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RESONANT KICKER FOR BUNCH-GAP PARTICLE REMOVAL*

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Abstract

A kicker modulated at the residual betatron tune of the beam is proposed for the removal of beam particles in the bunch gap, which, otherwise, will trap electrons leading to *e*-*p* coupled-beam instability. The beam dynamic is derived analytically and the implication discussed. Applications are made to the Los Alamos Proton Storage Ring (LANL PSR) and the Spallation Neutron Source Proton Storage Ring (SNS PSR).

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1 INTRODUCTION

In high intensity proton storage rings such as the LANL PSR and the SNS PSR, some protons experiencing large space charge may overcome the focusing provided by rf and drift into the bunch gap. Once inside the gap, these protons can trap electrons and initiate *e*-*p* instabilities [1, 2]. To clear these protons, we suggest the use of a kicker that gives a kick of a certain length of time to only those protons in the gap. To facilitate proton clearing, the kicker voltage should be modulated at frequency $[\nu_\beta]f_0$, where f_0 the revolution frequency and $[\nu_\beta]$ is the horizontal/vertical residual (or nonintegral part of) *bare* betatron tune of the storage ring if the kicker under consideration is a horizontal/vertical one. In this way, the protons in the gap will be kicked resonantly and will be deflected towards a collimator system in the beam pipe.

There is a spread of betatron tune in the beam particles because of amplitude detunings and chromaticities. To accommodate the tune variation, the modulated tune in the kicker should also have a spread. Here, we are going to study the particle response to the kicker having a modulated tune spread. Analytical expressions are derived, from which results can be understood easily and the kicker parameters can be chosen.

2 THE BEAM DYNAMICS

The transverse position of a particle is represented by its transverse displacement x and transverse angular displacement $x' = dx/ds$ with respect to the designed orbit. Here s is the longitudinal distance measured along the closed orbit from some point of reference. Let the normalized displacement coordinates be written as a complex number $\vec{X}$ so that betatron motion will be a circle in phase space [3]:‡

$$\vec{X} = \left(\frac{x}{\sqrt{\beta_x}}, \frac{\beta_x x' + \alpha_x x}{\sqrt{\beta_x}} \right) \ , \tag{2.1}$$

where β_x and α_x are the Twiss parameters at the location under consideration. In this convention, from location 1 to location 2, where the Floquet phase advance is Ψ,

‡In Ref. [3], only a single location at the ring is referenced. Therefore, the representation $\vec{X} = (x, \beta_x x' + \alpha_x x)$ can be used and it transforms as Eq. (2.2). Here, there is more than one location, the kicker and the collimator, and the representation in Eq. (2.1) must be used.

the complex displacement is transformed according to

$$\vec{X}_1 \longrightarrow \vec{X}_2 = \vec{X}_1 e^{-i\Psi} \ . \tag{2.2}$$

Let us start with the particle having displacement $\vec{a}_0$ just before entering the kicker. After passing through the kicker, the displacement is changed by $\Delta\vec{X}_0$. After one turn when the particle arrives back just before the kicker, the displacement becomes $\big(\vec{a}_0+\Delta\vec{X}_0\big)e^{-i2\pi\nu_{\beta 1}}$, where $\nu_{\beta 1}$ is the betatron tune of the particle in the first turn. The particle receives a displacement modification $\Delta\vec{X}_1$ when crossing the kicker and arrives at the collimator which has a phase advance ϕ upstream. The particle displacement $\vec{X}_1$ at the collimator after one turn with two kicker traversals is therefore

$$\vec{X}_1 = \Big(\big(\vec{a}_0 + \Delta\vec{X}_0\big)e^{-i2\pi\nu_{\beta 1}} + \Delta\vec{X}_1\Big)e^{-i\phi} \ . \tag{2.3}$$

Thus after N turns around the ring, the particle, after traversing the kicker $N+1$ times, arrives at the collimator with the displacement

$$\vec{X}_{\scriptscriptstyle N} = \bigg(\bigg(\cdots\bigg(\bigg(\vec{a}_0 + \Delta\vec{X}_0\bigg)\, e^{-i2\pi\nu_{\beta 1}} + \Delta\vec{X}_1\bigg)\, e^{-2\pi\nu_{\beta 2}}\cdots\bigg)e^{-i2\pi\nu_{\beta N-1}} + \Delta\vec{X}_{\scriptscriptstyle N}\bigg)e^{-i\phi} \,, \tag{2.4}$$

which can be written more conveniently as

$$\vec{X}_{\scriptscriptstyle N} = \vec{a}_0 e^{-i2\pi\sum_{j=1}^{N}\nu_{\beta j}-i\phi} + \sum_{\ell=0}^{N}\Delta\vec{X}_\ell e^{-i2\pi\sum_{j=\ell+1}^{N}\nu_{\beta j}-i\phi} \ . \tag{2.5}$$

