ON QUANTUM INTEGRABLE SYSTEMS

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Abstract

Many quantum integrable systems are obtained using an accelerator physics technique known as Ermakov (or normalized variables) transformation. This technique was used to create classical nonlinear integrable lattices for accelerators and nonlinear integrable plasma traps. Now, all classical results are carried over to a nonrelativistic quantum case.

I. INTRODUCTION

Chaotic motion is not desirable in particle accelerators or plasma traps because it leads to particle losses and beam blow-ups. The preferred lattices in use are those that conserve simple invariants of motion and in most of them the motion is linear. The invariants for a linear time-dependent oscillator were found by V. Ermakov [1] in the 1880s and became known as Courant-Snyder invariants in accelerators [2]. Also, there exists a simple change of coordinates and time that results in a time-independent motion in special (the so-called normalized) variables.

Surprisingly, this transformation (we will call it the Ermakov transform below) allows one to create a large class of nonlinear integrable systems for accelerators [3] and plasma traps [4]. In this paper we extend it to quantum systems. We will show that all classical time-dependent integrable systems, found with the help of Ermakov transform, generate quantum integrable systems with a similar transformation of nonrelativistic quantum equations. The Schrödinger and Pauli equations become separable in some coordinate systems and can be solved in analytic functions.

II. THE ERMAKOV TRANSFORM

The equations of motion in accelerators (in the uncoupled case) are written as:

$$\begin{cases} x'' + K_x(s)x = 0 \\ y'' + K_y(s)y = 0 \end{cases} \qquad (1)$$
$$K_{x,y}(s+C) = K_{x,y}(s)$$

where K_x and K_y are piecewise constant functions of s (the time-equivalent longitudinal coordinate), and C is the accelerator circumference and the longitudinal motion is negligible.

One can notice that these are two uncoupled Hill's equations. Such an equation was first solved by Ermakov [1] who obtained its invariant, which in accelerator physics is called the

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Courant-Snyder invariant [2]. This invariant can be understood by introducing the so-called normalized phase-space coordinates:

$$z_N = \frac{z}{\sqrt{\beta(s)}},$$
$$p_N = p\sqrt{\beta(s)} - \frac{\beta'(s)z}{2\sqrt{\beta(s)}}, \tag{2}$$

where z stands for either x or y, p is similarly either p_x or p_y, and $\beta(s)$ is either the horizontal or vertical beta-function (defined in, e.g. [5]). In these new normalized variables, the initial time-dependent Hamiltonian associated with Eqs. (1) becomes time-independent,

$$H = \frac{1}{2}\left(p_N^2 + z_N^2\right), \tag{3}$$

and thus leads to two invariants, the horizontal and vertical Hamiltonians. According to Eq. (3), in a linear lattice, all particles execute harmonic oscillations around the reference orbit with a frequency, known as the betatron tune, which is identical for all particles, regardless of their amplitude. Linear lattices have been considered attractive, in part because linear dynamics is easily understood. Now, we use the transform (2) for nonlinear systems. For any Hamiltonian of the form

$$H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + K(s)\left(\frac{x^2}{2} + \frac{y^2}{2}\right) + V(x,y,s), \tag{4}$$

the transformation (2) produces a new Hamiltonian

$$H_N = \frac{p_{xN}^2 + p_{yN}^2}{2} + \frac{x_N^2 + y_N^2}{2} + \beta(\mu)V\left(x_N\sqrt{\beta(\mu)}, y_N\sqrt{\beta(\mu)}, s(\mu)\right), \tag{5}$$

where $\mu' = \frac{1}{\beta(s)}$. Now if we choose the potential V such that it is independent of μ, we would obtain one invariant of motion – the new Hamiltonian (5). In addition, other invariants can be found to obtain a completely integrable system (see [3, 4]).

III. SCHRÖDINGER EQUATION

We rewrite the Schrödinger equation in some convenient form by redefining time and coordinates:

$$i\sqrt{2}\frac{\partial\psi}{\partial t} = -\nabla^2\psi + \frac{K(t)r^2}{2}\psi + U(\vec{r},t)\psi, \tag{6}$$

where time here is $t=\sqrt{2}T/\hbar$ (T is the real time), and coordinates $\vec{r}=\vec{R}\sqrt{2m}/\hbar$ ($\vec{R}$ stands for real coordinates), *K(t)* is an arbitrary focusing coefficient and $U(\vec{r},t)$ is an arbitrary potential.

