FERMILAB-FN-0841-AD

Comments on the slip factor and the relation $\Delta\phi = -h\Delta\theta$

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(September, 2009)

Abstract

The definition of the slip factor can be obtained from the phase equation. However, a derivation using the relation $\Delta\phi = -h\Delta\theta$ leads to a different slip-factor definition. This apparent paradox is examined in detail and resolved. Here $\Delta\phi$ is the rf phase difference and $\Delta\theta$ is the azimuthal phase difference around the accelerator ring between an off-momentum particle and the synchronous particle, while h is the rf harmonic.

1 Introduction

The phase equation in the longitudinal phase space must be compatible with the definition of the slip factor η. However, through the relation $\Delta\phi = -h\Delta\theta$, the phase relation readily reduces to a different definition of η. Here, $\Delta\phi$ is the difference in rf phase advance between an off-momentum particle and the synchronous particle, $\Delta\theta$ is the difference in azimuthal phase along the accelerator ring between the two particles, while h is the rf harmonic. An accurate definition of the slip factor is important, especially when the particle energy is close to transition, or in an isochronous storage ring. To ensure isochronicity for particles of all momentum in the bunch, sextupoles and octupoles are often employed to cancel, respectively, the terms in the slip-factor expansion that are linear and quadratic in momentum spread. If sextupole and octupole currents are applied according to an incorrect definition of the slip factor, the anticipated cancellation may not materialize.

We show in Sec. 2 the concept of time advance and how the definition of the slip factor can be obtained from the phase equation. To understand why the commonly used relation $\Delta\phi = -h\Delta\theta$ leads us to a different definition, we first examine in Sec. 3 the definition of $\Delta\phi$, the so-called rf phase difference. We next study the physical meaning of the relation $\Delta\phi = -h\Delta\theta$ and discover that it is only approximate. An exact alternative relation is presented in Sec. 4. This alternative relation between the rf phase and azimuthal phase leads us to the correct definition of the slip factor.

2 Time Advance

At time $t = 0$, the synchronous particle and the off-momentum particle have just clear the single thin rf cavity in the ring. When the synchronous particle returns to the cavity after exactly one revolution, the time past is T_0^s and the rf phase has advanced by $\phi^s = 2\pi h$. At this moment, the off-momentum particle has passed the cavity already; it passes at time $t = T_0^s + \Delta T_0$, Here, we assume the off-momentum particle is ahead of the synchronous particle, and therefore $\Delta T_0 < 0$. We say that the off-momentum particle has a time advance of $-\Delta T_0$, because it will take the synchronous particle a time $-\Delta T_0$ to catch up to this present location of the off-momentum particle. Let the period of the off-momentum particle be T_0, then

$$T_0 = T_0^s + \Delta T_0. \tag{2.1}$$

In the situation of a positive time advance ($-\Delta T_0 > 0$), $T_0 < T_0^s$.

We define the slip factor as

$$\frac{\Delta T_0}{T_0^s} = \eta\delta, \tag{2.2}$$

where δ is the fractional momentum spread, or

$$\frac{\text{change in period}}{\text{syn. period}} = \eta\delta. \tag{2.3}$$

This translates readily into

$$\frac{\text{time advance}}{\text{syn. period}} = -\eta\delta, \tag{2.4}$$

which is the phase equation in the longitudinal space space. The sign on the right-hand side can be checked as follows. Above transition, $\eta > 0$. A particle with $\delta > 0$ will arrive slower. So the time advance is negative. The conclusion is that the phase equation of Eq. (2.4) is compatible to the definition of the slip factor of Eq. (2.2) or (2.3).

3 Phase Advance

When the synchronous particle crosses the cavity again, the off-momentum particle is ahead of it. But at this moment time past since the beginning when both particles were at the cavity is T_0^s, and the rf clock advances by the phase angle of $2\pi h$ exactly. Then how do we define rf phase advance? We can first monitor the time $T_0 = T_0^s + \Delta T_0$ ($\Delta T_0 < 0$) when the off-momentum particle crosses the cavity. The rf phase at that moment is $\phi = 2\pi h - (-\Delta T_0 \omega_{\rm rf})$. When the synchronous particle crosses the cavity at time T_0, the rf phase is $\phi^s = 2\pi h$. We can then say the the off-momentum particle is *ahead* of the synchronous particle by rf phase $\Delta\phi = \phi - \phi^s = \omega_{\rm rf}\Delta T_0$. This phase advance is related to the time advance by $-\Delta T_0 = -\Delta\phi/\omega_{\rm rf}$, and the phase advance is negative.