Equation (2.5) indicates that the position of the particle at the collimator after N turn is just the linear superposition of the evolution of the original particle displacement, the evolution of the zeroth-turn kick of the particle, the evolution of the first-turn kick, etc. If we assume that the betatron tune seen by the particle is the same for all turns, i.e., $\nu_{\beta j} = \nu_\beta$ for all j, Eq. (2.5) can be simplied to

$$\vec{X}_{\scriptscriptstyle N} = \vec{a}_0 e^{-i2\pi N\nu_\beta - i\phi} + \sum_{\ell=0}^{N}\Delta\vec{X}_\ell e^{-i2\pi(N-\ell)\nu_\beta - i\phi} \ . \tag{2.6}$$

The Hill's equation governing the motion of the particle is

$$x'' + K_x(s) = \sum_{\ell=0}^{\infty}(\Delta x')_k \cos\left(\frac{\nu s}{R} + \psi_0\right)\delta(s - 2\pi\ell R) \ , \tag{2.7}$$

where s is the distance measured along the ring from the initial position of the particle, E is the total energy of the particle, and R is the radius of the ring. Here, $(\Delta x')_k$ is the amplitude of the angular kick provided by the kicker. For an electric kicker, we can imagine the kicker to be composed of two parallel plates with a transverse electric field $\mathcal{E}$, and be turned on for a short duration $\Delta/(\beta c)$ where βc is the particle velocity and c the velocity of light. Then

$$(\Delta x')_k = \frac{e\mathcal{E}\Delta}{E\beta^2} \; . \tag{2.8}$$

For a magnetic kicker with magnetic flux density B, we have instead

$$(\Delta x')_k = \frac{eBc\Delta}{E\beta} \; . \tag{2.9}$$

In Eq. (2.7), a short kick duration or $\Delta \ll 2\pi R$ has been assumed. However, this assumption is merely for the sake of simplicity so that some analytic expressions can emerge. The kicker shows a modulation tune ν and a modulated phase ψ_0. Thus, crossing the kicker for the ℓ-th time introduces a kick of

$$\Delta x' = (\Delta x')_k \cos(2\pi\nu\ell + \psi_0) \; . \tag{2.10}$$

The transverse displacement x is unchanged if the kicking time $\Delta/(\beta c)$ is infinitesimally short. Therefore, in the ℓ-th crossing of the kicker, the particle displacement is changed by

$$\Delta\vec{X}_\ell = i(\Delta x')_k \sqrt{\beta_k} \cos(2\pi\nu\ell + \psi_0) \; , \tag{2.11}$$

where β_k is the betatron function at the kicker. Now Eq. (2.5) takes the form

$$\vec{X}_{_N} = \vec{a}_0 e^{-i2\pi N\nu_\beta - i\phi} + i(\Delta x')_k \sqrt{\beta_k} \int d\nu f(\nu) \sum_{\ell=0}^{N} \cos(2\pi\nu\ell + \psi_0) e^{-i2\pi(N-\ell)\nu_\beta - i\phi} \; , \tag{2.12}$$

where we have introduced a spread in the modulation tune of the kick to accommodates particles with slightly different tunes due to amplitude detunings and chromaticities. The modulation tune distribution $f(\nu)$ is normalized to unity. This distribution is centered at ν_0 with spread $\pm\Delta\nu$, and ν_0 should be chosen to be the average tune of the particles in the bunch and $\Delta\nu$ should be chosen to be the tune spread $\Delta\nu_\beta$ in the particle bunch. In this way, the tune ν_β of a bunch particle will always be inside the distribution $f(\nu)$ so that the particle will be kicked resonantly.

It is clear from Eq. (2.12) that only the residual or nonintegral part of the particle tune $[\nu_\beta]$ is relevant. Therefore, the tune component of the kicker can be chosen as close to $[\nu_\beta]$ instead. Such a choice enables us to employ a much lower modulation frequency for the kicker. Thus, the particle tune ν_β we reference below should be interpreted as the residual tune.

3 GAUSSIAN TUNE SPREAD IN KICKER

Let us assume a Gaussian distribution in tune spread in the kicker, or

$$f(\nu) = \frac{1}{\sqrt{2\pi}\sigma_\nu} e^{-(\nu-\nu_0)^2/(2\sigma_\nu^2)} , \tag{3.1}$$

where σ_ν is the rms tune spread. The integration over ν gives

$$\vec{X}_{\scriptscriptstyle N} = i(\Delta x')_k \sqrt{\beta_k} \sum_{\ell=0}^{N} \cos(2\pi\nu_0\ell + \psi_0) e^{-(2\pi\ell\sigma_\nu)^2/2} e^{-i2\pi(N-\ell)\nu_\beta - i\phi} , \tag{3.2}$$

