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An integral for longitudinal phase space tomography on equilibrium distributions.

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Longitudinal phase space tomography is developing rapidly into a practical diagnostic technique. The numerical code written by Hancock, Lindroos, and Koscielniak [2] at CERN constructed a distribution function from a collection of changing beam profiles using a tracking function that included space charge effects. Montag, D'Imperio, Kewisch and Lee [6] wrote their own code and obtained remarkable images of an instability that developed during rebucketing at RHIC. Schlarb [7] described a procedure used at DESY, based on a thesis of Geitz [1], for reconstruction of transverse distributions. More recently, Huening [4] constructed longitudinal tomographic images to study wake field effects at the Tesla Test Facility.

When a longitudinal distribution is in equilibrium, one profile should be sufficient to reconstruct it. C.-Y. Tan developed a numerical procedure for doing this when the bucket is stationary (non-accelerating). [8] The procedure can be modified to work on nonstationary (accelerating) buckets. [5] In either case one obtains a lower triangular linear system of equations which must be inverted, not an unusual occurrence in tomographic problems.

The purpose of this paper is to present an analytical expression for obtaining an equilibrium distribution function from a beam profile, an integral transform whose numerical implementation is equivalent to inverting that triangular system of equations. In the first section, we go through a simple exercise to set up the solution; in the second, we extend the exercise to longitudinal phase space. The final section contains a few generally edifying comments.

1 A simple yet helpful exercise

We begin by reviewing a simple exercise, so as to reduce notational complexity later. Suppose that a distribution in the Euclidean plane depends only on radius, say,

$$\mathrm{Prob}\,[\,(X,Y)\in D\,] = \int_D dxdy\, f(r^2) \quad ,$$

where X and Y are random real variables, D is a (Borel) subset of R^2, and, of course, $r^2 = x^2 + y^2$. Consider the projections at fixed values of x.

$$g(x) \equiv \int_{-\infty}^{\infty} dy\, f(r^2) = 2\int_0^{\infty} dy\, f(r^2) \quad .$$

Because $dx = 0$,

$$dy = \frac{r}{y}\,dr = \frac{r}{\sqrt{r^2 - x^2}}\,dr \quad ,$$

and we can write $g(x)$ as,

$$\begin{aligned} g(x) &= 2\int_{|x|}^{\infty} r dr \frac{f(r^2)}{\sqrt{r^2 - x^2}} \\ &= \int_{x^2}^{\infty} du \frac{f(u)}{\sqrt{u - x^2}} \quad , \end{aligned} \tag{1}$$

with, of course, $u = r^2$.

This is much like an integral equation motivated by one of those "bead on a frictionless wire" problems: Suppose that a mass drops in a constant gravitational field along a contrained path, expressed as the set of points $\{\, (y(z), z) \mid z \in (0, \infty)\,\}$. You measure the time, $\tau(h)$, that it takes for the bead to reach $z = 0$ when it is dropped from the altitude $z = h$. From the function $\tau(h)$, is it possible to determine $y(z)$?

The answer, yes, was determined by the Norwegian mathematician, Niels Henrik Abel in 1823, about one year before he proved the impossibility of solving, "in closed form," polynomial equations of degree greater than four. The problem is expressed as an integral equation by using conservation of energy:

$$\frac{1}{2} m v^2 + mgz = mgh \quad \Rightarrow \quad v = \sqrt{2g(h - z)}$$

which means that

$$\tau(h) = \int_0^{\tau(h)} dt = \int \frac{ds}{v} = \int_0^h dz \, \frac{ds/dz}{\sqrt{2g(h - z)}} \quad , \tag{2}$$

where ds is diffential arc length along the path. The (by now) well known inversion[1] of Eq.(2) can be found in most textbooks on integral equations, such as [9].

$$\frac{ds}{dz} = \frac{\sqrt{2g}}{\pi} \left(\frac{\tau(0)}{\sqrt{z}} + \int_0^z dh \frac{\tau'(h)}{\sqrt{z - h}} \right) \quad . \tag{3}$$

Once ds/dz is known, the task of finding $y(z)$ has been reduced to quadrature.

The connection between Eq.(1) and Eq.(2) is accomplished by changing the integration coordinate:

$$uv = 1 \quad \Rightarrow \quad udv + vdu = 0 \quad .$$

so that, after a little algebra, Eq.(1) is rewritten as follows.

$$|x|\, g(x) = \int_0^{1/x^2} dv \left(\frac{f(1/v)}{v^{3/2}} \right) \frac{1}{\sqrt{1/x^2 - v}}$$

[1] There are other ways of expressing the answer.

