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Adiabatic Capture and Debunching*

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Abstract

In the study of beam preparation for the $g-2$ experiment, adiabatic debunching and adiabatic capture are revisited. The voltage programs for these adiabbatic processes are derived and their properties discussed. Comparison is made with some other form of adiabatic capture program.

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1 Introduction

The muon $g-2$ experiment at Fermilab calls for intense proton bunches for the creation of muons. A booster batch of 84 bunches[†] is injected into the Recycler Ring, where it is debunched and captured into 4 intense bunches with the 2.5-MHz rf. The experiment requires short bunches with total width less than 100 ns. The transport line from the Recycler to the muon-production target has a low momentum aperture of $\sim \pm 22$ MeV. Thus each of the 4 intense proton bunches required to have an emittance less than $\sim$ 3.46 eVs. The incoming booster bunches have total emittance $\sim$ 8.4 eVs, or each one with an emittance $\sim$ 0.1 eVs. However, there is always emittance increase when the 84 booster bunches are debunched. There will be even larger emittance increase during adiabatic capture into the buckets of the 2.5-MHz rf. In addition, the incoming booster bunches may have emittances larger than 0.1 eVs. In this article, we will concentrate on the analysis of the adiabatic capture process with the intention of preserving the beam emittance as much as possible. At this moment, beam preparation experiment is being performed at the Main Injector. Since the Main Injector and the Recycler Ring have roughly the same lattice properties, we are referring to adiabatic capture in the Main Injector instead in our discussions.

2 Adiabatic Capture

We want to examine adiabatic capturing of a coasting beam in the Main Injector at injection. Here, we set the criterion that the relative change in bucket height or bucket area should be much slower than the synchrotron frequency $\omega_s/2\pi$, or [1, 2]

$$\omega_s \gg \frac{1}{A}\frac{dA}{dt}. \tag{2.1}$$

Bucket area is proportional to $\sqrt{V}$ and ω_s is also proportional to $\sqrt{V}$, where V is the rf voltage. Let $a = \omega_s/\sqrt{V}$, which is a constant. Then

$$a \gg \frac{1}{2V^{3/2}}\frac{dV}{dt}. \tag{2.2}$$

Or

$$at \gg C - \frac{1}{\sqrt{V(t)}}, \tag{2.3}$$

[†]Actually, there are only about 80 bunches because of the extraction gap.

where C is a constant.

We cannot start from $V = 0$, because the constant will become infinite. So we start at $V = V_0$ at $t = 0$ and end at $V = V_1$ at $t = T$. Thus $C = 1/\sqrt{V_0}$. Define $\omega_{s0}/2\pi$ as the initial synchrotron frequency (at $V = V_0$) and evaluate a at $t = 0$, resulting in

$$\omega_{s0} t \gg 1 - \sqrt{\frac{V_0}{V(t)}}. \tag{2.4}$$

Suppose we say $(1/n_{\rm ad})^{\rm th}$ of the synchrotron frequency will be slow enough for the adiabatic change of the bucket area.[‡] Here, $n_{\rm ad}$, defined by

$$n_{\rm ad} = \frac{\omega_{s0} t}{1 - \sqrt{V_0/V(t)}} \tag{2.5}$$

at every time, is called the *adiabatic parameter*, which measures the slowness of the relative change of bucket height with respect to the synchrotron frequency. The larger the adiabatic parameter we choose, the more the adiabaticity is the capture. Solution of Eq. (2.5) gives the voltage changing program

$$V(t) = \frac{V_0}{(1 - \omega_{s0} t/n_{\rm ad})^2}. \tag{2.6}$$

Obviously, the capture time t_1 from V_0 to V_1 is given by

$$V_1 = \frac{V_0}{(1 - \omega_{s0} t_1/n_{\rm ad})^2}. \tag{2.7}$$

Thus $\omega_{s0}/n_{\rm ad}$ can be eliminated to give the rf voltage program for adiabatic capture[§]

$$V(t) = \frac{V_0}{\left[1 - \left(1 - \sqrt{\frac{V_0}{V_1}}\right)\frac{t}{t_1}\right]^2}. \tag{2.8}$$