where the initial displacement $\vec{a}_0$ has been set to zero. We are interested mostly in the envelope of the particle displacement from the resonant kicker. Thus, the summation over revolution turns can be approximated by an integral. We obtain, in the closed form,

$$\vec{X}_{\scriptscriptstyle N} = i\sqrt{\beta_k}\varepsilon_k \Big[F(a, y_{\scriptscriptstyle N}) e^{-i\psi_0} + F(b, y_{\scriptscriptstyle N}) e^{i\psi_0} \Big] e^{-i2\pi N\nu_\beta} e^{-i\phi} , \tag{3.3}$$

where the envelope function, $F(a, y_{\scriptscriptstyle N})$ is given by

$$F(a, y_{\scriptscriptstyle N}) = \frac{2}{\sqrt{\pi}} \int_0^{y_N} e^{-y^2 - i2ay} dy = w(a - iy_{\scriptscriptstyle N}) e^{(a-iy_{\scriptscriptstyle N})^2 - a^2} - w(a) , \tag{3.4}$$

with $w(z)$ being the complex error function,

$$a = \frac{\nu_0 - \nu_\beta}{\sqrt{2}\sigma_\nu} , \qquad b = \frac{-\nu_0 - \nu_\beta}{\sqrt{2}\sigma_\nu} , \tag{3.5}$$

representing the particle tune offset from the center kicker tune $\pm\nu_0$, and

$$y_{\scriptscriptstyle N} = \sqrt{2}\pi\sigma_\nu N \tag{3.6}$$

representing the turn number. The kicker strength is represented by the dimensionless variable

$$\varepsilon_k = \frac{(\Delta x')_k}{4\sqrt{2\pi}\sigma_\nu} . \tag{3.7}$$

Obviously, the kicker will be driving the betatron motion of the particle resonantly only when ν_0 is close to ν_β. Thus, we expect $F(a, y_{_N})$ to dominate over $F(b, y_{_N})$ in Eq. (3.3). For this reason, $F(b, y_{_N})$ will be dropped. The envelope of the transverse displacement at the collimator can therefore be written as

$$|x_c| = \sqrt{\beta_k \beta_c}\, \varepsilon_k \big| F(a, y_{_N}) \big| \ , \tag{3.8}$$

where β_c is the betatron function at the collimator. The normalized envelope of particle displacement is therefore given by $\big|F(a, y_{_N})\big|$ and depends on only two variables a and $y_{_N}$. It will be independent of the modulated phase ψ_0 as well as the phase advance ϕ of the collimator ahead of the kicker. Here, a measures the deviation of the kicker modulated tune from the betatron tune of the particle beam, while $y_{_N}$ measures the number of times the particle beam passes through the kicker. Notice that in Eq. (3.8), $|x_c|/\sqrt{\beta_c}$ is just the square root of the acceptance of the collimator system. Figure 1 shows the normalized envelopes for various values of a from 0 to 2.47. We have also performed simulations employing a kicker having center tune $\nu_0 = 0.203$ with rms spread $\sigma_\nu = 0.005$. The particle tune ν_β was varied from $|\nu_0 - \nu_\beta| = 0$ to 0.0175 or $3.5\sigma_\nu$. The simulation results are also plotted in Fig. 1. We see that the computed envelopes do agree very well with the simulated particle amplitudes although the term $F(b, y_{_N})$ has been discarded.

3.1 Particle tune right at kicker center tune

This is the situation when the particle betatron motion is driven resonantly by the kicker center tune. Here, $a = 0$ and the normalized envelope becomes

$$F(0, y_{_N}) = \mathrm{erf}(y_{_N}) = \begin{cases} \dfrac{2}{\sqrt{\pi}}\, y_{_N} = 2\sqrt{2\pi}\sigma_\nu N & \qquad y_{_N} \ll 1 \ , \\ 1 & \qquad y_{_N} \to \infty \ , \end{cases} \tag{3.9}$$

where $\mathrm{erf}(y_{_N})$ is the error function. This is depicted in the first plot of Fig. 1. The linear rise when $y_{_N} \ll 1$ signifies resonant driving, whereas the saturation when $y_{_N} \gg 1$ is a result of the tune spread in the kicker. At the beginning, all tune components of the kicker are kicking the particle turn-after-turn in phase, the particle absorbs energy from the kicker and its betatron amplitude therefore grows linearly with time. After some time, the kicks by different tune components are no longer in

