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POPULATING A BARRIER BUCKET

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Abstract

The population of a bunch in a barrier bucket is discussed, and the general method is presented. In order that the bunch matches the bucket, the population must be according to the invariant tori inside the bucket.

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1 INTRODUCTION

If we wish to study the gymnastics of barrier rf, for example, bunch compression, bunch merging, etc., we must first learn how to populate particles inside a stationary barrier bucket. If the populating is incorrect, we will see the distribution change immediately when the particles move according to the equations of motion, even when the rf waves are not changing and coupling impedance of the vacuum chamber has not been turned on.

2 THE HAMILTONIAN

The particles reside in two regions: the region outside the barrier pulses and the region inside the barrier pulses. For a given half energy spread $\widehat{\Delta E}$ and squared barrier pulses of voltage $\pm V_0$, it is easy to determine the boundary of the beam. The Hamiltonian is given by

$$H = -\frac{\eta\delta^2}{2} + \frac{eV_0}{\beta^2 E_0 T_0}\Big[(\tau-\tau_1)\theta(\tau-\tau_1) - (\tau+\tau_1)\theta(-\tau-\tau_1)\Big] \ , \tag{2.1}$$

where the inner edge of the barrier with voltage $-V_0$ is at τ_1 and the inner edge of the barrier with voltage $+V_0$ is at $-\tau_1$. We have chosen the particle arrival time τ ahead of some synchronous particle at a point on the ring midway between the barriers (positive/negative for head/tail) as the canonical coordinate and the momentum offset δ as the canonical momentum. In above, T_0 is the nominal revolution period, η is the nominal slip factor, E_0 is the nominal energy of the particles and βc is the nominal velocity with c being the velocity of light. The maximum momentum offset is $\hat{\delta} = \widehat{\Delta E}/(\beta^2 E_0)$. The setup of this barrier bucket, with positive barrier pulse on the left and negative barrier pulse on the right, is suitable for an operation below transition when $\eta < 0$. All the formulas, however, are valid for operation above transition too, when $\eta > 0$ and $V_0 < 0$. To contain the bunch within the barrier bucket, the integrated barrier strength must be large enough. For this reason, we simplify our problem by not specifying the outer edges of the barriers.

From the Hamiltonian, we obtain the penetration into the barrier,

$$\tau_0 = -\frac{\eta\hat{\delta}^2\beta^2 E_0 T_0}{2V_0} \ . \tag{2.2}$$

Note that $\tau_0 > 0$ is independent of operating above or below transition. The boundary of

the beam is therefore given by

$$\begin{cases} -\dfrac{\eta(\hat{\delta}^2-\delta^2)}{2} = \dfrac{V_0}{\beta^2 E_0 T_0}(\tau-\tau_1) & -\tau_1-\tau_0<\tau<-\tau_1 \\ |\delta| = \hat{\delta} & -\tau_1<\tau<\tau_1 \\ -\dfrac{\eta(\hat{\delta}^2-\delta^2)}{2} = -\dfrac{V_0}{\beta^2 E_0 T_0}(\tau+\tau_1) & \tau_1<\tau<\tau_1+\tau_0 \end{cases} \tag{2.3}$$

where the first/last line represents the penetration boundary of the tail/head of the beam inside the barrier pulse of positive/negative voltage, while the middle line represents the beam boundary in the barrier-free region.

3 UNIFORM DISTRIBUTION

The distribution in the longitudinal phase space is

$$\psi(\tau,\delta)d\tau d\delta = \psi(H)d\tau d\delta \ . \tag{3.1}$$

For uniform distribution, we let ψ equal to a constant independent of τ and δ. This implies equal number of particles in each area element $d\tau d\delta$. In other words, the particle density is the same anywhere inside the beam boundary. Thus, we just populate the region inside the beam boundary uniformly independent whether the particles are inside or outside the barrier pulses. One may suspect the validity of such population because particle density may not be the same in the barrier-free region and the region of penetration inside the barriers. For example, we show three particle trajectories inside the barrier bucket in the left plot of Fig. 1. In the right plot, the barrier voltages are raised. We see that the penetrations of the three trajectories become less deep. These trajectories are more crowded inside the barrier pulses. However, one must remember that when the barrier voltage is higher, particles gain more energy each turn and move faster along the trajectories inside the barrier. Thus although particles are more dense inside the penetration in τ, they are less dense in $\widehat{\Delta E}$. The resulting density in the two-dimensional longitudinal phase space is unchanged and Liouville's theorem has not been violated.

