad (2)$$

$$q = \frac{N_d}{n_e\sigma^3}.$$

Figures 1 and 2 show the bubble boundary for $q = 100$ and $q = 800$ respectfully. As one can see, the bubble is approximately spherical, except for its very frontal part, where the drive bunch is located. The bubble radius is seen to be well approximated by $R \cong \sigma q^{1/3} = (N_d / n_e)^{1/3}$, so it does not depend on the bunch length σ. Note that the related range of parameters corresponds to $R \gg \sigma$. The longitudinal electric field on the axis is given by [@Lu]

$$E_z = 2\pi n_e e r_b r_b' \approx 2\pi n_e e R\zeta = E_0\zeta \qquad (3)$$

where $\zeta = (z - z_0)/R$ is a dimensionless distance to the bubble centre, and the approximation assumes reasonable closeness to the centre. The transverse focusing field, under the same conditions, is given by [@Lu]

$$F_r = 2\pi n e r = E_0 r / R. \qquad (4)$$

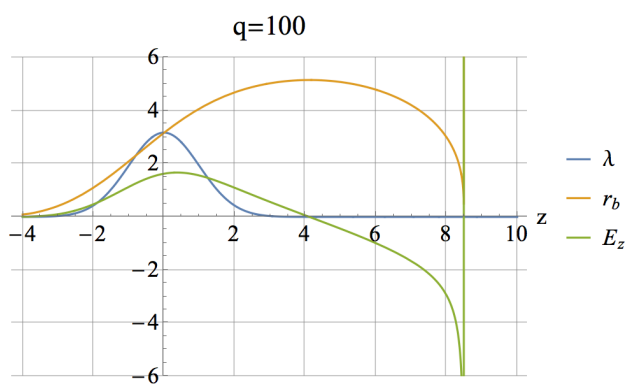

Fig. 1: Solution of the Lu equation for its parameter *q*=100. The line density λ is in arbitrary units, the bubble radius is in the units of the rms bunch length σ and the longitudinal field E_z is in units of $4\pi n_e e\sigma$.

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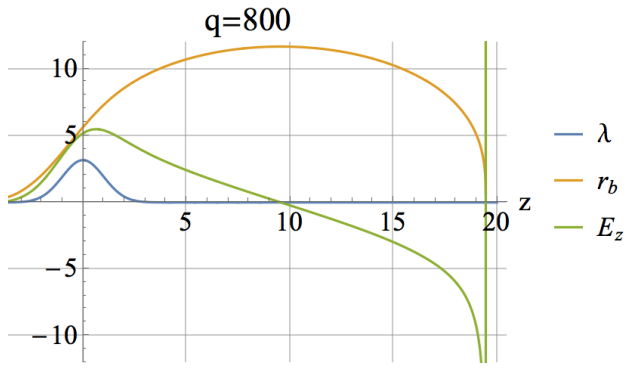

Fig. 2: Same as the Fig 1 but for q=800.

It is worth mentioning that, according to Eq. (2), the number of plasma ions in the bubble is proportional to the number of electrons in the drive bunch:

$$N_i = \frac{4}{3}\pi R^3 n_e \approx 4 N_d \tag{5}$$

Let us define the transformer ratio as

$$r_t = \frac{E_0}{\langle E_z \rangle_d} \approx \frac{11}{q^{1/4}}; \quad (q \geq 100)$$
$$\langle E_z \rangle_d = \frac{\int dz \lambda E_z}{\int dz \lambda}. \tag{6}$$

With that, the power transfer ratio can be expressed as

$$\frac{P_w}{P_d} = r_t \zeta \frac{N_w}{N_d}, \tag{7}$$

where N_w is a number of electrons in the witness bunch.