For example, if we start from $V_0 = 1$ kV and end at the 2.5MHz rf voltage $V_1 = 75$ kV, the capture time will be[¶] $t_1 = 187$ ms when $n_{\rm ad} = 8$. We have assumed beam energy 8.9383 GeV so that the relativistic parameters are $\gamma = 9.5263$, $\beta = 0.994475$, revolution period $T_0 = 11.133$ μs, slip factor $\eta = -0.008915$ (transition gamma $\gamma_t = 21.8$), synchroton tune $\nu_{s0} = 6.703 \times 10^{-5}$ at V_0, so that $\omega_{s0} = 6.021$ s^{-1}.

[‡]In other words, the $\gg$ sign of Eq. (2.1) becomes an equality when the right side is multiplied by $n_{\rm ad}$.

[§]Although we derived this adiabatic formula independently in 2002, [1] we notice lately that such formula had already implemented in the control pages of the Main Injector back in 1999. [2] Their definition of adiabatic parameter is $1/n_{\rm ad}$ instead, where $n_{\rm ad}$ is given by Eq. (2.5).

[¶]At this moment, the allowed capture time in the beam preparation is kept at less than ~ 180 ms.

3 Properties of Voltage Program

1. Some of us may think about an exponential increase in rf voltage during adiabatic capture. However, as depicted in Eq. (2.6), this voltage program is far from exponential. In fact, at start or when $\omega_{s0}t/n_{\rm ad} \ll 1$, it is linear

$$\frac{V(t)}{V_0} \approx 1 + \frac{2\omega_{s0}t}{n_{\rm ad}}. \tag{3.1}$$

It also shows that the larger the adiabatic parameter $n_{\rm ad}$, the slower the rise in capture voltage.

2. This voltage program obeys the chain rule. That is, keeping the adiabatic parameter $n_{\rm ad}$ constant, the capture from V_0 to V_1 followed by the capture from V_1 to V_2 is the same as a capture from V_0 to V_2. The proof is as follows: The capture time t_1 from V_0 to V_1 is given by

$$\sqrt{\frac{V_0}{V_1}} = 1 - \frac{\omega_{s0}t_1}{n_{\rm ad}}. \tag{3.2}$$

From V_1, the rf voltage $V(t)$ increases according to

$$\sqrt{\frac{V_1}{V(t)}} = 1 - \frac{\omega_{s1}(t-t_1)}{n_{\rm ad}}, \tag{3.3}$$

where $\omega_{s1}/2\pi$ is the synchrotron frequency at rf voltage V_1. Multiplying the two, we obtain

$$\begin{aligned}\sqrt{\frac{V_0}{V(t)}} &= \left[1 - \frac{\omega_{s1}(t-t_1)}{n_{\rm ad}}\right]\left[1 - \frac{\omega_{s0}t_1}{n_{\rm ad}}\right] \\ &= \left[1 - \frac{\omega_{s0}(t-t_1)}{n_{\rm ad}}\sqrt{\frac{V_1}{V_0}}\right]\left[1 - \frac{\omega_{s0}t_1}{n_{\rm ad}}\right],\end{aligned} \tag{3.4}$$

where we have used that fact that $\omega_{s1}/\omega_{s0} = \sqrt{V_1/V_0}$. Substituting Eq. (3.2), the above reduces to

$$\sqrt{\frac{V_0}{V(t)}} = 1 - \frac{\omega_{s0}t}{n_{\rm ad}}, \tag{3.5}$$

which agrees with the rf voltage program of Eq. (2.6).

