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Quasi-3D Space Charge Simulation

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Introduction

The longitudinal space charge effect is simulated by binning the longitudinal beam profile in order to calculate the force on the bins using the binned particle distribution *via* FFT, and applying momentum kick based upon this space charge force to macro-particles. Usually, the longitudinal space charge kick is calculated once per turn since the longitudinal profile doesn't change much in a single turn. Besides, the longitudinal profile is used as a weighting factor for the transverse space charge force. The transverse space charge effect is simulated by projecting the 3-D beam to a 2-D Gaussian distribution in order to use the complex error function to compute the transverse space charge force, and applying this space charge force to macro-particles. One transverse space charge calculation per scale length of the beam shape variation requires at least ten transverse space charge force calculations per betatron oscillation.

Longitudinal Space Charge Effect

The Fermilab Booster is used as an example. Longitudinally, the beam is bunched into h identical bunches with N_0 particles each.[1] Each rf period, which contains one bunch, is partitioned into 2^N bins, and the line density is calculated at these bins, λ_i (i =1..2^N). Afterwards, the beam current is calculated using equation (1).

$$Ib_i = \beta \cdot c \cdot \lambda_i \qquad (1)$$

Here, β is the Lorentz relativistic factor, and c is the speed of light.

An FFT is done on the binned beam current Ib_i,[2] and the beam current at the n^{th} rf harmonic is obtained, IB_n. Here, n = 1..2^{N-1}. The empirical choice of the macro-particle number is 5000 and the bin number is 256 per rf period in order to have an average macro-particle number per bin ~20 and a time resolution of ~0.1ns/bin in case of a short bunch. The longitudinal space charge impedance at the n^{th} rf harmonic is calculated using equation (2).[1]

$$\frac{Z_n}{n} = -i \times h \times Z_0 \times g \Big/ \left(2 \times \beta \times \gamma^2\right) \tag{2}$$
$$Z_n = Z_\omega, \quad \omega = n \times h \times 2 \times \pi \times f_0$$
$$g = 1 + 2 \times \ln(b/a)$$

Here, h is the harmonic number, Z_0 = 377Ω is the free-space impedance, f_0 is the revolution frequency, b and a are the radius of the beam pipe and the beam. The voltage generated by the beam current due to the longitudinal space charge impedance is calculated using equation (3).

$$V_n = -Z_n \cdot IB_n \tag{3}$$

Finally, an inverse FFT is done on the longitudinal space charge induced voltage V_n; the result is Vsp_i, i=1.. 2^N, and the real part of of Vsp_i, represented by $VRsp_i$, is used to apply the momentum kick to a macro-particle based upon the longitudinal bin where the macro-particle is.

Transverse Space Charge Effect

A 2-D Gaussian distribution, as shown by equation (4), is used to represent the transverse beam profile in the Booster, and it's consistent with experimental observations.

$$\rho(x,y) = \frac{Q}{2\pi\sigma_x\sigma_y} \cdot e^{-(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2})} \tag{4}$$

For the proton beam, it's a good approximation that the longitudinal and transverse space charge forces are decoupled. So the longitudinal profile obtained from the longitudinal space charge calculation is used as a weighting factor for the transverse space charge force; other than that, we can neglect the longitudinal dimension in the transverse space charge calculation.

σ_x and σ_y are obtained using equation (5).

$$\sigma_x = \sqrt{\sum_{i=1}^{N_0} \left(x_i - \bar{x}\right)^2 \Big/ (N_0 - 1)}, \quad \sigma_y = \sqrt{\sum_{i=1}^{N_0} \left(y_i - \bar{y}\right)^2 \Big/ (N_0 - 1)} \tag{5}$$
$$\bar{x} = \sum_{i=1}^{N_0} x_i \Big/ N_0, \qquad \bar{y} = \sum_{i=1}^{N_0} y_i \Big/ N_0$$

The x and y components of the electric field from a 2-D Gaussian beam $\rho(x,y)$ are represented by E_x and E_y, and their analytical expressions were derived by M. Bassetti and G. Erskine in 1980,[3] except there is a sign error. The correct formulas for E_x and E_y are shown as equation (6).