The best description is not to use the time t as the independent variable. Instead use s, the distance along the closed orbit of the synchronous particle. So instead of talking about the off-momentum particle ahead of the synchronous particle by a certain amount in distance at the moment of the synchronous particle crosses the cavity (description 1), we monitor everything at the cavity and talk about the synchronous particle crosses it at time T_0^s and the off-momentum particle crosses it at time $T_0 = T_0^s + \Delta T_0$ (description 2). The first description is called the snapshot view, which records the two particles at different

places but at the same time T_0^s, similar to taking a photograph from above. The second description is to record the two particles crossing the same location in different times. It will be easier to talk about phase advance in this second description, because the the rf phase ϕ will accumulate a phase larger than $2\pi h$ if the off-momentum particle passes through the cavity later than the synchronous particle. The phase advance $\Delta\phi = \phi - \phi^s > 0$ here means longer phase accumulated by the rf clock or the particle passes the cavity at a later time than synchronous particle, when the off-momentum particle is behind the synchronous particle. In this case, we define time advance by $-\Delta T_0 = -\Delta\phi/\omega_{\rm rf}$.

4 The Relation $\Delta\phi = -h\Delta\theta$

Let θ be the azimuthal angle an off-momentum particle covered around the accelerator ring, and ϕ be the phase accumulated in the rf clock. Sometimes, one writes

$$\phi = -h\theta, \tag{4.5}$$

and with a superscript s,

$$\phi^s = -h\theta^s, \tag{4.6}$$

for the synchronous particle. The difference of the two gives

$$\Delta\phi = -h\Delta\theta, \tag{4.7}$$

where $\Delta\phi = \phi - \phi^s$ and $\Delta\theta = \theta - \theta^s$. But what is the physical meaning of these equations?

4.1 Positions of 2 particles at specific moment

First, we compare the off-momentum and synchronous particle at some particular time t, the snapshot description. Since there is only a unique reading of the rf clock at one moment, we must have $\Delta\phi = 0$. Second, the two particles are not at the same azimuthal angle around the ring and therefore $\Delta\theta \neq 0$. Thus, the relation (4.7) cannot be satisfied at all. The problem comes about because there is always one ϕ for the rf clock in this interpretation, or we always have $\phi = \phi^s$. In this sense Eq. (4.5) is totally meaningless and we have only Eq. (4.6). In fact, the negative sign in Eq. (4.6) for the synchronous particle is also completely arbitrary in this interpretation, because we can choose the accumulation of the azimuthal phase in the same

direction of accumulation of the rf phase (hence positive sign) or in the opposite direction (hence negative sign). For this reason, Eq. (4.7) definitely does not imply the comparison of the two particles at any particular time.

4.2 Phases of two particles at same location

We next try an alternate interpretation. Assume the on-momentum and off-momentum particles leave the rf cavity at the same moment. Then one turn later when the synchronous particle crosses the cavity again, the off-momentum is at an azimuthal angle $\Delta\theta$ in front of the synchronous particle. This implies that the off-momentum particle crosses the rf cavity again at an earlier time than the synchronous particle. Then if the off-momentum particle is pushed back an azimuthal phase angle $\Delta\theta$ to the moment it crosses the cavity the second time, the rf phase accumulated at that moment is $\Delta\phi \approx -h\Delta\theta$. The approximation sign indicates that $\Delta\phi$ may not be exactly equal to $-h\Delta\theta$. Let us examine the possible deviation. The push-back time is $\Delta t = \Delta\theta/\omega_0$, where ω_0 is the angular frequency of the off-momentum particle after passing the cavity the second time. Then we will have $\Delta\phi = -\omega_{\rm rf}\Delta t = -\omega_{\rm rf}\Delta\theta/\omega_0$. However, $\omega_{\rm rf}/\omega_0^s = h$, where ω_0^s is the revolution frequency of the synchronous particle, and $\omega_{\rm rf}/\omega_0 \neq h$.