4 Other Capture Voltage Program

Kang studied adiabatic capture of a linac beam into buckets of the CIS booster of the IUCF Electron Cooler. [3] The voltage program used is

$$V(t) = \left[3\left(\frac{t}{t_1}\right)^2 - 2\left(\frac{t}{t_1}\right)^3\right][V_1 - V_0] + V_0. \tag{4.1}$$

This is plotted in Fig. 1 along side with the rf voltage of Eq. (2.6) suggested in this paper.

Kang's voltage program certainly differs very much from the one we derived according to adiabaticity. It has the following properties:

1. At start when $t/t_1 \ll 1$, the voltage increases quadratically with time.
2. The chain rule is certainly not satisfied.
3. This voltage program has a inverse symmetry. For simplicity, let $V_0 = 0$, measure time with respect to t_1, measure voltage with respect to V_1; i.e., Eq. (4.1) reduces to

$$V(t) = 3t^2 - 2t^3. \tag{4.2}$$

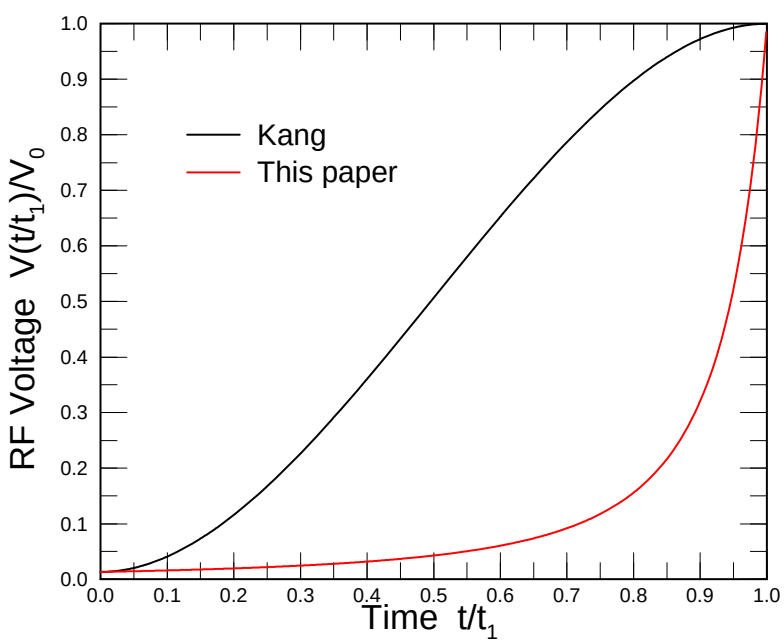

Figure 1: Plots of adiabatic rf voltage as function of time, using Kang's program and the program suggested in this paper.

Now make the substitution $t = 1 - t'$ and $V(t) = 1 - V'(t)$, we obtain

$$1 - V'(t) = 2(1 - t')^2 - 3(1 - t')3 = 1 - 2t'^2 + 3t'^2, \tag{4.3}$$

or

$$V'(t) = 2t'^2 - 3t'^2, \tag{4.4}$$

which is just the inversion of Eq. (4.2). It is possible that this voltage program was derived to satisfy the inversion symmetry. Unfortunately, the symmetry does not imply adiabaticity.

We try to compare the result of capture using these two different voltage programs. We capture a debunched beam with energy offset ± 3.5 MeV from $V_0 = 1$ kV to $V_1 = 75$ kV. The number of macro-particles has always been 4000 and they uniformly fill the debunched beam initially. The results are shown in Fig. 2. We see that when the capture time is 187 ms (corresponding to Ng's adiabatic parameter $n_{\rm ad} = 8$), the bunch is captured within ± 20 MeV using Ng's rf voltage program, whereas with Kang's rf voltage program, the bunch has a spread of ± 30 MeV instead. The reason is the much more rapid increase in rf voltage in Kang's program at the beginning of the capture. The next comparison is a capture in 468 ms or Ng's adiabatic parameter $n_{\rm ad} = 20$. We still see Ng's voltage program leads to a better capture than Kang's voltage program. We may even conclude that Ng's voltage program in 187 ms is better than Kang's voltage program in 468 ms.