$$Ex = \frac{Q}{2\varepsilon_0\sqrt{2\pi\left(\sigma_x^2-\sigma_y^2\right)}}\cdot \mathrm{Im}\left[W\left(\frac{x+iy}{\sqrt{2\pi\left(\sigma_x^2-\sigma_y^2\right)}}\right)-e^{-\left(\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2}\right)}\cdot W\left(\frac{x\cdot\frac{\sigma_y}{\sigma_x}+iy\cdot\frac{\sigma_x}{\sigma_y}}{\sqrt{2\pi\left(\sigma_x^2-\sigma_y^2\right)}}\right)\right]$$

$$Ey = \frac{Q}{2\varepsilon_0\sqrt{2\pi\left(\sigma_x^2-\sigma_y^2\right)}}\cdot \mathrm{Re}\left[W\left(\frac{x+iy}{\sqrt{2\pi\left(\sigma_x^2-\sigma_y^2\right)}}\right)-e^{-\left(\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2}\right)}\cdot W\left(\frac{x\cdot\frac{\sigma_y}{\sigma_x}+iy\cdot\frac{\sigma_x}{\sigma_y}}{\sqrt{2\pi\left(\sigma_x^2-\sigma_y^2\right)}}\right)\right]$$

(6)

Here, W(x) is the complex error function, as shown by equation (7).

$$W(Z) = e^{-Z^2}\cdot\left[1+\frac{2\mathrm{i}}{\sqrt{\pi}}\int_0^Z e^{\varsigma^2}\,d\varsigma\right] \qquad (7)$$

Equation (6) is for the case "$\sigma_x > \sigma_y$"; in the real situation, both "$\sigma_x > \sigma_y$" and "$\sigma_x < \sigma_y$" exist, and for the case "$\sigma_x < \sigma_y$", x and y are switched so that equation (6) still can be used, and afterwards, x and y are switched back.

Since the transverse space charge force includes both electric and magnetic forces, a factor of $1/\gamma^2$ is added to equation (6). Afterwards, the transverse space charge kick is calculated using equation (8).

$$\theta_x \approx \frac{e\cdot Ex}{\gamma^2}\cdot\frac{L}{\beta\cdot P\cdot c}$$
$$\theta_y \approx \frac{e\cdot Ey}{\gamma^2}\cdot\frac{L}{\beta\cdot P\cdot c} \qquad (8)$$

Here, $e = 1.602*10^{-19}$ *Coulomb*, γ is the Lorentz relativistic factor, P is the momentum of synchronous particle, and L is the path length of applying the transverse space charge kick. The program of the complex error function (CWERF) exists in CERN program library, and it's directly called. The rest of transverse and longitudinal space charge calculations have been programmed in Fortran.

Longitudinal Space Charge Simulations

The longitudinal code is used for the longitudinal space charge simulation. At the Booster batch size of $3.6*10^{12}$, about $4.3*10^{10}$ per bunch, the longitudinal beam current and the space charge induced voltage *vs*. rf phase are plotted at left and right separately in Figure 1, from top to bottom at turn 9000, 10000, and 12000. The transition happens at turn 9478.