Now we introduce a new "time" variable, τ, new coordinates, and a new wave function by using functions $f(t)$, $g(t)$ and $\beta(t)$ (to be defined below) with the following relation to the conventional time and coordinates:

$$d\tau=\frac{dt}{\beta(t)},\vec{r}_N=\frac{\vec{r}}{\sqrt{\beta(t)}},\psi=f(t)\exp\left[ig(t)r_N^2\right]\psi_N. \tag{7}$$

The equation (6) in these new coordinates and time for the new wave function becomes:

$$i\sqrt{2}\frac{\partial\psi_N}{\partial\tau}=-\nabla_N^2\psi_N+\frac{r_N^2}{2}\psi_N+\left(\beta(\tau)U(\vec{r}_N\sqrt{\beta(\tau)},t(\tau)\right)\psi_N, \tag{8}$$

provided $\beta(t)$ satisfies the following differential equation:

$$\left(\sqrt{\beta}\right)''+K(t)\sqrt{\beta}=\frac{1}{\beta^{3/2}} \tag{9}$$

and

$$g(t)=\frac{\beta'}{4\sqrt{2}}, \tag{10}$$

$$f(t)=\beta^{-1/4}. \tag{11}$$

One can see a complete analogy with the Hamiltonian (5). In addition, Eq. (9) is exactly the envelope function equation for the linear motion in accelerators (see [5]), implying that all classical results on integrability from references [3, 4] are applicable to the Schrödinger equation. Specifically, all integrable time-dependent classical systems from those references generate the time-independent Schrödinger equation in the new variables and can be solved analytically. In addition to the transformation (7), there exist another transformation of a simpler type (see the first publication in Ref. [3], Eq. 25) that produces a similar conversion of the Schrödinger equation to the same equation but with a different potential.

The simplest example of the conversion to a time-independent integrable system is given by $U(\vec{r},t)=A(\theta)/r^2$ in (6), where $A(\theta)$ an arbitrary function of a zenith angle in spherical coordinates. Then, by solving Eq. (9) and transforming the variables and the wave function using (7), (10) and (11), one obtains a time-independent equation (8) with the new potential (including the quadratic term) $V(\vec{r},t)=\frac{r_N^2}{2}+\frac{A(\theta)}{r_N^2}$, which is a classical integrable potential and separable in spherical coordinates [6]. Note that all results of a classic paper [7] are obtained in a single line (transformation (7)) and many more integrable systems can be found by this transform from classical results of papers [3, 4].

IV. PAULI EQUATION

This equation is more involved than the previous one and but the transformation from time-dependent systems to time-independent ones is similar. The Pauli equation reads:

$$i\sqrt{2}\frac{\partial\phi}{\partial t}=\left(\left(i\vec{\nabla}+d\vec{A}\right)^2+U\left(\vec{r},t\right)\right)\phi+\frac{e\hbar}{2m}\vec{\sigma}\cdot\vec{B}\left(\vec{r},t\right)\phi \tag{12}$$

where ϕ is a two-component spinor wavefunction, $\vec{\sigma}$ are Pauli matrices, $U,\vec{A}$ are electric and vector potentials, respectively, $\vec{B}$ is the magnetic field, $d=e/\sqrt{2m}$, and time and coordinates scaled as in (6). With the transformation (7) (with the bi-spinor ϕ_N instead of ψ_N) the Pauli equation is transformed to:

$$\begin{aligned} i\sqrt{2}\frac{\partial\phi_N}{\partial\tau}&=\left[i\vec{\nabla}+d\sqrt{\beta(\tau)}\vec{A}\left(\vec{r}_N\sqrt{\beta(\tau)},t(\tau)\right)\right]^2\phi_N+\frac{r_N^2}{2}\phi_N+\beta(\tau)U\left(\vec{r}_N\sqrt{\beta(\tau)},t(\tau)\right)\phi_N\\ &+\frac{e\hbar\beta(\tau)}{2m}\vec{\sigma}\cdot\vec{B}\left(\vec{r}_N\sqrt{\beta(\tau)},t(\tau)\right)\phi_N-2\sqrt{\beta(\tau)}g\left(\tau\right)x_N\vec{A}\left(\vec{r}_N\sqrt{\beta(\tau)},t(\tau)\right)\phi_N \end{aligned} \tag{13}$$

Equation (13) has a few new features with respect to the magnetic field. While the electric potential in new variables gains a factor of β after the transform, the vector potential gains a factor of $\sqrt{\beta}$ - they transform differently. It means, for example, that the electric potential $U\propto\frac{1}{r^2}$ is invariant under transformation (7), but for the vector potential the invariant function is $\vec{A}\propto\frac{\vec{a}}{r}$, where $\vec{a}$ is an arbitrary vector that does not change when all coordinates are multiplied by the same arbitrary factor. For the magnetic field, though, the invariant dependence on coordinates is $1/r^2$ as it should be because $\vec{B}=\vec{\nabla}\times\vec{A}$. One can check that if the term with the vector and electric potentials is independent of time, the last two terms in (13) is also independent of time, therefore we restore all the properties of Ermakov transformation even for Pauli equation – all the classical approaches on how to obtain time-independent integrable systems automatically work for this equation as well.

V. CONCLUSIONS

In this paper we have described an extension of the Ermakov-like transformation to the Schrödinger and Pauli equations. It is shown that these newly found transformations create a vast variety of time dependent quantum equations that can be solved in analytic functions, or, at least, can be reduced to time-independent ones.

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