When the capture is extremely slow with adiabatic parameter $n_{\rm ad} = 100$ or 2338 ms, the results using the two capture voltage programs become roughly similar. The similarity between the two results comes from the fact that the approximate limit of adiabaticity has already been reached long before 2338 ms when Ng's voltage program is used, but may have just been reached when Kang's voltage program is used. Careful examination reveals that Ng's voltage program is still better than Kang's program, that the bunch edge is much better defined with the former than with the latter programs. In short, we can conclude that adiabatic capture will take less time using Ng's voltage program than Kang's voltage program.

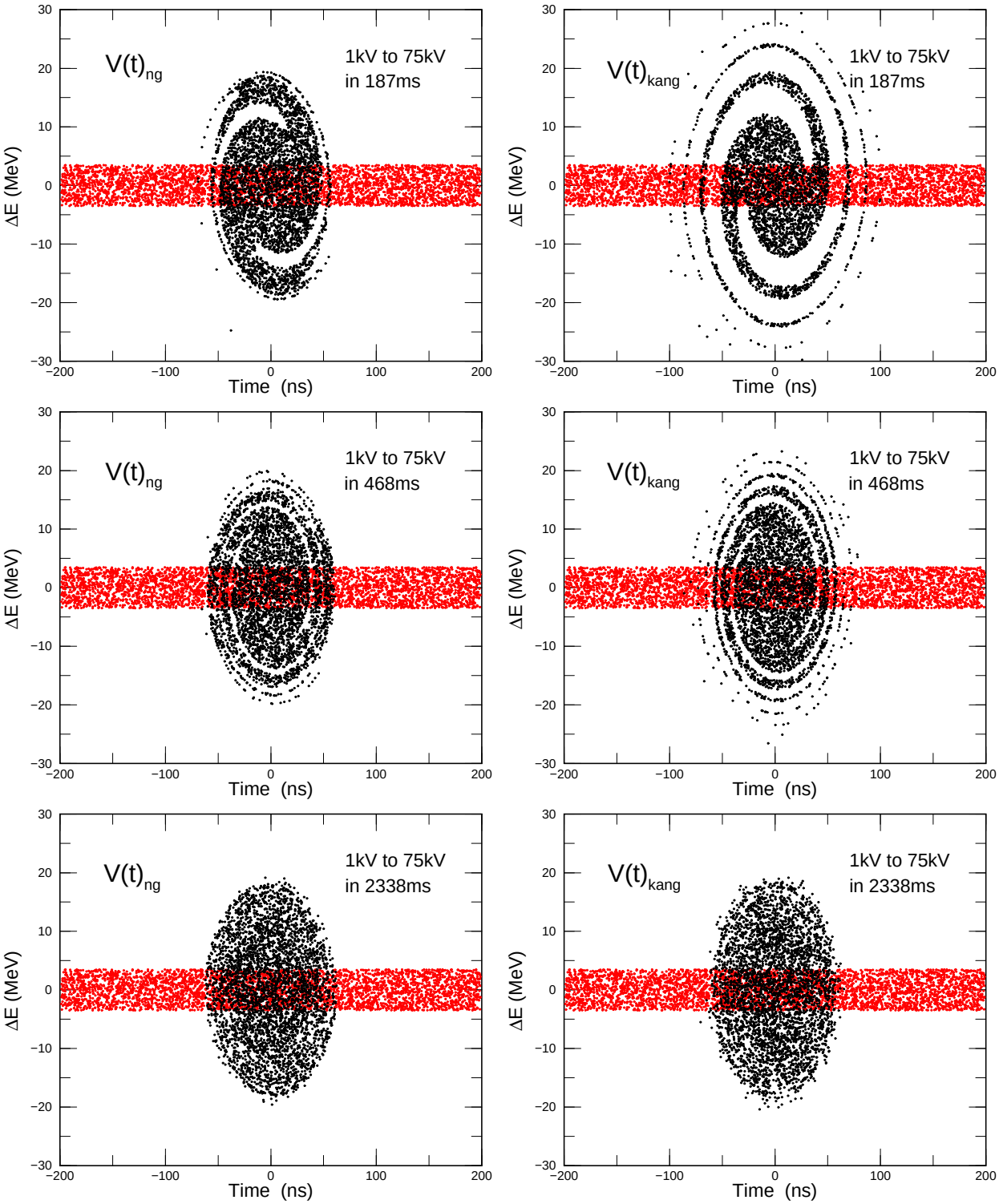

Figure 2: Results of adiabatic capture using Ng's rf voltage program (left) and Kang's rf voltage program (right) in 187 ms, 468 ms, and 2338 ms. In all cases, the rf voltage is ramped from $V_0 = 1$ kV to $V_1 = 75$ kV, and the initial half energy spread of the unbunched beam is 3.5 MeV.

5 The Initial RF Voltage

When the capture time is increased, the capture process becomes more adiabatic, and the captured bunch will have much better appearance, for example, smoother edges. This is supported by Fig. 2, where we see that as the captured time increases from 187 ms to 2238 ms, the empty phase-space portion inside the bunch diminishes drastically. However, there is still an increase in bunch area during the capture process. The original beam area is $2\Delta E T_0/h = 2.783$ eVs, where $T_0 = 11.13$ μs is the period of the Main Injector, $h = 28$ is the rf harmonic and $\Delta