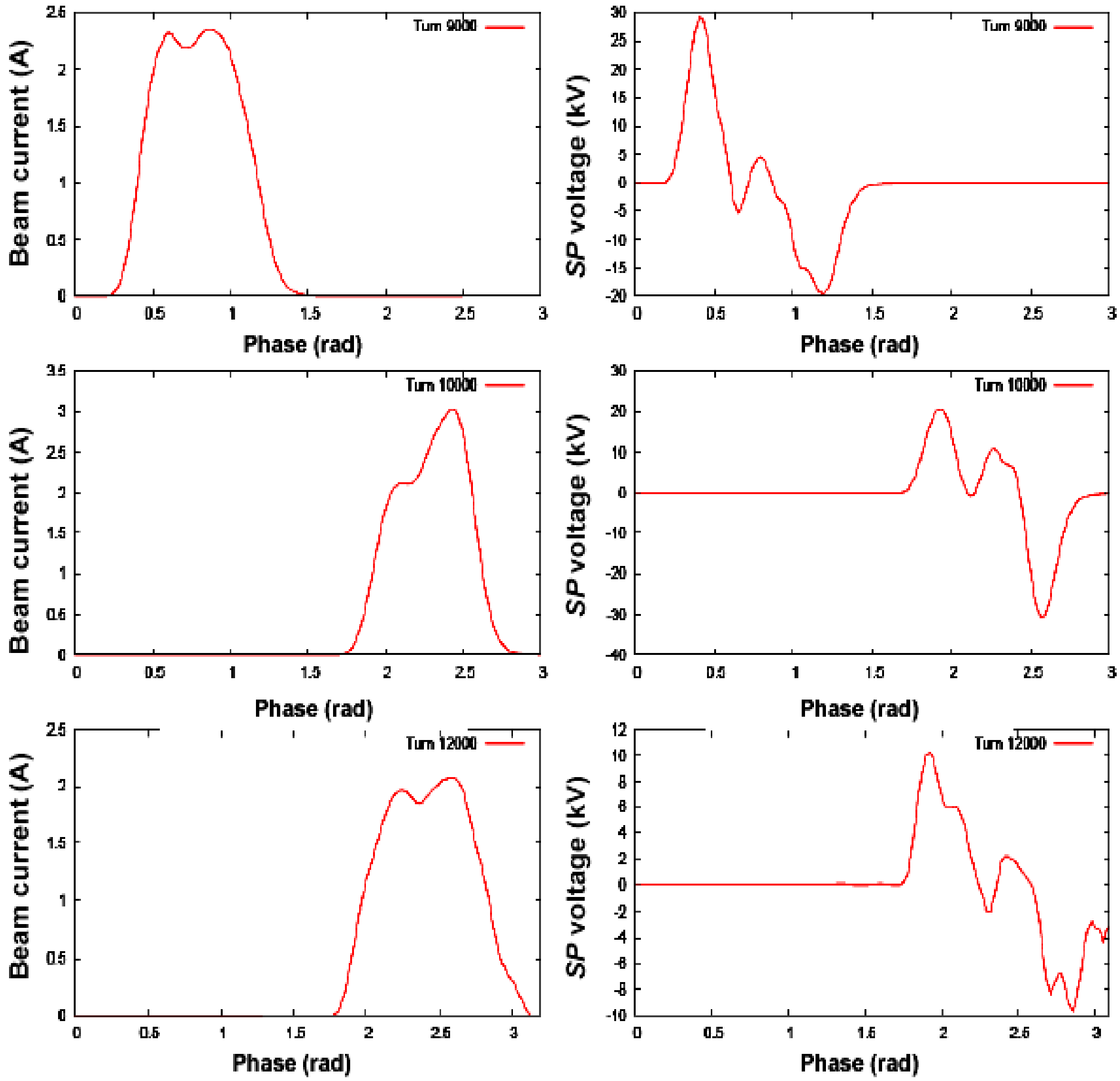

Figure 1

In Figure 2, the charge transmission *vs*. the beam intensity is simulated at four different batch sizes (red), and the results are compared to the experimental observation (green). They agree reasonably well.

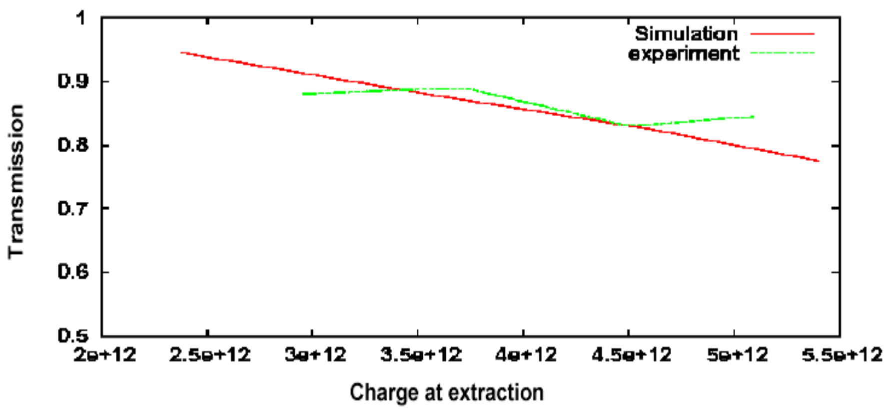

Figure 2

Transverse Space Charge Simulations

After the longitudinal space charge simulation is tested in the 1-D program, all the programs, which are responsible for the longitudinal and transverse space charge calculations, are added to STRUCT.[4]

At turn 2 after injection, with a beam intensity of $5.7*10^{12}$, using test particles with a x distribution of "-10σ_x to 10σ_x" and "y =0.0" (red) and test particles with a y distribution of "-10σ_y to 10σ_y" and "x =0.0" (green), the transverse space charge angle kick *vs*. position are plotted for the path length of 1m at Long-01 (top) and Short-02 (bottom) in Booster as Figure 3.