Before continuing the discussion, a comment on ω_0^s is necessary. Note that ω_0^s, which is defined by $\omega_0^s = v^s/R^s$, where v^s is the velocity of the synchronous particle and $2\pi R^s$ is the length of the closed orbit of the synchronous particle. This definition is important because ω_0^s is not the actual angular change in azimuthal angle per unit time measured from some center of the accelerator ring or it is not the change in angular direction per unit time of a radius vector sweeping along the closed orbit, because this latter definition makes ω_0^s not a constant around the accelerator ring; the ring is not a perfect circle but consists of dipole arcs and straight sections. The definition $\omega_0^s = v^s/R^s$ makes ω_0^s a constant in one revolution around the ring between two successive passages of the rf cavity (if there is only one thin cavity). In the same way, the azimuthal angle θ along the closed orbit of the synchronous particle is not really the phase angle of some radius vector, it is simply defined by $\theta = s/R^s$, with s being the distance along the closed orbit measured from some reference point. The same applies to the revolution frequency ω_0 of the off-momentum particle, with the definition $\omega_0 = v/R$, where v is the velocity of the particle and $2\pi R$ is circumference of the close orbit of that particle.

Although the revolution frequency ω_0 of the off-momentum particle does change after passing through the rf cavity due to the change in energy. In above, when we push the off-momentum particle back by $\Delta\theta$, the azimuthal in advance of the synchronous particle at the moment the synchronous particle passes the cavity the second time, to the moment the off-momentum particle passes the cavity the second time, the push-back time $\Delta t = \Delta\theta/\omega_0$ is computed with the ω_0 of the off-momentum particle between the second and third passages of the cavity. But independent of which ω_0 is used, it is not the same as ω_0^s, and therefore, we do not have exactly the relation $\Delta\phi = -h\Delta\theta$. Thus according to this second interpretation, $\Delta\phi = -h\Delta\theta$ cannot be established. Instead we have

$$\Delta\phi = -\frac{\Delta\theta}{\omega_0}\omega_{\rm rf} = -\Delta\theta\frac{\omega_{\rm rf}}{\omega_0^s}\frac{\omega_0^s}{\omega_0} = -h\Delta\theta\frac{\omega_0^s}{\omega_0}. \tag{4.8}$$

Also we have to remember that while ω_0^s is the angular frequency of the synchronous particle between first and second passages of the rf cavity, ω_0 is the angular frequency of the off-momentum particle between second and third passages of the rf cavity. This is not really a problem, because the rf frequency $\omega_{\rm rf}$ is usually the same turn after turn throughout the acceleration cycle, and so will be $\omega_0^s = \omega_{\rm rf}/h$, the angular frequency of the synchronous particle. Thus we just say ω_0/ω_0^s is the ratio of the angular frequency of the off-momentum particle and the synchronous particle in the next revolution around the ring.

It is educational that Eq. (4.8) can be checked easily. If we divide throughout by Δt, the push back time, we get for the left side $\Delta\phi/\Delta t = -\omega_{\rm rf} = -h\omega_0^s$ (the rf phase would have advanced by $-\Delta\phi = \omega_{\rm rf}\Delta t$ in time Δt), which is equal to the right side since $\Delta\theta/\Delta t = \omega_0$. However, if we divide both sides by $T_0^s = 2\pi/\omega_0^s$, the period of the synchronous particle, we will get something completely different, as will be shown in the next section.

4.3 Definition of η

We wish to answer the question: If the phase equation of motion is written as

$$\frac{d\Delta\phi}{dt} = \omega_{\rm rf}\eta\delta \quad \text{or} \quad \frac{d\Delta\phi}{dt} = h\omega_0^s\eta\delta, \tag{4.9}$$

what is the definition of η. Here $\Delta\phi$ is the phase angle of the rf clock at the moment an off-momentum particle passes the rf cavity in advance of the rf phase angle when the synchronous particle passes the cavity. The positive signs on the right come about as follows. Above transition $\eta > 0$ and $\delta > 0$, an off-momentum particle will arrive later and later than

synchronous particle turn after turn, so the rf clock needs to continue accumulate its phase until the off-momentum particle reaches the rf cavity again, and $\Delta\phi$ should therefore be positive.