Notice that multiplication by $|x|$ unfortunately suppresses the strongest part of the signal. This is now in the same form as Eq.(2), as can be seen via substitutions.[2]

$$
\begin{aligned}
& z \leftrightarrow v \\
& h \leftrightarrow 1/x^2 \\
& \frac{ds/dz}{\sqrt{2g}} \leftrightarrow \frac{f(1/v)}{v^{3/2}} \\
& \tau(h) = \tau(1/x^2) \leftrightarrow |x|\, g(x)
\end{aligned}
$$

The inversion can then be written immediately from Eq.(3) using the same substitutions.

$$
\frac{f(1/v)}{v^{3/2}} \quad = \quad \frac{1}{\pi}\int_0^v d(1/x^2)\frac{d(|x|g(x))}{d(1/x^2)} \cdot \frac{1}{\sqrt{v - 1/x^2}}
$$

In dropping the extra term, we have assumed (sensibly) that $\lim_{x\to\infty} |x|g(x) = 0$. To simplify a little, notice that

$$
d(1/x^2)\frac{d(|x|g(x))}{d(1/x^2)} = d(|x|g(x)) = dx\frac{d(|x|g(x))}{dx} \equiv dx\,(|x|g(x))'
$$

so that, upon choosing $x > 0$,

$$
\frac{f(1/v)}{v^{3/2}} = \frac{1}{\pi}\int_\infty^{1/\sqrt{v}} dx\,\frac{(xg(x))'}{\sqrt{v - 1/x^2}} \quad .
$$

As a final step, we go back and use $v = 1/u = 1/r^2$ to express this as follows.

$$
r^2 f(r^2) = \frac{1}{\pi}\int_\infty^r dx\,\frac{(xg(x))'}{\sqrt{1-(r/x)^2}} \tag{4}
$$

Abel's equation was first applied to tomography by Radon, as Allan Cormack, who shared the Nobel Prize in 1979 for inventing the CAT scan, noted in his acceptance speech.

> "[T]his seemed like a problem which would have been solved before, probably in the 19th century, but again a literature search and enquiries of mathematicians provided no information about it. Fourteen years would elapse before I learned that Radon had solved this problem in 1917. Again I had to tackle the problem from the beginning. The solution is easy for objects with circular symmetry... One has Abel's equation to solve, and its solution has been known since 1825. [sic]"

[2] One possible source of concern is whether it is necessary for $\tau(h)$ to be monotonically increasing, as is suggested by the physics problem. Fortunately, this is not the case: it is sufficient that $\tau(h)$ be continuous.[3] Physically, continuity precludes the possibility that the path possesses a roller coaster like dip, which, in turn, assures existence of the function $y(z)$.

2 Extension to longitudinal phase space

We will now extend this result to our problem of determining an equilibrium distribution in longitudinal phase space. For convenience, we write the Hamiltonian model (a) using coordinates centered at the synchronous phase and energy and (b) shifted so that $H = 0$ at the point of synchrony. Because of (a), in the corresponding map model, the kick imparted to a particle would change its energy according to,

$$\Delta E = eV \sin(\varphi_s + \varphi) \ ,$$

where, as usual, φ_s is the synchronous phase, and $eV > 0$ is the maximum possible energy imparted by the cavity. A Hamiltonian satisfying our two conditions is then written,

$$\begin{aligned} H &= -\frac{1}{2}\alpha W^2 + eV \cdot (\cos(\varphi_s + \varphi) - \cos\varphi_s + \varphi \sin\varphi_s) \\ &\equiv -\frac{1}{2}\alpha W^2 + U(\varphi;\varphi_s) \ \text{, where} \qquad (5) \\ \alpha &= \frac{h}{2\pi}\left(\left(\frac{\omega}{pc}\right)^2 \cdot E \cdot \left(\frac{1}{\gamma^2} - \frac{1}{\gamma_t^2}\right)\right)_s \ \text{, and} \\ W &= (E - E_s)(2\pi/\omega_s) = (E - E_s)\tau_s \end{aligned}$$

The subscript s always stands for "evaluated at the point of synchrony." The "potential" function U is effectively defined by Eq.(5).

The continuous signal is modeled as the function,[3]

$$S(\varphi) = \int_{-\infty}^{\infty} dW\, f(\varphi, W) = \int_{-\infty}^{\infty} dW g(H(\varphi, W)) = 2\int_{-\infty}^{0} dW g(H(\varphi, W)) \ .$$

We have used the equilibrium condition, $\partial f/\partial t = 0$, to write explicitly that the value of f depends only on the value of H: $f = g \circ H$. f (or g) is the distribution function that we seek, and I've used the symmetry of H to reduce the region of integration.