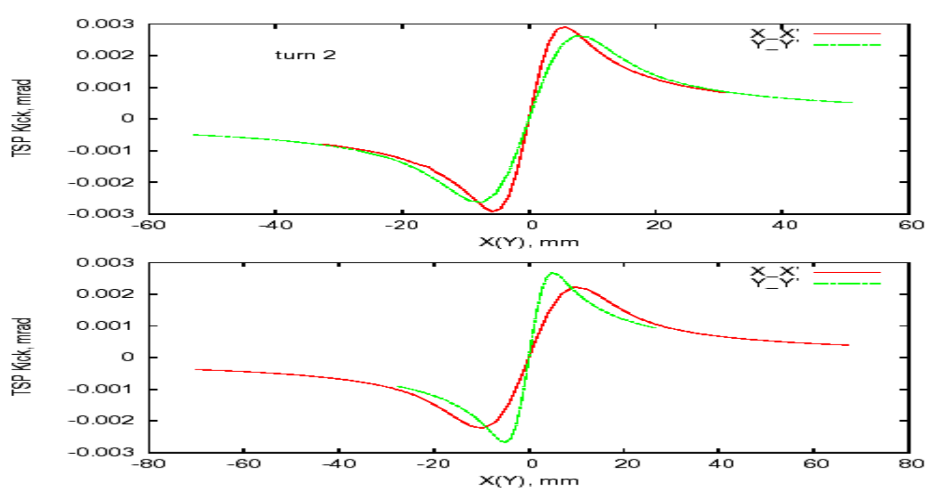

Figure 3

At turn 2 after injection, with a beam intensity of $5.7*10^{12}$, and the injected beam with a transverse emittance of $\varepsilon_x(95\%)= \varepsilon_y(95\%)=7.68$ mm·mrad, the transverse space charge angle kick *vs*. position are plotted for the path length of 1m at Long-01 (top) and Short-02 (bottom) as Figure 4, and their corresponding x-y plot are shown as Figure 5.

