Explicit Expressions of Impedances and Wake Functions

K.Y. Ng

_Fermilab, Batavia, IL 60510_

K.Bane

_SLAC, Stanford, CA 94309_

(October, 2010)

Abstract

Sections 3.2.4 and 3.2.5 of the Handbook of Accelerator Physics and Engineering on Landau damping are combined and updated. The new addition includes impedances and wakes for multi-layer beam pipe, optical model, diffraction model, and cross-sectional transition.

Submitted to
3rd edition of Handbook of Accelerator Physics and Engineering
3.2.4 Explicit Expressions of Impedances and Wake Functions

K.Y. Ng, FNAL, K. Bane, SLAC

See tables in the next pages.

References

[14] A. Burov and V. Lebedev, EPAC 02 p.1402
[18] I. Zagorodnov, K. Bane, Proc. EPAC 06, p.2859
[21] G. Stupakov, PAC95, p.3303
[22] K. Yokoya and K. Bane, PAC 99, p.1725
[26] M. Sands, SLAC note PEP-253 (1977); H.A. Bethe, PR 66 (1944) 163
[31] A. Novokhatski, A. Mosnier, PAC 97, p.1661
[34] K.Y. Ng, PRD 42 (1990) 1819; A. Burov and A. Novokhatski, HEAC 92, p.537
[37] J.B. Murphy et al, PAC 95, p.2980; PA 57 (1997) 9
[38] Y. Derbenev et al, DESY-TESLA-FEL 95-05 (1995)
[40] K. Bane, P. Morton, LINAC 86, p.490
[45] G. Dome, PAC 85, p.2531
[47] K.Y. Ng, R. Warnock, PAC 89, p.798; PRD 40 (1989) 231
[48] K.Y. Ng, PA 23 (1988) 93
General Remarks and Notation:

In cylindrically symmetric structures $W'_m(z)$ and $W_m(z)$ denote, respectively, $m$-th azimuthal multipole longitudinal and transverse wake functions, generated by point charge $Q$, at distance $z > 0$ behind. $W'_m(z) = dW_m(z)/dz$. $W'_m(z) = 0$ and $W_m(z) = 0$ when $z > 0$ when particle travels at the speed of light. $W'_m(0) = \frac{1}{\beta} \lim_{z \to 0} W'_m(z)$. Longitudinal and transverse momentum kicks on test charge $q$ near pipe axis: $\Delta p_L(z) = -q Q W'_0(z)/c$, $\Delta p_L(z) = -q Q t L W_1(z)/c$, where $t L$ is (small) offset of the source or exciting charge. The $m$-th multipole longitudinal impedance $Z_m^\parallel(k) = \int e^{-i k z / \beta} W_m^\parallel(z) dz / (i \beta c)$ is related to the $m$-th multipole transverse impedance, $Z_m^\perp(k) = i \int e^{-i k z / \beta} W_m^\perp(z) dz / (i \beta c)$, by $Z_m^\parallel = k Z_m^\perp (m \neq 0)$, where $k = \omega / c$. Note that $Z_m^\parallel(-k) = Z_m^\parallel (k)$, $Z_m^\perp(-k) = -Z_m^\perp(k)$. For periodic or translationally invariant structures: steady-state results are given per length $L$. Unless otherwise stated, structures are cylindrically symmetric with perfectly conducting metallic walls, and with beam pipes of radius $b$. In many cases, $\beta = v/c$ has been set to 1. $Z_0 = \sqrt{\mu_0 / \epsilon_0} \approx 377 \, \Omega$ is impedance, $\epsilon_0$ electric permittivity, and $\mu_0$ magnetic permeability of free space. ‘Pill-box cavity’ signifies a pill-box with beam pipes. Here $[\alpha \pm i |\beta|]^{1/n}$ (with $\alpha, \beta$ real, $n = 2$ or 3) is in the 1st/4th quadrant. $H(x) = 0, 1$ for $x \leq 0$. For 3D structures with mirror symmetry in $x$ and $y$, near axis momentum kick in $y$, $\Delta p_y = -q Q (y' W'_y + y W'_y)$, with $y'$ ($y$) offset of exciting (test) charge, and $W'_y (W'_y)$ dipole (quad) wake terms. Total $y$ wake $W_y = W'_y + W'_y$; total $y$ impedance $Z_y = Z'_y + Z'_y$.

<table>
<thead>
<tr>
<th>Description</th>
<th>Impedances</th>
<th>Wake</th>
</tr>
</thead>
<tbody>
<tr>
<td>Space-charge: $[1]$</td>
<td>$Z_0^\parallel / L = \frac{Z_0 k_0 g_0}{4 \pi \beta \gamma^2}$, $g_0 = 1 + 2 \ln b / a$, $W'_0 / L = \frac{Z_0 c}{4 \pi \gamma^2} [1 + 2 \ln b / a] \delta(z)$</td>
<td>$W_m^\parallel / L = \frac{Z_0 c}{4 \pi \gamma^2 m} [1 + 2 \ln b / a] \delta(z)$</td>
</tr>
<tr>
<td>Nonuniform distributions: $[2]$ $a_{\text{eff}}$ is equivalent-uniform-beam radius, $g_0 = 1 + 2 \ln (b / a_{\text{eff}})$, while $a_{\text{eff}} = [\pi \lambda(0)]^{-1/2}$ is the same when self-force part written as $1 / a_{\text{eff}}^{1/2}$, $\gamma_c \approx 0.57721$ is Euler’s constant.</td>
<td>Distribution $\lambda(r)$, $g_0 (m=0)$, $a_{\text{eff}} (m=0)$, $a_{\text{eff}} (m=1)$</td>
<td></td>
</tr>
<tr>
<td>Resistive wall: $[1$, $3]$ wall thickness $t$, dc and ac conductivities $\sigma_c$, $\bar{\sigma_c}$, relaxation time $\tau$; assume $</td>
<td>k</td>
<td>b \gg (s_0 / b)^3$, thick walls: $t \gg \delta_c = \sqrt{2 /</td>
</tr>
</tbody>
</table>