4 TRUNCATED GAUSSIAN DISTRIBUTION

Let us study the untruncated Gaussian and add in the truncation later. We want the

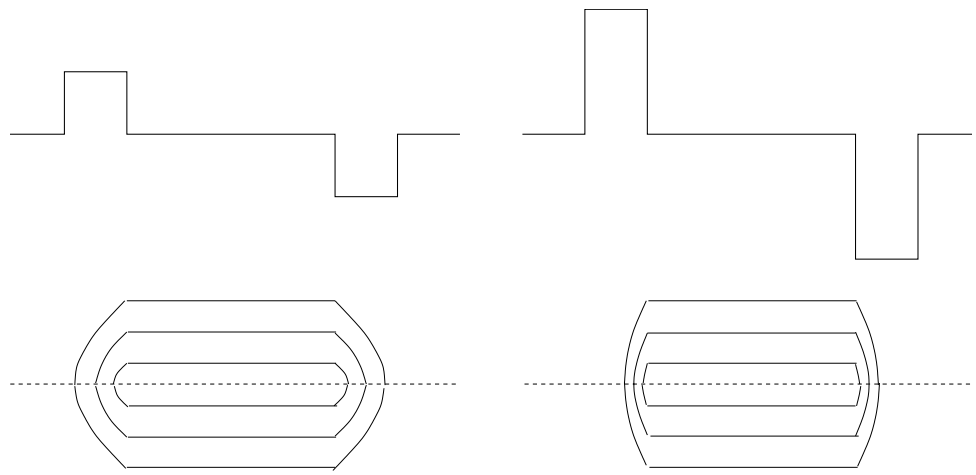

Figure 1: Schematic drawing illustrating that the particle trajectories are more crowded in the penetrations of the barrier pulses. The left plot shows three particles trajectories inside a barrier bucket established by a pair of barrier pulses of equal and opposite polarities. The right plot shows the same but with much higher barrier voltages. It is evident that because of the higher barrier voltages in the right plot, the penetrations into the barriers become smaller and the trajectories in the penetrations become more crowded. However, this does not imply that the particle density is higher, because particles will move faster along the crowded trajectories because the barrier voltages are higher.

distribution in energy offset to have Gaussian distribution or

$$\psi(\tau,\delta) \propto \exp\left\{-\frac{\delta^2}{2\sigma_\delta^2}\right\} \ , \tag{4.1}$$

where σ_δ is the rms momentum offset. To be a function of the Hamiltonian, the distribution in the longitudinal phase space must be in the form

$$\psi(\tau,\delta) = A\exp\left\{\frac{H}{\eta\sigma_\delta^2}\right\} \ , \tag{4.2}$$

where

$$A = \frac{1}{2\sqrt{2\pi}\sigma_\delta}\frac{v}{1+v\tau_1} \tag{4.3}$$

is the normalization constant, which can be determined easily by equating the integration of ψ to unity, and we have defined a reduced voltage

$$v = -\frac{eV_0}{\eta\sigma_\delta^2\beta^2E_0T_0} \ . \tag{4.4}$$

Note that $v > 0$ and is independent of operating above or below transition. The distribution $\psi(\tau, \delta)$ is separable in the sense that the distributions in τ and δ are statistically independent.[†] In other words,

$$\psi(\tau, \delta) = \rho(\tau) f(\delta) \tag{4.5}$$

is just the product of two one-dimensional distributions. The population can therefore be performed easily. First let us integrate over δ to get the linear distribution density

$$\rho(\tau) = \sqrt{2\pi}\sigma_\delta A \exp\left\{-v\Big[(\tau - \tau_1)\theta(\tau - \tau_1) - (\tau + \tau_1)\theta(-\tau - \tau_1)\Big]\right\} . \tag{4.6}$$

The distribution $R(\tau)$, which is the integration of ρ from $-\infty$ to τ, is given by

$$R(\tau) = \begin{cases} \sqrt{2\pi}\sigma_\delta A \dfrac{e^{v(\tau+\tau_1)}}{v} & \tau \le -\tau_1 \\ \sqrt{2\pi}\sigma_\delta A \left[\dfrac{1}{v} + \tau + \tau_1\right] & \tau_1 \le \tau \le \tau_1 \\ \sqrt{2\pi}\sigma_\delta A \left[\dfrac{2}{v} + 2\tau_1 + \dfrac{e^{-v(\tau+\tau_1)} - 1}{-v}\right] & \tau \ge \tau_1 \end{cases} \tag{4.7}$$