E = 3.5$ MeV is the half energy spread. The final bunch, even after the captured time of 2338 ms, has a half width of $\hat{\tau} = 62.3$ ns and half energy spread $\widehat{\Delta E} = 19.5$ MeV. The area is $A \approx \pi\hat{\tau}\widehat{\Delta E} \approx 3.82$ eVs, an increase of $\sim 37\%$.

To reduce the increase in bunch area, in addition to a long capture time, the initial rf voltage V_0 must be reduced as well. To demonstrate this, we lower V_0 from 3 kV to 0.01 kV, perform the simulation with 10000 macro-particles with adiabatic parameter $n_{\rm ad} = 100$. The final rf voltage is always $V_1 = 75$ kV. The area of the captured bunch is computed in two ways. First, the maximum half time spread $\hat{\tau}$ and half energy spread $\widehat{\Delta E}$ are monitored. Assuming that the final bunch fits the $V_1 = 75$ kV bucket, the energy spread of the bunch edge at time advance τ is given by[‖]

$$\Delta E = \sqrt{\frac{2eV_1\beta^2 E}{\pi h|\eta|}}\sqrt{\sin^2\frac{h\omega_0\hat{\tau}}{2} - \sin^2\frac{h\omega_0\tau}{2}}. \tag{5.1}$$

The bunch area is

$$A = 4\int_0^{\hat{\tau}} \Delta E d\tau = \sqrt{\frac{128eV_1\beta^2 E}{\pi h^3\omega_0^2|\eta|}}I, \tag{5.2}$$

where $\phi = h\omega_0\hat{\tau}/2$ and

$$I = \int_0^a \sqrt{\sin^2\phi - \sin^2 x}\, dx. \tag{5.3}$$

To evaluate the integral, let $\sin x = \sin\phi\sin\theta$. Then

$$\begin{aligned} I &= \int_0^{\pi/2} \frac{\sin^2\phi\cos^2\theta}{\sqrt{1-\sin^2\phi\sin^2\theta}}\, d\theta = \int_0^{\pi/2}\left[\sqrt{1-\sin^2\phi\sin^2\theta} - \frac{1-\sin^2\phi}{\sqrt{1-\sin^2\phi\sin^2\theta}}\right] d\theta \\ &= E(m) - (1-m)K(m), \end{aligned} \tag{5.4}$$