Image part of $Z_1^\parallel$ can be written in terms of Laslett’s electric image coefficients as $1 / b^2 - 2 (\xi_{1,x,y} - \epsilon_{1,x,y}) / h^2$ with $h$ denoting half height of vacuum chamber. See Sec. ??.
<table>
<thead>
<tr>
<th>Description</th>
<th>Impedances</th>
<th>Wakes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low frequency: ([1]) (k \ll 1/s_0), long range (</td>
<td>z</td>
<td>\gg s_0).</td>
</tr>
</tbody>
</table>

Note: \(Z_1' = \frac{2}{b \delta} Z_0'\), \(W_1 = \frac{2}{b^2} \int W_0' dz\). |

| Low frequency, thin wall: \([1]\) \(t \ll \delta_c\) and \(|k| \ll 1/\sqrt{bt}\). | \(Z_m' = \frac{Z_0 kt}{L} = -\frac{Z_0 t c \delta'(z)}{2\pi b}, \ Z_1' = -\frac{i Z_0 t}{\pi b^3}\) | \(W_m' = -\frac{Z_0 t c \delta'(z)}{2\pi b}, W_1 = -\frac{Z_0 t c \delta(z)}{\pi b^3}\) |

| High frequency: \([3]\) \(k \gg 1/s_0\), short range \(|z| \lesssim s_0\), with \(ct \gg s_0\). \(k_p = \sqrt{Z_0 \sigma_c / ct}\) is plasma frequency/c. | \(Z_m' = \frac{4 Z_0 e \tau (m + 1)}{\pi b_{m+1}^2} \times \frac{1 - 4 i k c \tau}{b (1 - 4 i k c \tau)^2 + 32 k_p (\alpha c \tau)^2}\) | \(W_m' = \frac{Z_0 c (m + 1)}{\pi b_{m+2}^2 - 1} e^{z/4 \alpha c \tau}\times \cos \sqrt{\frac{2 k_p}{b} \alpha z}\), for \(\alpha\) see above |

| Fineline, lossy inserts: \([4]\) of length \(L\), in lossless pipe | These formulae depend only on the plasma frequency of the metal. Effects of relative magnetic permeability have not been considered. |

| Displaced beam: \([5]\) at \(\bar{a} = (a_x, a_y)\), rms bunch length \(\sigma_z\), average current \(I_b\), and \((b/k^2, b - a) \gg \delta_c\) and \(\gamma \gg 1\). | Wall impedances in last section multiplied by \(f_z\) for \(Z_0^z\) and \(f_x, y\) for \(Z_1^z\) with \(f_x = \frac{b^2 + a^2}{b^2 - a^2}, f_y = \frac{b(b^2 - a^2 + 4 a_0^2)}{(b^2 - a^2)^3}\), \(Z_0^z = 2^{3/2} Z_0 \sqrt{\frac{\pi L}{k}}\), else \(Z_0^z\) as given above |

| Displaced beam between two infinite plates: \([5]\) at \(y = \pm h/2\). \(\gamma \gg 1\). \(h/k^2, h - 2y_0 \gg \delta_c\). Thin dielectric coating of thickness \(\Delta h\). | \(Z_0^z = \frac{1 - \text{sgn}(\omega)i}{\pi h} \sqrt{|\omega| \mu_0 Z_0 f_x}, Z_1^z = \frac{\pi (\text{sgn}(\omega) 1 - i)}{2 |\omega| \sigma_c / (\epsilon_c \mu_0 Z_0)} f_x\) |

| Metallic coating on ceramic pipe: \([6]\) compared with all metal pipe \(Z_0^z\) (metal). \(t_{c, \text{met}} = \text{metal/ceramic thickness} \ll b\), \(\gamma \gg 1\). \((\mu - 1) k^2, (1 - \mu^{-1}) b t_c\) \ll \(\sigma_c^2\). Loss \(P/L\) is max. at \(V = 0.82\). |

| Metallic coating on ceramic pipe: \([6]\) compared with all metal pipe \(Z_0^z\) (metal). \(t_{c, \text{met}} = \text{metal/ceramic thickness} \ll b\), \(\gamma \gg 1\). \((\mu - 1) k^2, (1 - \mu^{-1}) b t_c\) \ll \(\sigma_c^2\). Loss \(P/L\) is max. at \(V = 0.82\). |

| Elliptical beam pipe: | Low frequency, see \([7, 8, 5]\), high frequency, see \([9, 10]\). |

| Rectangular beam pipe: | Low frequency, see \([8]\), high frequency, see \([9, 11]\). |
**Multi-layer pipe wall impedances:** [12, 13] Cylindrical beam pipe with $N$ layers, $p$th layer between $b^{(p-1)} < r < b^{(p)}$ and $b^{(N)} \to \infty$. Layer 1 is vacuum, $a < r < b^{(1)}$, with particle beam of charge $Q$ at $r = a$ and $\theta = 0$. $r < a$ is called the 0-th layer. Each layer has its own wavenumber $\nu = k\sqrt{1-\beta^2 z_{1}^{2}}$, $k = \omega/v$ and own properties $\epsilon = \epsilon_{0}\varepsilon_{1} = \epsilon_{0}\varepsilon_{r}(1 + i\tan\theta_{e}) - \frac{\gamma_{0}}{\omega^2(1-\omega^2)}$, $\mu = \mu_{0}\mu_{1} = \mu_{0}\mu_{r}(1+i\tan\theta_{m})$; $\theta_{e}$, $\theta_{m}$ are loss angles, $\sigma_{ac}$ dc conductivity, and $\tau$ relaxation time. Actually any frequency dependent $\epsilon$, $\mu$, and conductivity can be assumed. Inside vacuum, $\nu = k/\gamma$; inside conducting metal of skin depth $\delta_{c}$, $\nu \approx (1-i)/\delta_{c}$. A user-friendly Mathematica code for computation is available [12]. The derivation is outlined briefly below.