WAKE FUNCTIONS

The Lu equation (1) allows for obtaining the longitudinal wake function for witness beam particles. Indeed, an addition of the line density perturbation

$$\delta\lambda(z) = \frac{\delta(z - z_1)}{2\pi e n_e} \tag{8}$$

to Eq.(1) yields a step in the longitudinal field

$$\delta E_z = W_{\parallel}(-0) = \frac{4}{r_b^2} \tag{9}$$

which is, by definition, the longitudinal wake function at distances $|\Delta z| \ll R, k_p^{-1}$. Note that this wake value is identical to the one for a hollow cylindrical plasma channel of the radius r_b, as found in Ref. [@Schroeder99]. The same value of the short-range wake is valid for dielectric channels and resistive walls as well [@BurNov, @Chao]. For any channel, the Panofsky-Wenzel relation between the longitudinal and transverse wakes is given by [@Bane]:

$$W_{\perp} = \frac{2}{r_b^2}\int W_{\parallel} dz = \frac{8\Delta z}{r_b^4}. \tag{10}$$

Below, we are assuming that this transverse wake is valid for the particles of a witness bunch, which length is small enough, $\Delta z << R, k_p^{-1}$.

INSTABILITY

The transverse instability of an accelerated short homogeneous bunch a zero momentum spread in a plasma channel was described in Ref.[@Schroeder]; its result for a slow instability is reproduced below. Assuming that focusing scales with the energy as $k_\beta \propto \gamma^{-1/2}$, which is valid for the plasma density being kept constant, the equation of motion has been solved with the following result for the local transverse offsets along the bunch:

$$\frac{X(l,s)}{X_0} = \frac{3^{1/4}}{2^{3/2}\pi^{1/2}}\left(\frac{\gamma_0}{\gamma}\right)^{1/4}\frac{\exp(A)}{A^{1/2}}\cos\psi; \tag{11}$$
$$\psi = \theta - A/3^{1/2} + \pi/12.$$

Asymptotically, $A \to (s/L)^{1/6}$, with the instability growth length

$$L = \frac{2^{10}}{3^9}\left(\frac{I_0}{I}\right)^2\left(\frac{k_0\gamma_0^{1/2} r_b^4}{8l^2}\right)^2 \gamma';$$
$$k_0\gamma_0^{1/2} = k_\beta \gamma^{1/2} = k_p / 2^{1/2}; \tag{12}$$
$$I_0 = mc^3/e; \; I = N_w ec/l.$$

For the plasma bubble, where

$$k_0^2\gamma_0 = k_p^2/2; \quad \gamma' = k_p^2 R\zeta/2;$$
$$r_b^2 = R^2(1-\zeta^2); \quad R = (N_d/n_e)^{1/3}; \tag{13}$$

the instability length is simplified to

$$L \approx 0.03\left(\frac{N_d}{N_w}\right)^2 \frac{R^3}{l^2}\zeta(1-\zeta^2)^4. \tag{14}$$

The growth length L reaches its maximum at $\zeta = 1/3$:

$$L_{\max} \approx 0.006\left(\frac{N_d}{N_w}\right)^2 \frac{R^3}{l^2}. \tag{15}$$

Assuming, for example, $N_d / N_w = 10$, and $R/l = 10$, we get $L_{\max} \approx 60R$. Due to the extremely slow growth of the exponent power A, the available acceleration length $s_{\max}$ can exceed the instability length

$L_{\max}$ by several orders of magnitude. Indeed, let's assume that the offset growth shown in Eq.(11) is limited by a factor of 10: $X / X_0 = 10(\gamma_0 / \gamma)^{1/4}$. This limitation yields a maximum for the exponent power $A \le A_{\max} \approx 4.0$, and correspondingly,

$$s_{\max} = A_{\max}^6 L_{\max} \approx 4 \cdot 10^3 L_{\max} \approx 2 \cdot 10^5 R \quad (16)$$

for this numerical example, leading to the maximal energy

$$\gamma_{\max} = A_{\max}^6 L_{\max} \gamma' \approx 2 \cdot 10^5 \Delta\gamma_R; \quad \Delta\gamma_R = \gamma' R = k_p^2 R^2 / 6. \quad (17)$$