We can now solve τ as a function of R and obtain

$$\tau = \begin{cases} -\tau_1 + \dfrac{1}{v}\ln\left[2(1 + v\tau_1)R\right] & R \le \dfrac{1}{2(1 + v\tau_1)} \\ \dfrac{2(1 + v\tau_1)}{v}\left(R - \dfrac{1}{2}\right) & \dfrac{1}{2(1 + v\tau_1)} \le R \le \dfrac{1 + 2v\tau_1}{2(1 + v\tau_1)} \\ \tau_1 - \dfrac{1}{v}\ln\left[2(1 + v\tau_1)(1 - R)\right] & R \ge \dfrac{1 + 2v\tau_1}{2(1 + v\tau_1)} \end{cases} \tag{4.8}$$

where

$$R(-\tau_1) = \frac{1}{2(1 + v\tau_1)} , \qquad R(\tau_1) = \frac{1 + 2v\tau_1}{2(1 + v\tau_1)} , \tag{4.9}$$

and Eq. (4.3) has been used.

To populate the bunch with N particles and truncate the distribution at $\hat{\delta} = n\sigma_\delta$ (n need not be an integer), we proceed as follows:

1. Choose a random number F on [0,1], and compute δ from the inverse of cumulative Gaussian distribution function [1]

$$F = \int_{-\infty}^{\delta/\sigma_\delta} f(x)dx \tag{4.10}$$

[†]This is only true for the untruncated Gaussian without a beam boundary. A truncated Gaussian has a finite boundary for the beam and this boundary couples the distributions in τ and δ making them statistically dependent.

where

$$f(t) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \tag{4.11}$$

is the standard normal distribution density function. The inverse of F is usually a standard routine in many software libraries. If $|\delta| < \hat{\delta}$, we go on to the next step, otherwise this step is repeated by generating another random number.

2. Generate a random number R on [0,1], and compute τ according to Eq. (4.8). If the point (τ, δ) falls inside the beam boundary, this is a valid particle. The particle counter is increased by one.

3. If the particle counter reaches N, the required number of particles has been generated. Otherwise, go back to step 1.

COMMENTS

1. When populating a beam with a truncated Gaussian distribution using the formulas derived from an untruncated Gaussian, σ_δ no longer represents the rms momentum offset.

2. To have a distribution that is far away from uniform, we choose a small σ_δ and a large n so that the beam boundary $\hat{\delta} = n\sigma_\delta$ remains as specified. In this way, the number of rms covered by the beam will be large.

   On the other hand, to have a distribution close to uniform, we choose $n \ll 1$ and therefore a large σ_δ. To populate such a beam according to the procedure given above will be very inefficient because most of the random points generated will fall outside the beam boundary.

3. Here, we would like to see whether the distribution would become uniform as $\sigma_\delta \to \infty$. Notice from Eq. (4.4) that as $\sigma_\delta \to \infty$, $v \to 0$. Since the distribution spreads out to a very large extent, only those random numbers in the vicinity of $R \approx \frac{1}{2}$ will fall within the beam boundary. Actually from Eq. (4.9),

$$R(\pm\tau_1) \approx \frac{1}{2}\left(1 \pm v\tau_1\right) . \tag{4.12}$$

We see that

$$\frac{1}{v}\ln\left[2(1+v\tau_1)R = \frac{1}{v}\ln\left[1+v\tau_1\right] + \frac{1}{v}\ln\left[1+2\left(R-\frac{1}{2}\right)\right]\right.$$
$$\approx \tau_1 + \frac{2}{v}\left(R-\frac{1}{2}\right) \ . \tag{4.13}$$

In this way, we can show that in all three regions of R in Eq. (4.8), we obtain the same result

$$\tau \approx \frac{2}{v}\left(R-\frac{1}{2}\right) \ . \tag{4.14}$$

Since $R-\frac{1}{2}$ is a random number, the distribution in τ is random or uniform in all the three regions, inside or outside the barrier pulses.