In terms of Bessel and Kelvin functions, $m$th multipole longitudinal fields inside $p$-th layer:

$$E_{s}^{(p)} = \cos m\theta e^{iks} C_{1e}^{(p)} I_{m}(\nu^{(p)} r) + C_{K_{e}}^{(p)} K_{m}(\nu^{(p)} r), \quad \bar{E} \text{ is electric field}$$

$$C_{s}^{(p)} = \sin m\theta e^{iks} C_{1g}^{(p)} I_{m}(\nu^{(p)} r) + C_{K_{g}}^{(p)} K_{m}(\nu^{(p)} r), \quad \bar{G} = Z_{0}\tilde{H}, \tilde{H} \text{ is magnetic field}$$

Matching $E_{s}$, $E_{o}$, $G_{s}$, and $G_{o}$ at boundary $r = b^{(p)}$ between $p$-th and $(p+1)$-th layers gives

$$\begin{bmatrix} C_{1e}^{(p+1)} \\ C_{K_{e}}^{(p+1)} \\ C_{1g}^{(p+1)} \\ C_{K_{g}}^{(p+1)} \end{bmatrix} = M^{p+1} = \begin{bmatrix} C_{1e}^{(p)} \\ C_{K_{e}}^{(p)} \\ C_{1g}^{(p)} \\ C_{K_{g}}^{(p)} \end{bmatrix} \text{ iteratively} \begin{bmatrix} C_{1e}^{(1)} \\ C_{K_{e}}^{(1)} \\ C_{1g}^{(1)} \\ C_{K_{g}}^{(1)} \end{bmatrix} = \mathcal{M}$$

Where $\mathcal{M} = M_{N-1}^{N}M_{N-2}^{N-1}\cdots M_{1}^{2}$

See [12] for explicit expression of $M^{p+1}$

Since the last layer goes to infinity, $C_{1e}^{(N)} = C_{1g}^{(N)} = 0$. From the beam region, $C_{1e}^{(1)} = 0$ and

$$C_{K_{e}}^{(1)} = -i k Q z_{0} I_{m}(ka/\gamma) / [\pi\beta^{2}(1+\delta_{m})],$$

one can easily solve for $C_{1e}^{(1)} = -\alpha_{1}(k_{m}^{(1)} I_{m}^{(1)}) / I_{m}^{(1)}$, with $I_{m}^{(1)} = I_{m}(\nu^{(1)} b^{(1)})$, $I_{o}^{(1)} = I_{o}(\nu^{(1)} b^{(1)})$.

With beam at $r = a_{1}$, $\theta = 0$, reduced forces on a unit test charge at $r = a_{2} > a_{1}$ and $\theta = \theta_{2}$ are

$$Z_{||} = -i \int ds [E_{s}(a_{2}, \theta_{2}, s; \omega) e^{-iks} + Z_{x} = -i \int ds [E_{s}(a_{2}, \theta_{2}, s; \omega) - \beta G_{o}(a_{2}, \theta_{2}, s; \omega)] e^{-iks}$$

Space-charge contributions for all multiplies ($\alpha_{1} = 1$ or perfectly conducting at $r = b^{(1)}$):

$$Z_{||}^{sc} = \sum_{m=0}^{\infty} i k Z_{0} L \cos m\theta_{2} \frac{m\gamma}{a_{2}^{2}} I_{m}(x_{2}) K_{m}(x_{1}), \quad K_{m}(x_{i}) = [K_{m}(x_{i}) - K_{m}^{(1)} I_{m}(x_{i})] / K_{m}^{(1)}, x_{i} = ka_{i} / \gamma$$

The rest are from wall impedances. To any order $a_{1}^{n_{1}} a_{2}^{n_{2}}$, they are

$$Z_{||}^{W,n_{1},n_{2}} = \frac{i L_{o} \mu_{o}(ka_{1})^{n_{1}}(ka_{2})^{n_{2}}}{\pi\beta^{2} \gamma^{2}} \sum_{n=0}^{\infty} \cos m\theta_{2} \cos m\theta_{2} K_{m}(x_{2}) + m\gamma / a_{2}^{2} \sin m\theta_{2} K_{m}(x_{2})$$

The usual monopole and dipole pipe-wall impedances are

$$Z_{||}^{W,0,0} = \frac{i k Z_{0} L \bar{\alpha}_{1} K_{m}^{(1)}}{2\pi\beta^{2} \gamma^{2} I_{m}^{(1)}}$$

**Multi-layer special cases:** [13] Pipe wall: $b^{(1)} < r < b^{(2)} = b^{(1)} + t$.

**Thin wall:** Good for low frequencies. $t \to 0$ and $E_{s}$ does not change across wall. At $r = b^{(3)}$,

Case PC: perfectly conducting, Case PM: perfectly magnetic, and Case INF: $b^{(2)} \to \infty$.

$$\bar{\alpha}_{1} = -\frac{\gamma^{2} \beta^{2}(1-\alpha_{2}) + 2ix_{2} \beta / m\gamma}{1 + x_{2}^{2} / m - \frac{2}{\gamma(1-\alpha_{2})} + \frac{2}{\gamma(1-\alpha_{2})}}$$

$$\bar{\alpha}_{2}^{PC} = -\bar{\alpha}_{2}^{PM} = \frac{K_{m}(y) I_{m}(x)}{K_{m}(x) I_{m}(y)} \approx \left(\frac{b^{(1)}}{b^{(3)}}\right)^{2}, \quad \bar{\alpha}_{2}^{INF} = 0$$

$x = kb^{(1)} / \gamma$, $y = kb^{(3)} / \gamma$, $\zeta = Z_{0}\sigma_{c} t$, and $m \neq 0$. 