[‖]The Hamiltonian is $H = -\dfrac{\eta\Delta E^2}{2\beta^2 E} + \dfrac{eV}{2\pi h}\left[1 - \cos h\omega_0\tau\right]$.

with $m = \sin^2\phi$. The bunch area is therefore

$$A = \sqrt{\frac{128eV_1\beta^2E}{\pi h^3\omega_0^2|\eta|}}\Big[E(m) - (1-m)K(m)\Big], \tag{5.5}$$

where $\omega_0 = 2\pi/T_0$, $K(m)$ and $E(m)$ are complete elliptical functions of the first and second kind. The area is thus evaluated using the half bunch length $\Delta\tau$ as an input. We next start from the half energy spread of the bunch $\widehat{\Delta E}$. From Eq. (5.1), the half bunch length $\hat{\tau}$ can be obtained via

$$\widehat{\Delta E} = \sqrt{\frac{2eV_1\beta^2E}{\pi h|\eta|}}\sin\frac{h\omega_0\hat{\tau}}{2}. \tag{5.6}$$

The bunch area is computed again by substituting this new $\hat{\tau}$ into Eq. (5.5). The results of the two computations are shown in the left plot of Fig. 3 in black and red, respectively. The average of the two is shown in blue. We see that the bunch area approaches the initial area (2.783 eVs) of the beam only when the initial rf voltage approaches zero. Because of the ruggedness of the bunch edge with finite number of macro-particles, it may not be accurate to consider the maximum extents of the bunch as the half bunch length and half energy spread. We therefore try to compute the rms bunch length σ_τ and rms energy spread σ_E

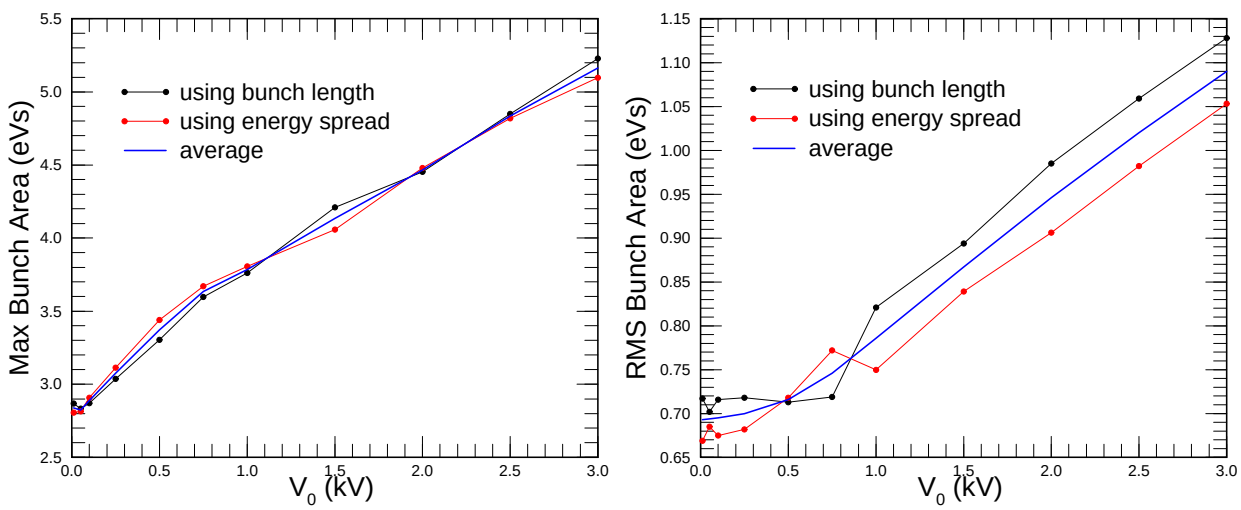

Figure 3: Left plot: Emittance of captured bunch when captured from various initial voltage V_0. Initial emittance is 2.783 eVs. Black curve shows emittance computed from bunch length and red curve shows emittance computed from energy spread, while blue curve shows the average. Right plot: Same as left plot, but with rms emittance depicted instead. Black curve shows rms emittance computed from rms bunch length and red curve shows rms emittance computed from rms energy spread, while blue curve shows the average. In all cases, adiabatic parameter has been kept constant at $n_{\rm ad} = 100$, and the highest captured voltage is 75 kV.