We are using φ to parametrize the signal. However, it is recorded as a function of time, t, where $\varphi = \omega_{\mathrm{rf}}(t - t_s)$. To make matters worse, shortly we will want to use $\sqrt{U(\varphi)}$ as the parametrization coordinate. To avoid a confusing jumble of terminology – an all but impossible task, under the circumstances – we introduce notation,

$$S(\varphi) \equiv S_\Phi(\varphi) \equiv S_T(t) \equiv S_U(\sqrt{U}) \ ,$$

and anticipate using the appropriate representation as needed.

We have reduced the domain of integration over W to one for which the maps $W \mapsto H$ and $H \mapsto W$ would be one-to-one. Thus, over this domain — i.e., $W \in (-\infty, 0)$ — H can be used as a legitimate coordinate.[4] Accordingly, change integration coordi-

[3] An integration over a small "bin width," $\Delta\varphi$, is subsumed in the definition of S.

[4] Of course, we could just as well have chosen $W \in (0, \infty)$ and obtained equivalent results.

nate from W to H.

$$S(\varphi) = 2\int_\Gamma dH \left(\frac{\partial W}{\partial H}\right)_\varphi g(H)$$
$$= 2\int_\Gamma dH \left(-\frac{1}{\alpha W}\right) g(H) \ ,$$

The range of integration, Γ, depends on whether we are above or below transition. It will be written explicitly below.

Now use Eq.(5) to substitute for W.

$$(\alpha W)^2 = -2\alpha(H - U)$$

Over our domain, $W \in (-\infty, 0)$, $W < 0$, and

above transition:	$\alpha < 0$,	$1/\alpha W > 0$,	and	$0 \le U(\varphi) \le H(\varphi, W)$.
below transition:	$\alpha > 0$,	$1/\alpha W < 0$,	and	$H(\varphi, W) \le U(\varphi) \le 0$.

where U and H are evaluated *inside the separatrix*. We then write,[5]

$$\text{above transition}: \ S_U(\sqrt{U}) = 2\int_\infty^{U(\varphi;\varphi_s)} dH(-1/\alpha W)g(H)$$
$$= 2\int_U^\infty dH g(H)/\sqrt{(-2\alpha)(H-U)}$$
$$\text{below transition}: \ S_U(\sqrt{-U}) = 2\int_{-\infty}^{U(\varphi;\varphi_s)} dH(-1/\alpha W)g(H)$$
$$= 2\int_{-\infty}^{U} dH g(H)/\sqrt{(-2\alpha)(H-U)}$$
$$= 2\int_{-U}^{\infty} d(-H) g(-(-H))/\sqrt{(2\alpha)((-H)-(-U))}$$

In order to write S in terms of *coordinate* $\sqrt{U}$, we use only half the signal. The maps $\varphi \mapsto U$ and $U \mapsto \varphi$ are one-to-one over restricted domains that do not include the fixed points.

Both of these equations are of the correct form for application of Eq.(1), *provided we use the fact that both S and g vanish outside the separatrix.* We only need to make

[5] I am using the symbol H to represent (a) the Hamiltonian function and (b) a real-valued coordinate. While this may be a little confusing, context should separate the two meanings.

the substitutions:

$$
\begin{array}{ll}
\text{above transition}: & x \leftrightarrow \sqrt{U} \\
& u = r^2 \leftrightarrow H \\
& f(u) \leftrightarrow g(H)/\sqrt{-2\alpha} \\
& g(x) \leftrightarrow S_U(\sqrt{U}) \\
\text{below transition}: & x \leftrightarrow \sqrt{-U} \\
& u = r^2 \leftrightarrow -H \\
& f(u) \leftrightarrow g(-(-H))/\sqrt{2\alpha} \\
& g(x) \leftrightarrow S_U(\sqrt{-U})
\end{array}
$$

An application of Eq.(4) then yields the results, above transition,

$$
\begin{aligned}
Hg(H)/\sqrt{-2\alpha} &= -\frac{1}{\pi}\int_{\sqrt{H}}^{\infty} \frac{d(\sqrt{U})}{\sqrt{1-H/U}}\,\frac{d}{d(\sqrt{U})}\left(\sqrt{U}\,S_U(\sqrt{U})\right) \\
&= \frac{1}{\pi}\int_{\infty}^{\varphi_1(H)} \frac{d\varphi}{\sqrt{1-H/U}}\,\frac{d}{d\varphi}\left(\sqrt{U}\,S_\Phi(\varphi)\right) \quad . \qquad (6)
\end{aligned}
$$

Obviously, a corresponding expression can be written below transition. The lower limit is a bit formal. It actually need extend no farther than the separatrix, since $S = 0$ outside the separatrix. $\varphi = \varphi_1(H)$ is the point at which the orbit whose Hamiltonian has value H intersects the $W = 0$ axis.