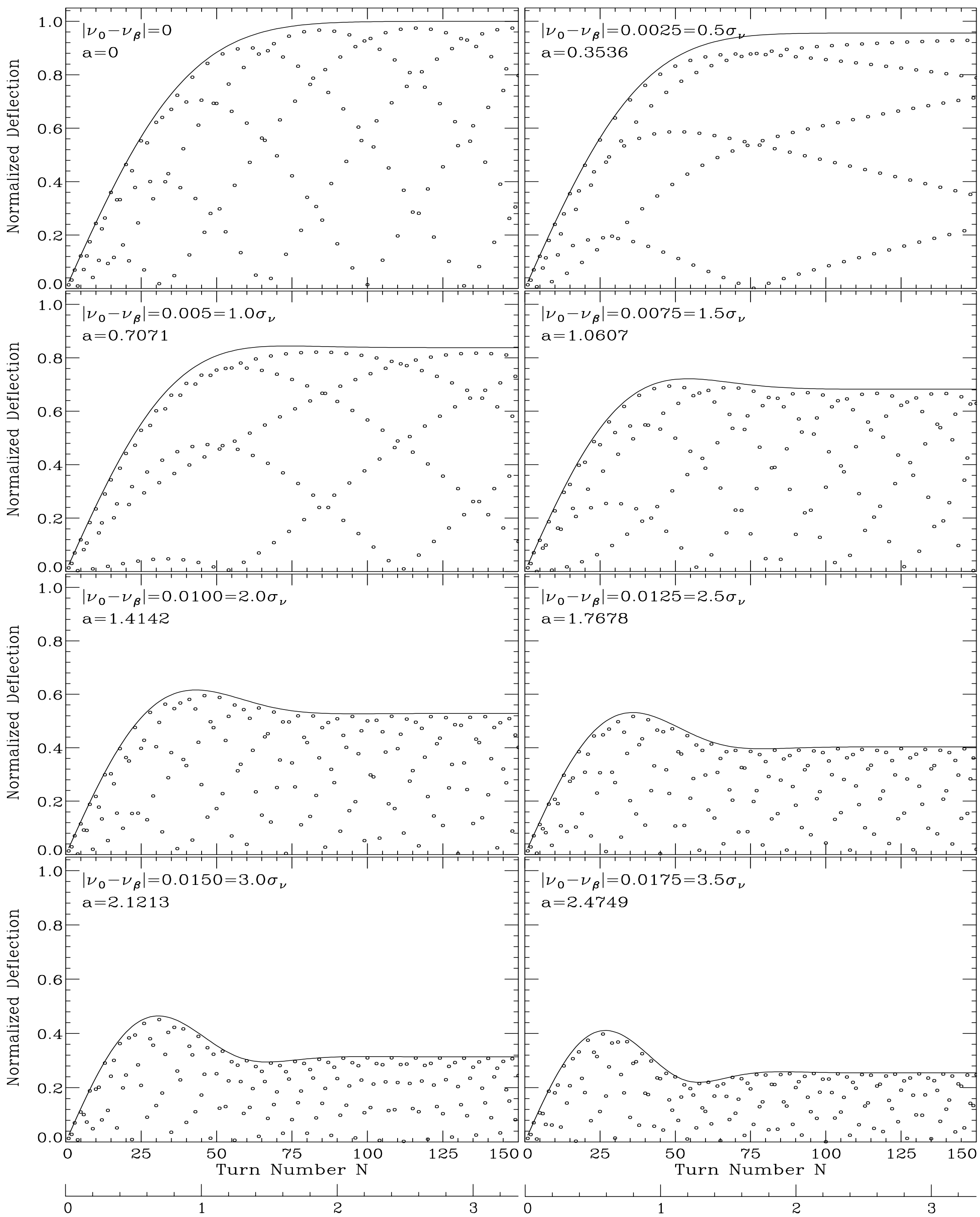

Figure 1: Plots showing the simulated turn-by-turn normalized particle deflection together with the computed envelope for various particle tune offset from the kicker center tune.

phase because of the finite tune spread σ_ν in the kicker. The particle is not gaining energy from all the kicker tune components and, instead returning energy back to some kicker tune components. For this reason, the particle amplitude increases much slower and finally saturates. This phenomenon is called Landau damping. The turning point is around

$$y_{_N} \sim 1 \qquad \text{or} \qquad N \sim \frac{1}{\sqrt{2}\pi\sigma_\nu} \ . \tag{3.10}$$

Thus, to maintain an efficient resonant kick, the tune spread σ_ν should be made as small as possible. However, it must be broad enough to encompass the tune spread in the particle beam.