5 ELLIPTICAL-LIKE DISTRIBUTIONS

Let us consider a generalized elliptical distribution. Without barrier rf, the distribution in the longitudinal phase space is

$$\psi(\tau,\delta) = A\left(\hat{\delta}^2-\delta^2\right)^n \ , \tag{5.15}$$

where n need not be an integer and A is the normalization constant. With the barrier rf included, ψ becomes

$$\psi(\tau,\delta) = A\left\{\hat{\delta}^2-\delta^2-\frac{2eV_0}{\beta^2|\eta|E_0T_0}\left[(\tau-\tau_1)\theta(\tau-\tau_1)-(\tau+\tau_1)\theta(\tau+\tau_1)\right]\right\} \ , \tag{5.16}$$

since it must be a function of the Hamiltonian. Integration over δ from $-\mathcal{A}$ to $\mathcal{A}$ with

$$\mathcal{A} = \left\{\hat{\delta}^2-\frac{2eV_0}{\beta^2|\eta|E_0T_0}\left[(\tau-\tau_1)\theta(\tau-\tau_1)-(\tau+\tau_1)\theta(\tau+\tau_1)\right]\right\}^{1/2} \ , \tag{5.17}$$

gives the distribution density function in τ,

$$\rho(\tau) = \int_{-\mathcal{A}}^{\mathcal{A}}\psi(\tau,\delta)d\delta = 2A\gamma_n\mathcal{A}^{2n+1}$$
$$= 2A\gamma_n\left\{\hat{\delta}^2-\frac{2eV_0}{\beta^2|\eta|E_0T_0}\left[(\tau-\tau_1)\theta(\tau-\tau_1)-(\tau+\tau_1)\theta(\tau+\tau_1)\right]\right\}^{(2n+1)/2} \ , \tag{5.18}$$

where

$$\gamma_n = \int_0^1(1-t^2)^n dt = \int_0^{\pi/2}\cos^{2n+1}\theta d\theta = \frac{\pi\Gamma(2n+2)}{2^{2n+1}\Gamma^2(n+\frac{3}{2})} \ . \tag{5.19}$$

We can now determine the distribution function in τ,

$$F(\tau) = \int_{-\tau_e}^{\tau} \rho(\tau')d\tau' \ , \tag{5.20}$$

where

$$\tau_e - \tau_1 = \frac{\beta^2|\eta|E_0T_0\hat{\delta}}{2eV_0} \ . \tag{5.21}$$

is the penetration into the barrier. We obtain

$$\frac{F(\tau)}{2A\gamma_n} = \begin{cases} \dfrac{\beta^2|\eta|E_0T_0}{(2n+3)eV_0}\left[\hat{\delta}^2 + \dfrac{2eV_0}{\beta^2|\eta|E_0T_0}(\tau+\tau_1)\right]^{(2n+3)/2}, & -\tau_e \le \tau \le -\tau_1 \ , \\ \dfrac{\beta^2|\eta|E_0T_0}{(2n+3)eV_0}\hat{\delta}^{2n+3} + (\tau+\tau_1)\hat{\delta}^{2n+1}, & -\tau_1 \le \tau \le \tau_1 \ , \\ \dfrac{2\beta^2|\eta|E_0T_0}{(2n+3)eV_0}\hat{\delta}^{2n+3} + 2\tau\hat{\delta}^{2n+1} - & \\ \quad - \dfrac{\beta^2|\eta|E_0T_0}{(2n+3)eV_0}\left[\hat{\delta}^2 - \dfrac{2eV_0}{\beta^2|\eta|E_0T_0}(\tau-\tau_1)\right]^{(2n+3)/2}, & \tau_1 \le \tau \le \tau_e \ . \end{cases} \tag{5.22}$$

Since $F(\tau_e) = 1$, the normalization constant can be determined. The distribution function can be inverted easily to obtain $\tau = \tau(F)$, from which τ can be populated by letting F equal to a random number between 0 and 1.

For each random variable τ determined, $\mathcal{A}$ is computed according to Eq. (5.17), and then the distribution function in δ,

$$G(\delta) = \int_{-\mathcal{A}}^{\delta} A\left[\mathcal{A}^2 - \delta'^2\right]^n d\delta' \ , \tag{5.23}$$

from the inverse function, $\delta = \delta(G)$, a random variable δ is obtained through the substitution of G by a random number between 0 and 1.

6 CONCLUSION

The method to populate a barrier bucket according to the invariant tori has been presented when the momentum-offset distribution is uniform or a truncated Gaussian. For other distribution of the momentum offset, the population can be carried out in a similar way. [1]

References

[1] See, for example, K.Y. Ng, *Particle Population Inside a Tight Stationary Bucket*, Fermilab Report FN-695, 2000.