3 A few comments

Eq.(6) is our principal result. It provides a linear filter from which to construct an equilibrium distribution function from the profile signal. Notice that:

(a) The value of the distribution function at the point of synchrony, $g(0)$, is left undetermined, apart from continuity:

$$g(0) = \lim_{H\to 0} g(H).$$

This is true as well when one inverts the signal numerically rather than analytically. It is a general issue in tomography.

(b) Remember: H is just a number in the integrand. In particular, it satisfies, $H < U(\varphi)$ within the range of integration. This does *not* contradict the inequalities written on the previous page, which refer to H as a function.

(c) In the intermediate steps I used the fact that U was convex, up or down, between the stable fixed point and the separatrix. However, in the answer's final form, that condition does not appear explicitly.

(d) Apart from convexity of the potential – which is guaranteed inside a separatrix –

U can be arbitrary. It is certainly not necessary to assume that it provides a pure sinusoidal field.

(e) Only half the signal is used. Breaking the signal into two parts, on each side of the synchronous phase, and processing them separately should give identical results. (This is *not* the same as claiming that $S_\Phi(\varphi) = S_\Phi(-\varphi)$, which will certainly not be the case, in general.) This can serve to check (i) that we have identified t_s or, equivalently, φ_s correctly from the signal and (ii) our key assumption that the distribution is in equilibrium is valid.

(f) Another check can be done by finding the cumulative distribution as a function of action coordinate, I, rather than the Hamiltonian, H: say, $G(I)$. This is possible if the distribution is in equilibrium. If acceleration is adiabatic, $G(I)$ should not change throughout the ramp, since action is an adiabatically invariant coordinate.

(g) The square root singularity will be annoying when it comes down to numerical integration, but it can be handled. For example, it can be removed via an integration by parts. Using,

$$\frac{1}{\sqrt{1-H/U}} = \frac{\sqrt{U}}{\sqrt{U-H}} = \frac{2\sqrt{U}}{dU/d\varphi}\frac{d}{d\varphi}(\sqrt{U-H}) \ ,$$

we get,

$$Hg(H)/\sqrt{-2\alpha} = \frac{1}{\pi}\int_{\varphi_1(H)}^{\infty} d\varphi \, \sqrt{U-H}\,\frac{d}{d\varphi}\left(\frac{2\sqrt{U}}{dU/d\varphi}\left(\frac{d}{d\varphi}(\sqrt{U}S(\varphi))\right)\right) \ .$$

This removes the square root singularity, but now requires taking two derivatives of the signal. A better numerical trick may be to introduce subtractions and cutoffs.

$$\begin{aligned} Hg(H)/\sqrt{-2\alpha} &= \frac{1}{\pi}\int_{\infty}^{\varphi_1(H)} \frac{d\varphi}{\sqrt{1-H/U}}(\cdots) \\ &= \frac{1}{\pi}\int_{\infty}^{\varphi_1(H)} \frac{d\varphi}{\sqrt{1-H/U}}\left[(\cdots) - (\cdots)|^{\varphi=\varphi_1(H)}\right] + \frac{1}{\pi}\int_{\infty}^{\varphi_1(H)} \frac{d\varphi}{\sqrt{1-H/U}}(\cdots)|^{\varphi=\varphi_1(H)} \ . \end{aligned}$$

The first integrand has no singularity; the second can be handled with a cutoff.

$$\frac{1}{\pi}\int_{\infty}^{\varphi_1(H)} \frac{d\varphi}{\sqrt{1-H/U}} = \left(\frac{1}{\pi}\int_{\infty}^{\varphi_1(H)-\varepsilon} \frac{d\varphi}{\sqrt{1-H/U}} + \frac{1}{\pi}\int_{\varepsilon}^{\varphi_1(H)} \frac{d\varphi}{\sqrt{1-H/U}}\right) \ ,$$

where ε is chosen sufficiently small so that we can approximate $1-H/U \propto (\varphi-\varphi_1(H)) + O((\varphi-\varphi_1(H))^2)$. We reserve further discussion of numerical issues for another time.[5]

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