4
**Thick wall**: Good for high frequencies. At \( r = b^{(2)} \), Case PC: perfectly conducting, Case PM: perfectly magnetic, and Case INF: \( b^{(2)} \to \infty \). For \( m \geq 1 \),

\[
\alpha_1 = \frac{1}{1 - 2i \beta^2} \left[ 1 - \frac{(1+i) \Delta Q_n}{2m \mu_\gamma^2} \right] \quad \beta = \frac{k^2 \delta_0^2}{2}, \quad \Delta = \mu_1 \beta k \delta_c,
\]

\[
Q_n = \frac{k b^{(1)}}{1 - \alpha^2} \frac{Q_2 - \alpha Q_2}{1 - \alpha}, \quad Q_n = \frac{k b^{(1)}}{1 - \alpha^2} \frac{Q_2 - \alpha Q_2}{1 - \alpha}, \quad Q_2 = \frac{K^{(1)}_n}{K^{(2)}_n}, \quad P_2 = \frac{I^{(2)}_m}{I^{(2)}_m}
\]

Boundary conditions require \( \alpha^2_{PM} = \eta^2_{PM} = \frac{K^{(2,3)}_m I^{(2)}_m}{I^{(2,3)}_m K^{(2)}_m} \), \( \eta^2_{PC} = \alpha^2_{PC} = \frac{K^{(2,3)}_m I^{(2)}_m}{I^{(2,3)}_m K^{(2)}_m} \).

\( \alpha^2_{INF} = \eta^2_{INF} = 0 \), with \( I^{(p+1)}_m = I_m(\nu(b^{(p)}) \nu(b^{(p)})) \) and similar definitions for \( I^c_m \), \( K_m \), and \( K'_m \).

**Electric- and magnetic-dipole approximation**: \( \alpha_1 \) can also be derived [14] by approximating beam dipole motion as a superposition of oscillating electric and magnetic dipoles.

<table>
<thead>
<tr>
<th>Description</th>
<th>Impedances</th>
<th>Wakes</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>High frequency optical model</strong>: [15] High frequency ( k \gg 1/h ), short-range ( -z \ll h ), transition length ( L \ll kh^2 ), ( h ) is minimum aperture. For tapered transition of angle ( \theta ), need ( k \gg 1/h \theta ).</td>
<td>( Z^\parallel ) and ( kZ^\perp ) are both constants similar for ( kZ_0^d ), ( kZ_0^q ), ( W_0^d ), ( W_0^q )</td>
<td>( W_\parallel = -Z^\parallel c\delta(z) ) ( W_\perp = -kZ^\perp cH(-z) )</td>
</tr>
</tbody>
</table>
| Transitions, shallow cavities, collimators, iris(es): (a) Axially symmetric examples: [16]-[18] (i) step-in transition (from \( d \) to \( b \)) \( Z_0^d = \frac{Z_0}{\pi} \ln \frac{d}{b} \), \( kZ_1^d = \frac{Z_0}{\pi} \left( \frac{1}{b^2} - \frac{1}{d^2} \right) \)
| (ii) step-out (from \( b \) to \( d \)), long collimator, shallow cavity with gap \( g \), (iii) thin iris \( b/k \ll 1 \) \( Z_0^q = \frac{Z_0}{\pi} \ln \frac{d}{b} \), \( kZ_1^q = \frac{Z_0}{\pi} \left( \frac{1}{b^2} - \frac{1}{d^2} \right) \) \( Z_0^d = \frac{Z_0}{\pi} \ln \frac{d}{b} \), \( kZ_1^d = \frac{Z_0}{\pi} \left( \frac{1}{b^2} - \frac{1}{d^2} \right) \) \( Z_0^q = \frac{Z_0}{\pi} \ln \frac{d}{b} \), \( kZ_1^q = \frac{Z_0}{\pi} \left( \frac{1}{b^2} - \frac{1}{d^2} \right) \) \( \) where \( b \) is small iris or pipe radius, \( d \) is large pipe radius. Note: for shallow cavity, waves reflect from outer wall \( \Rightarrow g \gtrsim k(d - b)^2 \); for collimator, bottom length \( \gg kb^2 \). |
| (b) 3D, mirror symmetric in \( x \), \( y \): [15] (i) flat step-out transition, aperture \( 2b \) to \( 2d \), (ii) any step-in transition; iris with small \( \text{(iii)} \) flat \( 2b \), (iv) elliptical (axes \( w \) by \( b \)), aperture \( kZ_y = \frac{\pi}{2\pi b^2} \), \( kZ_y = \frac{Z_0}{2\pi b^2} \), \( Z_y = Z_d = \frac{1}{2}Z_y \) |
| \( kZ_y = \frac{Z_0}{2\pi b^2} \), \( kZ_y = \frac{Z_0}{2\pi b^2} \), \( Z_y = Z_d = \frac{1}{2}Z_y \) |
| **High frequency diffraction formulæ**: \( k \gg 1/b \) (a) Deep cavity (Fresnel diffraction) \( [19, 1] \), cavity radius \( d \) and gap \( g \). | \( Z_m^d = \frac{\sqrt{2}Z_0^d}{(1 + \delta_m \pi)^\gamma} b^{2m+1} \pi \sqrt{\frac{ig}{k}} \), \( Z_m^d = \frac{2}{b^2k}Z_0^d \) | \( W_m' = \frac{\sqrt{2}Z_0^d c}{(1 + \delta_m \pi)^\gamma} b^{2m+1} \pi \sqrt{\frac{g}{k}} \) |
| Note: no reflections from outer wall \( \Rightarrow g \ll k(d - b)^2 \). | \( W_1' = \frac{2}{b^2} \int W_0 dz \) |
### Impedances

\[ \frac{Z_0}{L} = \frac{iz_0}{\pi \kappa b^2} \left[ 1 + \frac{\alpha(g/L)L}{b} \sqrt{\frac{2\pi i}{kg}} \right]^{-1} \]

\(\alpha(\zeta) \approx 1 - 0.465\sqrt{\zeta} - 0.070\zeta\)

\[ Z_1^\perp = \frac{2}{b^2k} Z_0^\parallel \]

\[ W_0' = \frac{Z_0 e^{\eta(z)^2}}{\pi b^2} \text{erfc}[\eta(z)] \]

\[ \eta(z) = \frac{\alpha L}{b} \sqrt{\frac{2\pi z}{g}} \]

\[ W_1 = \frac{2}{b^2} \int W_0' \, dz \]

### Numerical fit:

[24, 25] valid over larger \(z\) range: \(-z/L \leq 0.15, 0.34 \leq b/L \leq 0.69, 0.54 \leq g/L \leq 0.89\).