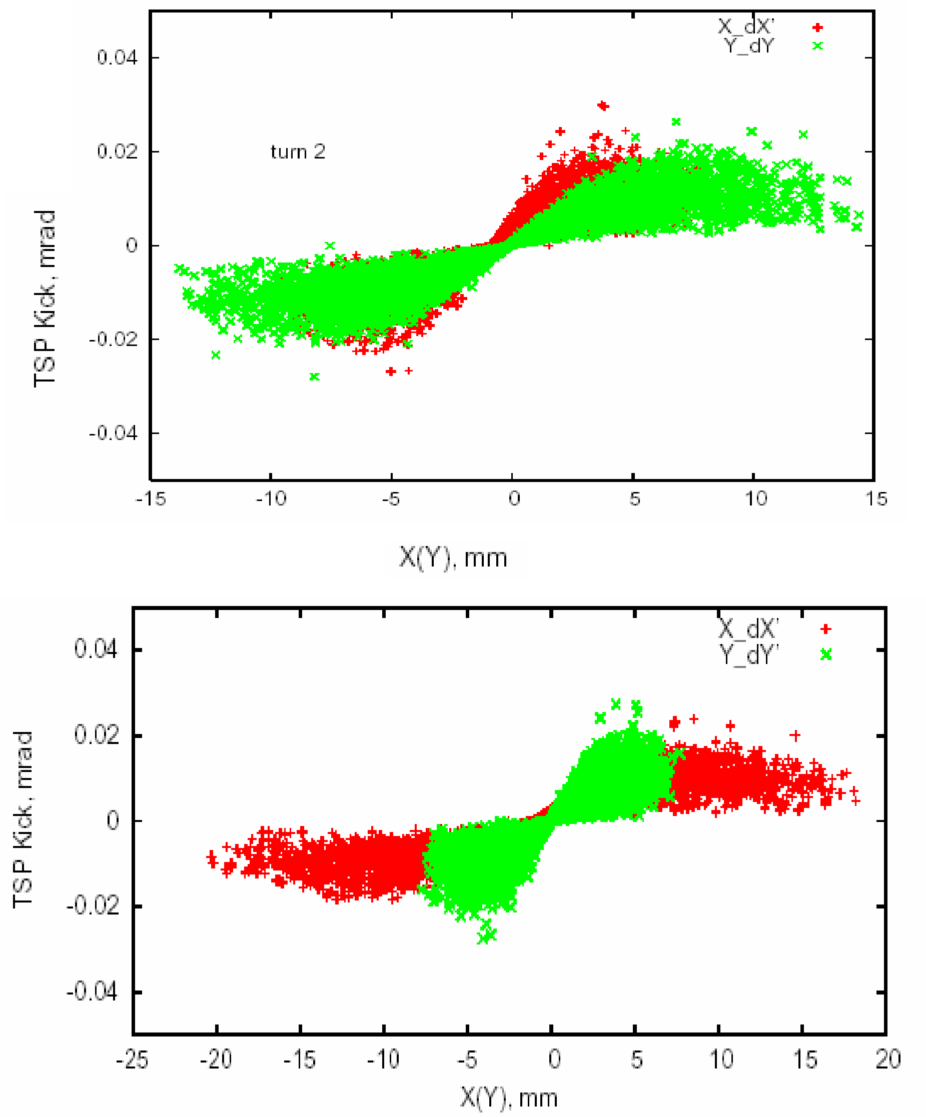

Figure 4

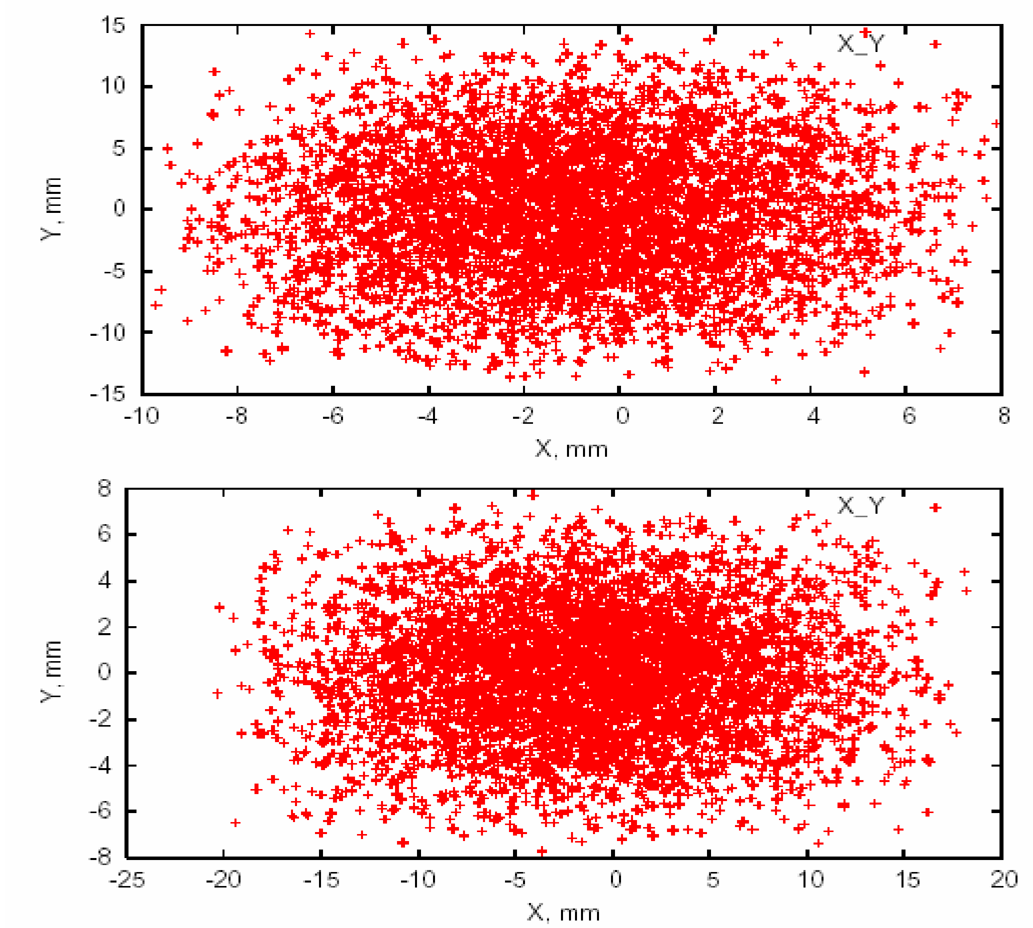

Figure 5

Conclusions

The transverse space charge calculation shown in Figure 4 has been compared to the result using Brute-Force PIC (particle in cell) method, and they agree quite well. Since the simplified space charge model assumes the decoupling between the longitudinal and transverse space charge effects, and also assumes the beam with a 2-D Gaussian distribution in x-y, which allows the transverse space charge force to be calculated *via* the complex error function, it's much faster than the 3-D PIC space charge method and also it's sufficient for most of the problems that we want to look into at Booster.

Acknowledgement

Francois Ostiguy for his help in calling functions from CERN program library.

References:

- [1] J. A. MacLachlan, “Longitudinal Phase Space Tracking With Space Charge and Wall Coupling Impedance”, FN-446.
- [2] W. Press etc., “Numerical Recipes” (Cambridge Press, New York, 1990).
- [3] M. Bassetti and G. Erskine, “Closed Expression for the Electrical Field of a Two-dimensional Gaussian Charge”, CERN-ISR-TH/80-06.
- [4] A. Drozhdin, “The STRUCT Program: User’s Reference Manual”.