If one sticks to the incorrect relation

$$\Delta\phi = -h\Delta\theta, \tag{4.10}$$

after dividing by T_0^s the revolution period of the synchronous particle, one gets

$$\frac{d\Delta\phi}{dt} = -h\Delta\omega_0 = \omega_{\rm rf}\eta\delta, \tag{4.11}$$

where $\Delta\phi = -h\Delta\theta$ has been used in the first step and the phase equation of (4.9) has been used in the last step. Or

$$\Delta\omega_0 = -\frac{1}{h}\omega_{\rm rf}\eta\delta = -\omega_0^s\eta\delta. \tag{4.12}$$

We therefore have the definition for η,

$$\frac{\omega_0 - \omega_0^s}{\omega_0^s} = -\eta\delta, \tag{4.13}$$

which is obviously different from Eq. (2.2).

We next start with the correct relation

$$\Delta\phi = -h\Delta\theta\frac{\omega_0^s}{\omega_0}. \tag{4.14}$$

Division by T_0^s leads to

$$\frac{d\Delta\phi}{dt} = -h\Delta\omega_0\frac{\omega_0^s}{\omega_0} = \omega_{\rm rf}\eta\delta. \tag{4.15}$$

We have

$$\frac{\Delta\omega_0}{\omega_0} = -\eta\delta \tag{4.16}$$

This becomes

$$\frac{\omega_0 - \omega_0^s}{\omega_0} = -\eta\delta, \tag{4.17}$$

which is different from Eq. (4.13) by having ω_0, the angular frequency of the off-momentum particle, in the denomination rather than ω_0^s, the angular frequency of the synchronous particle. Dividing the numerator and denominator of the left side by $\omega_0\omega_0^s$, we obtain

$$\frac{\frac{1}{\omega_0^s} - \frac{1}{\omega_0}}{\frac{1}{\omega_0^s}} = -\eta\delta, \tag{4.18}$$

or

$$\frac{T_0^s - T_0}{T_0^s} = -\eta\delta, \tag{4.19}$$

which is the same as Eq. (2.3).

The revolution period T_0 of the off-momentum particle can be expressed as

$$T_0 = \frac{C}{\beta c}\ , \tag{4.20}$$

where C is the closed orbit of the off-momentum particle and βc denotes its velocity. We can easily expand T_0 as a Taylor series in δ, from which each higher-order of the slip factor can be identified. For example, we have

$$\begin{aligned}
\frac{T_0^{s\prime}}{T_0^s} &= \frac{C_s'}{C_s} - \frac{\beta_s'}{\beta_s}\ , \\
\frac{T_0^{s\prime\prime}}{T_0^s} &= \frac{C_s''}{C_s} - \frac{2\beta_s' C_s'}{\beta_s C_s} - \frac{\beta_s''}{\beta_s} + \frac{2{\beta_s'}^2}{\beta_s^2}\ , \\
\frac{T_0^{s\prime\prime\prime}}{T_0^s} &= \frac{C_s'''}{C_s} - \frac{3\beta_s' C_s''}{\beta_s C_s} - \frac{3\beta_s'' C_s'}{\beta_s C_s} + \frac{6{\beta_s'}^2 C_s'}{\beta_s^2 C_s} - \frac{\beta_s'''}{\beta_s} + \frac{6\beta_s'\beta_s''}{\beta_s^2} - \frac{6{\beta_s'}^3}{\beta_s^3}\ ,
\end{aligned} \tag{4.21}$$

where the *prime* denotes differentiation with respect to δ and all variables are evaluated at the synchronous particle, which explains why all the variables above carry the subscriptions or superscripts s. The derivatives of C can be read off easily from the expansion