### Bethe’s dipole moments

Electric and magnetic dipole moments when wavelength \(\gg a\):

\[ \vec{d} = \frac{2Ze_0}{3} \vec{E}, \quad \vec{m} = -\frac{4}{3\mu_0} a^3 \vec{B} \]

\(\vec{E}\) and \(\vec{B}\) are electric and magnetic flux density at hole when hole is absent. This is a diffraction solution for a thin-wall pipe.

### Small 3D obstacle

on beam pipe: [27, 28] size \(\ll b\), low freq. \(k \ll 1/\text{(size)}\); \(\phi\) azimuthal angular position of object.

### Elliptical hole:

major and minor radii are \(a\) and \(d\). \(K(m)\) and \(E(m)\) are complete elliptical functions of the first and second kind, with \(m=1-m_1\) and \(m_1=(d/a)^2\). For long ellipse perpendicular to beam, major axis \(a \ll b\), beam pipe radius, because the curvature of the beam pipe has been neglected here [29].

\[ Z_0^\parallel = -i k e \mathcal{L}, \quad Z_1^\perp = \frac{4}{b^2} Z_0^\parallel \cos \phi \]

\[ W_0' = -e^2 \mathcal{L} \delta'(z) \]

\[ W_1 = \frac{4}{b^2} \int W_0' \cos \phi \, dz \]

**Inductance** \(\mathcal{L} = \frac{Z_0(\alpha_e + \alpha_m)}{4\pi^2 b^2 c}\)

\(\alpha_e\) is electric polarizability, \(\alpha_m\) magnetic susceptibility

**Circular hole** \(a \ll b\)

\[ \alpha_e + \alpha_m = \begin{cases} \frac{\pi a^3 m^2}{3} \left[ K(m) - E(m) \right] & \text{m = beam} \\ \frac{3E(m)}{3K(m)} - E(m) - m_1 K(m) & \text{long ellipse} \end{cases} \]

\[ \pi d^4 \left[ \ln(4a/d) - 1 \right] \quad a \ll b \]

Above are for \(t \ll a\). When \(t \geq a, \times 0.56\) when hole is circular and \(0.59\) when hole is long-elliptic.

For higher frequency correction, add to \(\alpha_e + \alpha_m\) the extra term,

\[ + \frac{2\pi a^3}{3} \left[ 11k^2a^2 \right] \text{circular,} \]

\[ + \frac{2\pi a^3}{3} \left[ 2k^2a^2 \right] \text{long ellipse} \]

**Rectangular slot:** length \(L\), width \(w\).

\[ \alpha_e + \alpha_m = w^3 (0.1814 - 0.0344w/L) \quad t \ll a, \quad \times 0.56 \text{ when } t \geq a \]