3.2 Particle tune not equal to kicker center tune

The other plots in Fig. 1 are for the situations when the particle tune deviates from the kicker center tune by $|\nu_0 - \nu_\beta| = 0.5\sigma_\nu$, $1.0\sigma_\nu$, $1.5\sigma_\nu$, $2.0\sigma_\nu$, $2.5\sigma_\nu$, $3.0\sigma_\nu$, and $3.5\sigma_\nu$. Although the envelope becomes smaller and smaller as a increases, the initial growth rate is the same. By performing an expansion of the integral in Eq. (3.4), the envelope initial growth rate is given by

$$\big|F(a,y_{_N})\big| \longrightarrow \frac{2}{\sqrt{\pi}}\, y_{_N}\!\left[1-\left(\frac{1}{3}+\frac{a^2}{6}\right)y_{_N}^2\right] \ , \tag{3.11}$$

which is valid for all a when $y_{_N} \ll 1$. This clearly demonstrates that the envelope growth rate will become smaller when a^2 is larger or when the particle tune deviates more from the kicker center tune. Equation (3.11) also illustrates that the envelope is independent on the sign of a. This is to be expected so long as the kicker tune distribution $f(\nu)$ is symmetric about the center tune ν_0. In any case, the envelope saturates at large turn number. This asymptotic behavior can be derived from the integral in Eq. (3.4) by setting $y_{_N} \to \infty$. We obtain, for all a,

$$F(a,\infty) = w(a) \ , \tag{3.12}$$

and

$$\big|F(a,\infty)\big| \longrightarrow 1-\left(1-\frac{2}{\pi}\right)a^2 \qquad \text{when} \quad a \ll 1 \ . \tag{3.13}$$

Again, we see that the asymptotic envelope is reduced when a^2 increases.

This asymptotic envelope, however, only serves as a reference, because the envelope overshoots this value whenever $a \neq 0$. In fact, the envelope may make many oscillations as illustrated in Fig. 1 before it settles down to the asymptotic value. This first maximum is important because we would like the particles to hit the collimator there. Unfortunately, it is not possible to express this first maximum in simple analytic form. Instead, we solve for it numerically with the results depicted in Figs. 2 and 3. This first maximum of the envelope decreases and occurs earlier when the particle tune deviates from the kicker center tune. Notice that the first maximum of the envelope is reduced roughly by a factor of two when the particle tune deviation is $\pm\sigma_\nu$, and this value should be taken into account when designing the kicker parameters.

3.3 Kicker strength

The dimensionless kicker strength ε_k is given by Eq. (3.7). Notice that it contains all the properties of the kicker and is normalized to the rms modulation tune spread σ_ν. In the situation that the modulation tune spread is very small and the particle tune is inside the spread, we expect the particle amplitude to grow linearly for a large number of turns $N \lesssim 1/(\sqrt{2}\pi\sigma_\nu)$. Then, the normalized envelope of displacement $\big|F\big| \sim 2y_{_N}/\sqrt{\pi} = 2\sqrt{2\pi}\sigma_\nu N$ is still less than unity according to the plots in Fig. 1. The actual growth is given by the multiplication with ε_k which exhibits the enhancement by having σ_ν in the denominator.

4 APPLICATIONS

4.1 SNS PSR

The SNS PSR has a circumference of 248.00 m, or 945.39 ns at the injection and storage kinetic energy of 1.0 GeV or $\beta = 0.8750$. The beam occupies about $\frac{2}{3}$ of the ring, leaving a gap of ~ 315 ns. Thus, there is plenty of space to install the resonant kicker.

Since there are only very small amount of beam particles that wander into the bunch gap, the space charge force plays no role here and we should use the bare betatron tunes of the ring, which are $\nu_\beta = 6.23$ in the horizontal and 6.20 in the vertical. For those particles that make their way into the bunch gap, they must have

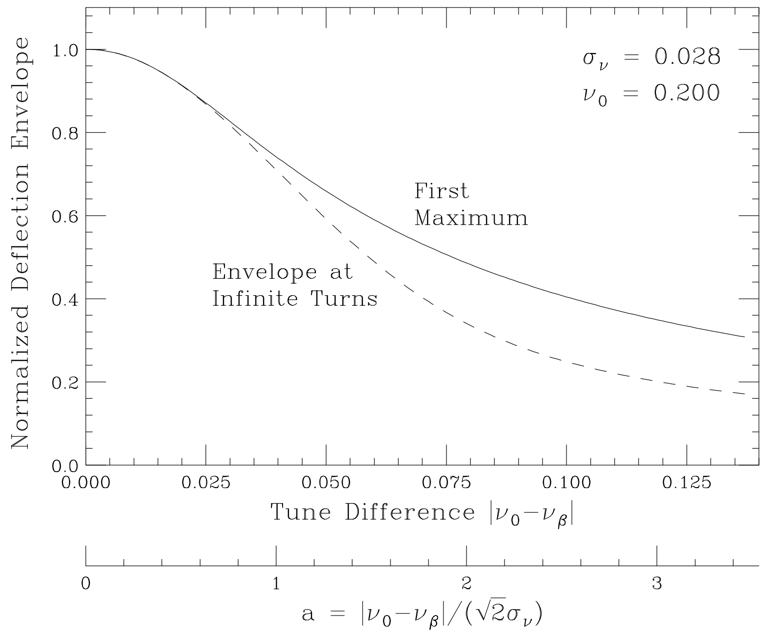

Figure 2: The first maximum of the response envelope is shown as solid. It decreases as the particle tune ν_β deviates from the kicker center tune ν_0. The envelope at infinite turns is also plotted in dashes and is always less than the first maximum except when $\nu_0 = \nu_\beta$.