instead. They are then used separately to compute the rms area of the bunch. The results are shown in the right plot of Fig. 3. We see that the computation using the rms bunch length (black) does differ from the the computation using the rms energy spread (red). The reason is unknown. Probably, this is a signal that the captured bunch does not fit the bucket well. Anyway, the average is shown in the blue curve. Here we see that the bunch rms emittance** (0.695 eVs) is preserved only when the initial rf voltage is less than ~ 0.2 kV.

It is plausible to believe that if the rf bucket created by the initial rf voltage V_0 is well within the initial unbunched beam, the emittance of captured bunch will be preserved. This is reasonable because the structure of the phase space set up by the rf voltage matches the initial unbunched beam as closely as possible. The bucket height is given by Eq. (5.1) with V_1 replaced by V_0, $h\omega_s\hat{\tau} = \pi$, and $\tau = 0$, or

$$\Delta E\Big|_{\text{bucket}} = \sqrt{\frac{2eV_0\beta^2 E}{\pi h|\eta|}}. \tag{5.7}$$

With $\Delta E|_{\text{bucket}} = 3.5$ MeV, we obtain $V_0 = 0.543$ kV. The emittance growth in our simulation is 2.9% when $V_0 = 0.5$ kV, reduces to 0.63% when $V_0 = 0.25$ kV, and almost zero when $V_0 = 0.01$ kV, using the rms computation. Thus $\Delta E|_{\text{initial bucket}} \ll \Delta E|_{\text{intial beam}}$ does give us a guide line of how small the initial rf voltage should be. It is important, however, to point out that lowering the initial rf voltage implies lengthening the capture time if the same degree of adiabaticity is maintained. According to Eq. (2.7), the increase in capture time t_1 is proportional to $V_0^{-1/2}$. In practice, the allowable time to perform this type of rf maneuvering is often limited. For this reason, the capture result must be compromised.

6 Adiabatic Debunching

Similar logic can result in the voltage program of adiabatic debunching. We start from the same requirement that the relative change bucket area is much slower than the synchrotron frequency, or

$$\omega_s = -n_{\text{ad}}\frac{1}{A}\frac{dA}{dt}, \tag{6.1}$$

where $n_{\text{ad}} \gg 1$ is the adiabatic parameter. Notice that there is a negative sign on the right side because the bucket area is decreasing during the debunching process. Solving the

**For a uniformly distributed circular bunch, the rms area is a quarter of the area of the circle. Thus without emittance increase the rms bunch area should be $2.783/4 = 0.695$ eVs.

differential equation exactly as before, we arrive at

$$\omega_{s0} t = n_{\rm ad} \left[\sqrt{\frac{V_0}{V(t)}} - 1 \right], \tag{6.2}$$

or

$$V(t) = \frac{V_0}{(1 + \omega_{s0} t / n_{\rm ad})^2}, \tag{6.3}$$

where V_0 is the initial rf voltage and $\omega_{s0}/2\pi$ is the corresponding synchrotron frequency. For a debunching duration t_1, the final rf voltage is

$$V_1 = \frac{V_0}{(1 + \omega_{s0} t_1 / n_{\rm ad})^2}, \tag{6.4}$$

We can therefore eliminate ω_{s0} to arrive at

$$V(t) = \frac{V_0}{\left[1 + \left(\sqrt{\frac{V_0}{V_1}} - 1 \right) \frac{t}{t_1} \right]^2}, \tag{6.5}$$

which is to be compared with the capture voltage program of Eq. (2.8).

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