**Rounded-end slot:** length \(L\), width \(w\).

\[ \alpha_e + \alpha_m = w^3 (0.1334 - 0.0500w/L) \quad t \ll a, \quad \times 0.59 \text{ when } t \geq a \]
<table>
<thead>
<tr>
<th>Description</th>
<th>Impedances</th>
<th>wake</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annular-ring-shaped cut: inner and outer radii $a$ and $d = a + w$ with $w \ll d$.</td>
<td>$\alpha_c + \alpha_m = \frac{\pi^2 d^2 a}{2 \ln(32d/w) - 4} - \frac{\pi^2 w^2 (a + d)}{16}$</td>
<td>$t \ll d$</td>
</tr>
<tr>
<td>Half ellipsoidal protrusion with semi axes $h$ radially, $a$ longitudinally, and $d$ azimuthally. $2F_1$ is the hypergeometric function.</td>
<td>$\alpha_c + \alpha_m = 2\pi ah d \left[ \frac{1}{I_b} + \frac{1}{I_c - 3} \right]$&lt;br&gt;$I_b = 2F_1\left(1, 1; \frac{3}{2}; 1 - \frac{L^2}{h^2} \right)$, $I_c = 2F_1\left(1, 1; \frac{5}{2}; 1 - \frac{L^2}{h^2} \right)$, if $a = d$</td>
<td>$\alpha_c + \alpha_m = \pi a^3 \frac{8h^3}{3} \left[ 1 + \left( \frac{4}{\pi} - \frac{\alpha}{h} \right) \frac{a}{h} \right]$ if $a = d \ll h$&lt;br&gt;$\alpha_c + \alpha_m = \pi a^3 \frac{8\pi h^4}{3a} \left[ \frac{\ln \left( \frac{2a}{h} - 1 \right)}{1} \right]$ if $a \gg h = d$</td>
</tr>
<tr>
<td>Small inductive objects-2D: [27, 30] small cavities, shallow irises, and transitions at low freq. ($h \ll b, k \ll 1/h$); $h$ is height of object, $g$ is gap of cavity or length of iris; $L$ is inductance. For tapered transition pair: $\theta$ is taper angle.</td>
<td>$Z_0^\parallel = -ik\mathcal{L}$, $Z_1^\parallel = \frac{2}{b^2k}Z_0^\parallel$</td>
<td>$W_0^\parallel = -c^2\mathcal{L}\delta'(z)$, $W_1 = \frac{2}{b^2} \int W_0^\parallel dz$</td>
</tr>
<tr>
<td>Wall roughness inductive model: [35] 1-D axisymmetric bump on beam pipe, $h(z)$ or 2-D bump $h(z, \theta)$. Valid for low frequency $k \ll \text{bump length or width})^{-1}$, $h \ll b$, and $</td>
<td>\nabla h</td>
<td>\ll 1$. See also [36]</td>
</tr>
</tbody>
</table>
| Small periodic corrugations: (a) [31, 32] $L \lesssim h \ll b$, $k \ll 1/h$; $L$ period, $h$ depth, $g$ gap, $\varphi$ principal value; $\beta_g$ group velocity. | $Z_0^\parallel = \frac{Z_0}{\pi b^2} \left[ \pi k_r \delta(k^2 - k_r^2) + i \varphi \left( \frac{k}{k^2 - k_r^2} \right) \right]$, $\frac{W_0^\parallel}{L} = \frac{Z_0 c}{\pi b^2} \cos k_r z$
$$Z_1^\parallel = \frac{2}{b^2k}Z_0^\parallel, \quad k_r = \sqrt{\frac{2L}{bg}} \times (1 - \beta_g) = \frac{4h\theta}{bL}, \quad W_1 = \frac{2}{b^2} \int W_0^\parallel dz$$ | $Z_0^\parallel = \frac{Z_0 h^{3/2}k_{\perp}^{3/2}}{8\pi b}(-ik)^{1/2}$<br>$\frac{W_0^\parallel}{L} = -\frac{Z_0 c h^2 k_{\perp}^3}{16\pi^{3/2} b (-k_{\perp} z)^{3/2}}$ |
<p>| (b) [33] $L \gg h$, $L \ll b$, $k \ll 1/h$; $k_L = 2\pi/L$. | | |</p>
<table>
<thead>
<tr>
<th>Description</th>
<th>Impedances</th>
<th>Wakes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thin dielectric or ferrite layer on pipe: [34] thickness $h \ll b$.</td>
<td>Like small periodic corrugations (a), but $k_r = \left[ \frac{2\epsilon_r}{(\epsilon_r\mu_r - 1)bh} \right]^{1/2}$, with relative dielectric constant $\epsilon_r$ and magnetic permeability $\mu_r$.</td>
<td></td>
</tr>
<tr>
<td>Coherent synchrotron radiation (CSR): [37]-[39]</td>
<td>$Z_0^\parallel = \frac{Z_0}{L} = \frac{Z_0}{2 \cdot 3^{1/3} \pi} \Gamma \left( \frac{2}{3} \right) \left[ \frac{ik\gamma}{R^2} \right]^{1/3}$</td>
<td>$W_0^\prime = -\frac{Z_0 c}{2 \cdot 3^{4/3} \pi R^{2/3}} \frac{1}{z^{4/3}}$</td>
</tr>
<tr>
<td>Bunch moves in free space on a circle of radius $R$; $k \ll \gamma^3/R$. See Sec. 23.</td>
<td>$\Gamma(2/3) \approx 1.3541$. Note: non-zero wake for test particle ahead of driving particle. $W_0^\prime(0^+)/L \approx 0.1 Z_0 c k^4/R^2$. This is also used to approximate effect at high $k$ for beam in beam pipe; shielded (suppressed) for $k \lesssim R^{1/2} b^{-3/2}$.</td>
<td></td>
</tr>
<tr>
<td>Round collimator: (a) [40] low frequency $k \ll 1/d$.</td>
<td>$Z_1^\perp = -0.3i \frac{Z_0}{d}$ collimator radius $d \ll b$.</td>
<td>$W_1 = -0.3 \frac{Z_0 c}{d} \delta(z)$ collimator radius $d \ll b$.</td>
</tr>
<tr>
<td>(b) High frequency $k \gg 1/d$; if tapered, angle $\theta \gg 1/(kd)$.</td>
<td>See optical model formulae (a) above</td>
<td></td>
</tr>
<tr>
<td>(c) [41] For any frequency, small angle, $d'(s) \ll 1$, $kdd' \ll 1$, with $d(s)$ pipe profile versus longitudinal position $s$, and $d'$ is derivative of $d$ with respect to $s$.</td>
<td>$Z_1^\parallel = -iZ_0 k \frac{1}{4\pi} \int ds \left( d' \right)^2$</td>
<td>$W_0^\prime = \frac{Z_0 c}{4\pi} \int ds \left( d' \right)^2 \delta'(z)$</td>
</tr>
<tr>
<td>$Z_1^\perp = -iZ_0 \frac{2\pi}{2\pi} \int ds \left( \frac{d'}{d} \right)^2$</td>
<td>$W_1 = -\frac{Z_0 c}{2\pi} \int ds \left( \frac{d'}{d} \right)^2 \delta(z)$</td>
<td></td>
</tr>
<tr>
<td>$\Rightarrow$ symm. tapers of angle $\theta \ll 1$:</td>
<td>$Z_1^\perp = -iZ_0 \frac{1}{\pi} \theta \left( \frac{1}{d} - \frac{1}{b} \right)$</td>
<td>$W_1 = -\frac{Z_0 c}{\pi} \theta \left( \frac{1}{d} - \frac{1}{b} \right) \delta(z)$</td>
</tr>
<tr>
<td>Flat collimator: [42] low frequency, small angle, $h'(s) \ll 1$, $h \ll w \ll \ell$, with $h(s)$ vertical profile, $w$ width, $\ell$ length</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Z_y = -iZ_0 w \frac{1}{4} \int ds \frac{(h')^2}{h^3}$</td>
<td>$W_y = -\frac{Z_0 c w}{4} \int ds \frac{(h')^2}{h^3} \delta(z)$</td>
<td></td>
</tr>
<tr>
<td>Pill-box cavity — low frequency: [43] cavity radius $d$, gap $g$; $S = d/b$. When $g \gg 2(d-b)$, replace $g$ by $d-b$. Valid for $k \ll 1/d$.</td>
<td>$Z_0^\parallel = -i k \frac{Z_0 g}{2\pi} \ln S$</td>
<td>$W_0^\prime = -\frac{Z_0 c g}{2\pi} \ln S \delta'(z)$</td>
</tr>
<tr>
<td>$Z_0^\perp = -iZ_0 g \frac{S^2 - 1}{\pi b^2 S^2 + 1}$</td>
<td>$W_1 = -\frac{Z_0 c g}{\pi b^2} \frac{S^2 - 1}{S^2 + 1} \delta(z)$</td>
<td></td>
</tr>
<tr>
<td>Resonator model: [1] for $m$-th azimuthal mode, with shunt impedance $R_s^{(m)}$, quality factor $Q$, and resonant frequency $k_r$.</td>
<td>$Z_{m}^\parallel = \frac{R_s^{(m)}}{1 + iQ (k_r / k - k/k_r)}$</td>
<td>$W_{m} = \frac{R_s^{(m)} c k_r}{Q k_r} e^\alpha z \sin k_r z$</td>
</tr>
<tr>
<td>$Z_{m}^\perp = \frac{R_s^{(m)} / k}{1 + iQ (k_r / k - k/k_r)}$</td>
<td>$\alpha = k_r / (2Q)$</td>
<td>$k_r = \sqrt[k_r^2 - \alpha^2]$</td>
</tr>
<tr>
<td>Valid only close to $k_r$. As $k \to \infty$, $Z_{m}^\parallel \to k^{-1/2}$ for non-periodic cavities and $- k^{-3/2}$ for an infinite array of cavities. [16, 46]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Closed pill-box cavity: [44] resonant frequencies