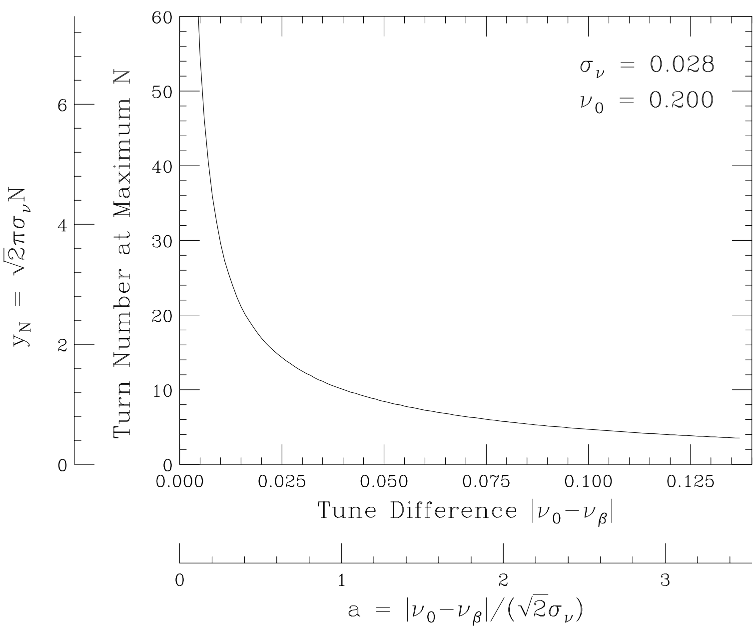

Figure 3: The first maximum of the response envelope is shown to occur at earlier turns as the particle tune deviates from the kicker center tune (or $|a|$ increases), in accord to the fact that the initial envelope growth is a-independent.

undergone a rather large change in momentum. As a result, chromaticity effect should contribute mostly to the tune spread. The bucket height in the longitudinal phase space has a momentum spread of $\delta \sim 0.1$. With the horizontal/vertical chromaticity $-7.9/-6.9$, the horizontal/vertical rms tune spread is roughly $\sigma_\nu = 0.0324/0.0282$. Thus, the resonant kicker should be modulated at $[\nu_\beta]f_0 = 243/211$ kHz with a rms spread of $\sigma_\nu f_0 = 34.1/29.8$ kHz.

Looking into Fig. 1, it is reasonable to run the kicker at such a strength that particles will be removed at $y_{_N} \sim 1$ when the envelope function is $|F| \sim \frac{1}{2}$. This implies particle removal in $N = y_{_N}/(\sqrt{2}\pi\sigma_\nu) \sim 7$ to 8 turns.

The SNS PSR is equipped with collimators with acceptance at about 140 πmm-mr. If the horizontal/vertical kicker is situated at a location where the corresponding betatron function is at a maximum (27.9 m horizontally and 15.7 m vertically), the strength of the kicker required turns out to be $(\Delta x')_k = 4\sqrt{2\pi}\sigma_\nu\varepsilon_k = 0.0014/0.0017$.

4.2 LANL PSR

The LANL PSR has a circumference of 90.2 m. It receives chopped proton beams from a linac in 1000 to 2000 turns at the kinetic energy of 797 MeV. The beam is bunched by a rf buncher to the desired length and is then extracted for experimental use. The beam usually occupies less than $\frac{2}{3}$ of the ring. At high intensity, the strong space charge force overcomes the rf force and pushes particles into the bunch gap. These are the particles we would like to clean up. Since *e*-*p* coupled-beam instability occurs mostly in the vertical direction, we should design a resonant kicker in the vertical direction.

The bare vertical betatron tune is $\nu_\beta = 2.19$; the kicker should be modulated at $[\nu_\beta]f_0 = 531$ kHz. where $f_0 = 2.796$ MHz is the revolution frequency. The maximum vertical betatron function is 14.5 m and the beam pipe has a vertical radius of 4.8 cm. If we assume collimation takes place at 4.0 cm and also the collimators and kicker are situated at locations where the vertical betatron function is at the maximum, the required reduced kicker strength is $\varepsilon_k = 0.0055$, where again an envelope function of $|F| = \frac{1}{2}$ has been used. An ACCSIM [4] simulation shows that [5], at the intensity of 7.3 μC, the rf bucket height is reduced to ± 3.65 MeV (Fig. 4, implying that beam particles leaking into the bunch gap will have such energy offset. The vertical chromaticity has been measured to be $\xi_y = -2.4$. This gives an rms spread of the