$$C(\delta) = C_s\big[1 + \alpha_0\delta(1 + \alpha_1\delta + \alpha_2\delta^2 + \cdots\ )\big], \tag{4.22}$$

where α_0, $\alpha_0\alpha_1$, $\alpha_0\alpha_2$, $\cdots$ are different orders of the momentum compaction factor of the accelerator ring. The derivatives of β can be computed straightforwardly. They are:

$$\begin{aligned}
\frac{\beta_s'}{\beta_s} &= \frac{1}{\gamma_s^2}, \\
\frac{\beta_s''}{\beta_s} &= -\frac{3\beta_s^2}{\gamma_s^2}, \\
\frac{\beta_s'''}{\beta_s} &= -\frac{3\beta_s^2(1 - 5\beta_s^2)}{\gamma_s^2}.
\end{aligned} \tag{4.23}$$

With the expansion of the slip factor

$$\eta = \eta_0 + \eta_1\delta + \eta_2\delta^2 + \cdots\ , \tag{4.24}$$

we obtain the expressions for the higher-order components of the slip factor:*

$$\begin{aligned}
\eta_0 &= \alpha_0 - \frac{1}{\gamma_s^2}, \\
\eta_1 &= \alpha_0\alpha_1 + \frac{3\beta_s^2}{2\gamma_s^2} - \frac{\eta_0}{\gamma_s^2}, \\
\eta_2 &= \alpha_0\alpha_2 - \frac{\eta_1}{\gamma_s^2} + \frac{(1-5\beta_s^2)\beta_s^2}{2\gamma_s^2} + \frac{3\beta_s^2\eta_0}{2\gamma_s^2}.
\end{aligned} \tag{4.25}$$

On the other hand if the incorrect definition of η in Eq. (4.13) is used, we will be expanding $\omega_0 = \beta c/C$ in powers of δ. The result is the same for the lowest order component η_0. But the higher order components become

$$\begin{aligned}
\eta_1 &= \alpha_0\alpha_1 + \frac{3\beta_s^2}{2\gamma_s^2} - \frac{\eta_0}{\gamma_s^2} - \eta_0^2, \\
\eta_2 &= \alpha_0\alpha_2 - \eta_1\left(2\eta_0 + \frac{1}{\gamma_s^2}\right) + \frac{(1-5\beta_s^2)\beta_s^2}{2\gamma_s^2} + \frac{3\beta_s^2\eta_0}{2\gamma_s^2} - \frac{\eta_0^2}{\gamma_s^2} - \eta_0^3.
\end{aligned} \tag{4.26}$$

We wish to point out that although Eq. (4.26) is different from Eq. (4.25), however, they are exactly the same when $\eta_0 = \alpha_0 - \gamma_s^{-2} = 0$.

5 Conclusion

1. The correct definition of the slip factor is

$$\frac{\Delta T}{T_0^s} = -\eta\delta, \tag{5.27}$$

where ΔT is the time advance in arrival of the off-momentum particle and T_0^s is the period of the synchronous particle. It can also be written as

$$\frac{\Delta T_0}{T_0^s} = \eta\delta, \tag{5.28}$$

where $\Delta T_0 = T_0 - T_0^s$ is the difference in revolution period for the two particles.

*One must be careful about the factorials in the Taylor expansion. For example, here we have

$$\eta_1 = \frac{1}{2!}\frac{T_0^{s\prime}}{T_0^s}, \quad \eta_2 = \frac{1}{3!}\frac{T_0^{s\prime\prime}}{T_0^s}, \quad \cdots$$

2. The relation $\Delta\phi = -h\Delta\theta$ is only approximate. The accurate relation is given by

$$\Delta\phi = -h\Delta\theta\frac{\omega_0^s}{\omega_0}, \tag{5.29}$$

   where ω_0 and ω_0^s are, respectively, the revolution frequencies of, respectively, the off-momentum particle and synchronous particle.

3. The above relation, Eq. (5.29), cannot be discussed in the snapshot view, that is monitoring the off-momentum and synchronous particles at different locations but at the same time, like taking a photograph from above. The relation is valid only when the observation is made at a fixed location along the accelerator ring, monitoring the arrival of the off-momentum and synchronous particles at difference times.