The impedance and “circuit” parameters $k_{mnp}$ and $(R_s/Q)_{mnp}$ [45], where $m$, $n$, $p$, are azimuthal, radial, longitudinal mode numbers. Cavity radius $d$ and length $g$, $x_{mn}$ is $n^{\text{th}}$ zero of Bessel function $J_m$.

\[
\begin{align*}
\left[\frac{R_s}{Q}\right]_{0np} &= \frac{Z_0}{x_{2n}^2} \left\{ \frac{\sin^2 gk_{0np}}{2\beta} \times \frac{1}{1 + \delta_{0p}} \right\} \quad \text{p even} \\
&= \frac{X_0}{J_1^2(x_{1n})} \left\{ \frac{\sin^2 gk_{1np}}{2\beta} \times \frac{1}{1 + \delta_{0p}} \right\} \quad \text{p odd}
\end{align*}
\]

### Curvature impedance: [47] Smooth toroidal chamber of rectangular cross section, width $b-a$, height $h$, inner radius $a$, outer radius $b$, and $R = \frac{1}{2}(a+b)$. As Lorentz factor $\gamma \to \infty$, a contribution remains.

Valid from zero frequency up to just below synchronous resonant modes, i.e., $0 < \nu < \sqrt{R/h}$ with $\nu = kh$.

\[
Z_0^{\parallel} = \frac{ikZ_0h^2}{\pi^2 R} \left\{ 1 - e^{-2\pi(b-R)/h} - e^{-2\pi(R-a)/h} \right\} \left[ 1 - 3 \left(\frac{\nu}{\pi}\right)^2 \right]
\]

\[
+ 0.05179 - 0.01355 \left(\frac{\nu}{\pi}\right)^2 + \rho k R
\]

\[
\approx \frac{ikZ_0h^2}{\pi^2 R} \left[ A - 3B \left(\frac{\nu}{\pi}\right)^2 \right].
\]

where $\rho$ is quadratic in $\nu$. As $(b-a)/h$ increases, $\rho$ vanishes exponentially and $A \approx B \approx 1$. In general, $A/B \approx 1$ implying $\text{Im} Z_0^{\parallel}$ changes sign (a node) near $\nu = \pi/\sqrt{3}$.

### Kicker with window-frame magnet: [49]

With window-frame magnet width $a$, height $b$, length $L$, beam offset $x_0$ horizontally, and all image current carried by conducting current plates.

\[
Z_0^{||} = \frac{k^2 \mu_0^2 L^2 x_0^2}{4a^2 Z_k}
\]

\[
Z_1^+ = \frac{kc^2 \mu_0^2 L^2}{4a^2 Z_k}
\]

\[
W_0' = -\frac{e^3 \mu_0^2 L^2 x_0^2}{4a^2 Z_k} \delta'(z)
\]

\[
W_1 = -\frac{e^3 \mu_0^2 L^2}{4a^2 Z_k} \delta'(z)
\]

\[
Z_k = -ikcL + Z_g \quad \text{with } L \approx \mu_0 bL/a \text{ the inductance of the windings and } Z_g \text{ the impedance of the generator and the cable. If the kicker is of C-type magnet, } x_0 \text{ in } Z_0^{||} \text{ should be replaced by } (x_0 + b).
\]

### Traveling-wave kicker [49]

With characteristic impedance $Z_c$ for the cable, and a window magnet of width $a$, height $b$, and length $L$. Valid for frequency below cutoff.

\[
\begin{align*}
Z_0^{||} &= \frac{Z_c}{4} \left[ 2 \sin^2 \theta - i \sin \theta \right] \\
Z_1^+ &= \frac{Z_c L}{4ab} \left[ 1 - \cos \theta - i \sin \theta \right]
\end{align*}
\]

\[
\begin{align*}
W_0' &= \frac{Z_c}{4} \left[ \delta(z) - \delta \left( z + \frac{L}{\beta_{ph}} \right) \right] \\
W_1 &= \frac{Z_c \beta_c}{4ab} \left[ H(z) - H \left( z + \frac{L}{\beta_{ph}} \right) \right]
\end{align*}
\]

$\theta = kL/\beta_{ph}$ denotes the electrical length of the kicker windings and $\beta_{ph} = Z_c ac/(Z_0 b)$ is the matched transmission-line phase velocity of the capacitance-loaded windings. Here, $\beta_{ph} \ll \beta \to 1$, the beam velocity.
### Wakes for a Gaussian Bunch:

The bunch wakes of a bunch with longitudinal charge distribution $\lambda_z$, are given by $W'_m(z) = \int_{-\infty}^0 W'_m(x) \lambda_z(z-x) \, dx$, $W_m(z) = \int_{-\infty}^0 W_m(x) \lambda_z(z-x) \, dx$. In the following we give bunch wakes of a Gaussian bunch $[\lambda_z = e^{-(z/\sigma_z)^2/2}/(\sqrt{2\pi}\sigma_z)]$, with $\sigma_z$ the rms bunch length] for wakefield forms found in the tables above, and also give their first moments $\langle W \rangle = \int_{-\infty}^\infty W(z) \lambda_z(z) \, dz$ and the rms $W_{\text{rms}} = \sqrt{\langle W^2 \rangle - \langle W \rangle^2}$. Here the z dependence alone is considered and the wake coefficient is scaled out; for a specific problem, the appropriate coefficients, found in the tables above, need to be included at the end.