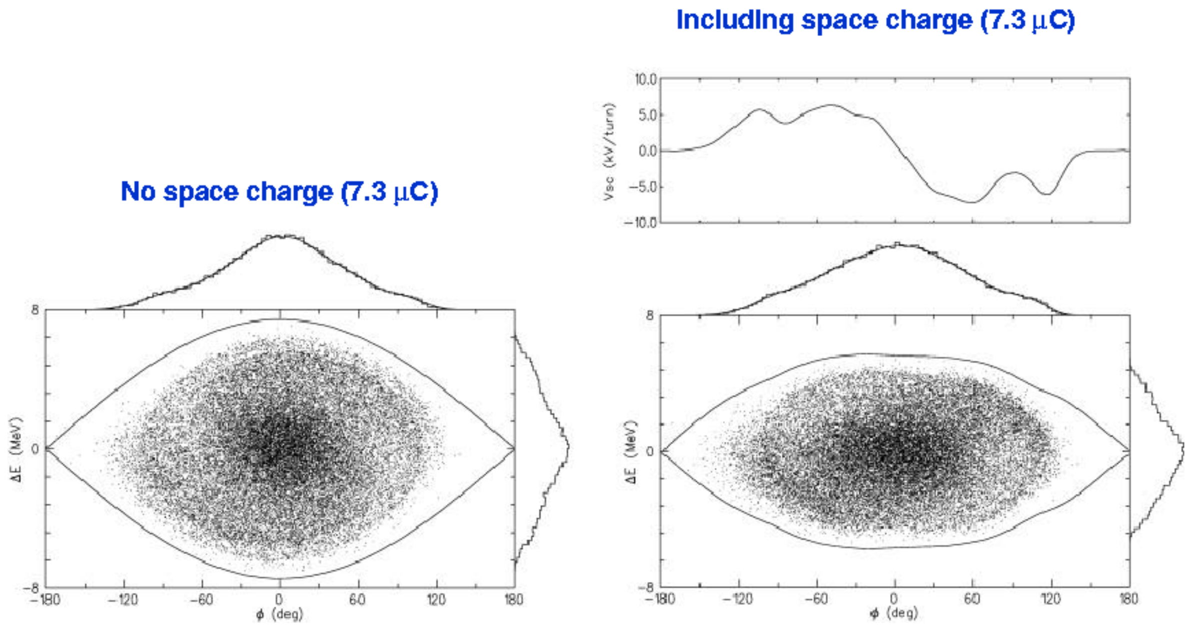

Figure 4: Simulation of a PSR bunch with an intensity of 7.3 μC at the buncher voltage of 13 kV using the code ACCSIM. The left plot is the result without space charge while the right plot is the result with space charge included. Notice that in the presence of space charge the bucket height is reduced from ± 4.80 MeV by 24% to ± 3.65 MeV. The top curve on the right shows the space charge voltage per turn (proportional to the spatial derivative of the proton line density).

vertical betatron tune as $\sigma_\nu = 0.0045$. The actual required kicker strength is therefore $(\Delta x')_k = 4\sqrt{2\pi}\sigma_\nu \varepsilon_k = 0.00025$.

4.3 Transverse-Momentum Following

The frequency modulation of the resonant kicker is rather slow, and is almost constant during a passage of the particles in the bunch gap. Thus, instead of modulating the kicker strength, one can also change the polarity of the kicker turn by turn following the centroid of the beam and kick in the same direction as the transverse momentum. For example, when the residual tune is 0.2, one follows the sequence $+$, $+$, $-$, $-$, $+$, $\cdots$. Such a method has been studied by Cousineau and Holmes [6].

5 ALTERNATE CONSIDERATION

In the previous section, we integrated over ν first in Eq. (2.12) before summing over the turn number ℓ. As an alternate, here we proceed to perform the summation over turn number ℓ first, leading to

$$\vec{X}_{_N} = \frac{i}{2}\sqrt{\beta_k}\,(\Delta x')_k\, e^{-i\phi}\int d\nu f(\nu)\left\{\frac{\sin[\pi(\nu_\beta-\nu)(N+1)]}{\sin\pi(\nu_\beta-\nu)}\, e^{-i\pi(\nu_\beta+\nu)N-i\psi_0} + \frac{\sin[\pi(\nu_\beta+\nu)(N+1)]}{\sin\pi(\nu_\beta+\nu)}\, e^{-i\pi(\nu_\beta-\nu)N+i\psi_0}\right\}\ , \tag{5.1}$$

where the initial particle displacement $\vec{a}_0$ has been discarded.