Note that

$$\gamma_{\max} \propto N_d^{10/3} N_w^{-2} l^{-2} n_e^{-1/3} \quad (18)$$

For $n_e = 10^{17}\,\mathrm{cm}^{-3}$ and $N_d = 6 \cdot 10^{10}$, this results in $R \approx 100\,\mu\mathrm{m}$, $k_p R \approx 5$ and $\gamma_{\max} \approx 10^6$. This limit is very sensitive to the maximally tolerable exponent power *A*, which, in its turn, depends on the accuracy of beam transfer from one plasma section into another. For example, if this exponent factor was a bit lower, $A_{\max} = 3$, the maximal energy would be about 5 times lower.

In principle, a proper momentum spread in the witness bunch could suppress the instability, by means of the BNS damping mechanism [@BNS]. This option is examined in the following section.

BNS DAMPING

In this section, the bunch is treated as consisting of two macroparticles with $N_w / 2$ electrons each. The longitudinal forces acting on them follow:

$$F_1 = eE_0\zeta_1 - N_w e^2 W_{\|} / 4; \quad F_2 = eE_0\zeta_2 - N_w e^2 W_{\|} / 4 - N_w e^2 W_{\|} / 2. \quad (19)$$

Due to the difference of these two forces, the particles are getting unequal momenta, which relative difference can be presented as

$$\frac{\delta p}{p} = 2\frac{F_1 - F_2}{F_1 + F_2} = \frac{-\delta\zeta + \rho}{\zeta - \rho}; \quad \zeta = (\zeta_1 + \zeta_2)/2; \quad \delta\zeta = \zeta_2 - \zeta_1; \quad (20)$$

$$\rho = \frac{N_w e^2 W_{\|}}{2eE_0} = \frac{N_w}{\pi n R^3 (1-\zeta^2)} \approx 0.3 \frac{N_w}{N_d (1-\zeta^2)}.$$

To provide BNS damping, the momentum spread has to be matched with the transverse wake [@Chao]:

$$\left.\frac{\delta p}{p}\right|_{BNS} = \frac{N_w r_0 W_{\perp}}{4 k_\beta^2 \gamma} = \frac{\rho\delta\zeta}{1-\zeta^2}. \quad (21)$$

From here, a condition for the distance between the macroparticles $\delta\zeta$ follows:

$$\frac{\rho - \delta\zeta}{\zeta - \rho} = \frac{\rho\delta\zeta}{1-\zeta^2}. \quad (22)$$

The relative momentum spread required by Eq.(21) can be much smaller than 100% only if the intensity parameter

$$\rho = 1. \quad (23)$$

With this condition, Eq.(22) yields:

$$\delta\zeta = \rho; \quad \left.\frac{\delta p}{p}\right|_{BNS} = \frac{\rho^2}{1-\zeta^2}. \quad (24)$$

Taking into account Eq.(20) and Eq.(7), the required momentum spread can be also presented as

$$\left.\frac{\delta p}{p}\right|_{BNS} \approx 0.1 r_t^{-2} \left(\frac{P_w}{P_d}\right)^2 \frac{1}{\zeta^2 (1-\zeta^2)^3}. \quad (25)$$

Minimal value of the energy spread is achieved when the witness bunch is positioned at half-radius, $\zeta = 1/2$, which leads to a limitation:

$$\left.\frac{\delta p}{p}\right|_{BNS} \ge \frac{1}{r_t^2} \left(\frac{P_w}{P_d}\right)^2. \quad (26)$$

Thus, high power efficiency entails large energy spread.

CONCLUSIONS

The transverse instability of the witness bunch sets a limit on the maximal energy up to which the bunch could be accelerated in the nonlinear regime of the plasma wake acceleration with a higher-intensity driving bunch. For the BNS damping, limitations on the acceptable momentum spread inevitably entail a corresponding reduction of the power efficiency of the acceleration.

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