Note: for power law wakes with $-2 < \alpha < -1$, $W$ is obtained using integration by parts [38]. It is assumed that in the range $|z| \ll \sigma_z$ the wake form changes so that $\int_{-\infty}^\infty W(z) \, dz = 0$. Consequently, $W$ can be obtained without knowing the details of $W$ at very short range.

<table>
<thead>
<tr>
<th>Description</th>
<th>Impedances</th>
<th>Wakes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strip-line BPMs (pair): [48] length $L$, angle each subtending to pipe axis $\phi_0$, forming transmission lines of characteristic impedance $Z_c$ with pipe.</td>
<td>$Z_0^\parallel = 2Z_c \left[ \frac{\phi_0}{2\pi} \right]^2 \left[ 2 \sin^2 kL - i \sin 2kL \right]$</td>
<td>$W'_0 = 2Z_c \left[ \frac{\phi_0}{2\pi} \right]^2 \left[ \delta(z) - \delta(z+2L) \right]$</td>
</tr>
<tr>
<td></td>
<td>$Z_0^\perp = \left[ \frac{Z_0}{k} \right]_{\text{pair}} \frac{1}{\pi b^2} \left[ \frac{4}{\phi_0} \right]^2 \sin^2 \left( \frac{\phi_0}{2} \right)$</td>
<td>$W_1 = \frac{8Z_c}{\pi^2 b^2} \sin^2 \left( \frac{\phi_0}{2} \right) \left[ H(z) - H(z+2L) \right]$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>The strip-lines are assumed to terminate with impedance $Z_c$ at the upstream end.</td>
</tr>
</tbody>
</table>

#### Wakeform, $W$

<table>
<thead>
<tr>
<th>Wakeform, $W$</th>
<th>Bunch wake, $W$</th>
<th>$\langle W \rangle$</th>
<th>$W_{\text{rms}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Circuit Models:</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Resistive: $\delta(z)$</td>
<td>$\frac{1}{\sqrt{2\pi}\sigma_z} e^{-(z/\sigma_z)^2/2}$</td>
<td>$\frac{1}{2\sqrt{\pi}\sigma_z}$</td>
<td>$\frac{\sigma_z}{2}$</td>
</tr>
<tr>
<td>Inductive: $\delta'(z)$</td>
<td>$\frac{z}{\sqrt{2\pi}\sigma_z^2} e^{-(z/\sigma_z)^2/2}$</td>
<td>$\frac{1}{\sqrt{2\pi}\sigma_z}$</td>
<td>0</td>
</tr>
<tr>
<td>Capacitive: $H(z)$</td>
<td>$\frac{1}{\sqrt{2\pi}\sigma_z^2} \left[ 1 + \text{erf} \left( -\frac{z}{\sqrt{2}\sigma_z} \right) \right]$</td>
<td>$\frac{1}{\sqrt{2\pi}}$</td>
<td>$\frac{1}{2\sqrt{3}}$</td>
</tr>
<tr>
<td><strong>Power Law: $(-z)^\alpha$</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low freq, resistive wall ($W'_m$) and Fresnel diffraction ($W'_m$): $\alpha = -\frac{1}{2}$</td>
<td>$f(-z/\sigma_z)\sigma_z^\alpha$, with $f(x)$ given by (upper/lower sign for $x \geq 0$):</td>
<td>0.723</td>
<td>0.292</td>
</tr>
<tr>
<td></td>
<td>$\sqrt{\frac{\pi</td>
<td>x</td>
<td>}{8}} e^{-x^2/4} \left[ I_{-\frac{1}{4}} \pm I_{\frac{1}{4}} \right]_{x^2/4}$</td>
</tr>
<tr>
<td>Fresnel diffraction ($W_m$): $\alpha = \frac{1}{2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\sqrt{\frac{\pi}{32}} \int_{-\infty}^x</td>
<td>y</td>
<td>^{1/2} e^{-y^2/4} \left[ I_{-\frac{1}{4}} \pm I_{\frac{1}{4}} \right]_{y^2/4} , dy$</td>
</tr>
<tr>
<td>Low freq, resistive wall ($W'_m$) and small periodic corrugations ($W'_m$): [50] $\alpha = -\frac{3}{2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\sqrt{\frac{\pi</td>
<td>x</td>
<td>^3}{8}} e^{-x^2/4} \left[ I_{1/4} - I_{-3/4} \pm I_{-1/4} + I_{3/4} \right]_{x^2/4}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{\sigma_z^{3/2}}{\sigma_z^{3/2}}$</td>
<td></td>
</tr>
<tr>
<td>CSR ($W'_0$): $z^\alpha$ with $\alpha = -\frac{1}{3}$ (note: $z &gt; 0$)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$-\frac{3}{\sqrt{2\pi}} \int_0^\infty (x+y) e^{-(x+y)^2/4} , dy$</td>
<td>-0.758</td>
<td>0.532</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{\sigma_z^{4/3}}{\sigma_z^{4/3}}$</td>
<td></td>
</tr>
<tr>
<td><strong>Resonator Model:</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$W = f(-z/\sigma_z)$, with $f(x) = \frac{1}{2} e^{-(k_r^2-\alpha_r^2)\sigma_z^2/2-\alpha_r \sigma_z x}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\times \left{ \begin{array}{ll} \text{Im} &amp; e^{ik_r \sigma_z (x-\alpha_r \sigma_z)} { 1 + \text{erf} \left( \frac{(ik_r - \alpha_r) \sigma_z + x}{\sqrt{2}} \right) } \ \text{Re} &amp; e^{-2k_r \alpha_r \sigma_z^2} { 1 + \text{erf} \left( (ik_r - \alpha_r) \sigma_z \right) } \end{array} \right}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>