The modulation tune distribution $f(\nu)$ is normalized to unity. This distribution should be centered at ν_0 with spread $\pm\Delta\nu$, and ν_0 should be chosen to be the average tune of the particles in the bunch and $\Delta\nu$ the tune spread $\Delta\nu_\beta$ in the particle bunch. In this way, the tune ν_β of a bunch particle will always be inside the distribution $f(\nu)$ so that the particle will be kicked resonantly. We can therefore set the requirement

$$\left|\nu-\nu_\beta\right| < \Delta\nu \ll 1\ . \tag{5.2}$$

As a result, for $\sin[\pi(\nu_\beta-\nu)(N+1)]/\sin\pi(\nu_\beta-\nu)$ in the first term of the curly brackets in Eq. (5.1), it is sufficient to consider only the contribution near $\nu\sim\nu_\beta$.

5.1 No Spread in kicker tune

First, let us consider the situation of no spread in the kicker modulation tune but the particle tune is inside $f(\nu)$; i.e., $f(\nu)=\delta(\nu-\nu_\beta)$. It is easy to arrive at

$$\vec{X}_{_N} = \frac{i}{2}\sqrt{\beta_k}\,(\Delta x')_k\, e^{-i\phi}\left\{(N+1)e^{-i2\pi N\nu_\beta-i\psi_0} + \frac{\sin[2\pi\nu_\beta(N+1)]}{\sin 2\pi\nu_\beta}\, e^{i\psi_0}\right\}\ . \tag{5.3}$$

The factor $(N+1)$ in the first term inside the curly brackets represents the linear growth of the particle displacement as a result of resonant driving. The second term is nonresonant, like $F(b,y_{_N})$ in Eq. (3.3), is small and can be neglected here and below. The amplitude of displacement at the collimator agrees with what we obtain in the first expression of Eq (3.9).

5.2 Kicker tune spread $\Delta\nu \ll 1/N$

Let us consider the function

$$g(\nu) = \frac{\sin[\pi(\nu-\nu_\beta)(N+1)]}{\sin\pi(\nu-\nu_\beta)}\, e^{-i\pi(\nu-\nu_\beta)N} \tag{5.4}$$

in the first term inside the curly brackets of Eq. (5.1). The function is periodic. When N is large, it has sharp peaks of mganitude $N+\frac{1}{2}$ at $\nu-\nu_\beta=k$, where k is an integer and half widths $\sim 1/(2N+1)$. Because of the restriction of Eq. (5.2) provided by $f(\nu)$, the tune spread distribution of the kicker, all the contribution of $g(\nu)$ will vanish except for the one near $\nu-\nu_\beta=0$.

When the tune spread of the kicker $\Delta\nu$ is very much less than $1/(N+1)$, we can treat the kicker tune distribution $f(\nu)$ as a δ-function, namely $f(\nu)=\delta(\nu-\nu_0)$. The integration over ν in Eq. (5.1) will pick up the value of $g(\nu)$ at ν_0. The result is

$$\vec{X}_{N} \approx \frac{i}{2}\sqrt{\beta_k}\,(\Delta x')_k\, g(\nu_0)\, e^{-i2\pi\nu_\beta N-i\psi_0-i\phi}\ . \tag{5.5}$$

Thus, when the particle tune ν_β is equal to the center tune ν_0 of the kicker, $g(\nu_0)=N+1$, and we recover the result of resonant driving in Eq. (5.3). When ν_β deviates from ν_0, the accumulated kicked displacement drops sharply according to $g(\nu_\beta)$ and becomes insignificant when $|\nu_\beta-\nu_0| \gtrsim 1/N$.

5.3 Kicker tune spread $\Delta\nu \gg 1/N$

This is the situation when $N \gg 1$ and we should obtain the asymptotic bound of the displacement envelope. When the spread of $f(\nu)$ is very much larger than the spread of $g(\nu)$, the latter approaches[§] in the vicinity of $\nu=\nu_\beta$,

$$g(\nu) = \frac{1}{2}\,\delta(\nu-\nu_\beta) - \frac{i}{\pi}\frac{\mathcal{P}}{\nu-\nu_\beta}\ , \tag{5.7}$$

[§]It is important that the factor $e^{-i\pi(\nu-\nu_\beta)N}$ must be included in $g(\nu)$ as a whole when the limit $N\to\infty$ is taken. This is because $e^{-i\pi(\nu-\nu_\beta)N}$ is *not* a smoothly varying function in the vicinity of $\nu=\nu_\beta$. If it were not included in $g(\nu)$, one would obtain instead the incorrect result

$$\lim_{N=\infty}\frac{\sin[\pi(\nu-\nu_\beta)(N+1)]}{\sin\pi(\nu-\nu_\beta)} \overset{\nu\sim\nu_\beta}{=} \delta(\nu-\nu_\beta)\ , \tag{5.6}$$

which is twice as big.

where $\mathcal{P}$ represents the principal value. The displacement at the collimator after N turns becomes

$$\vec{X}_{_N} \approx \frac{i}{4}\sqrt{\beta_k}\,(\Delta x')_k\, f(\nu_\beta)\; e^{-i2\pi\nu_\beta N - i\psi_0 - i\phi}\;. \tag{5.8}$$

For a flat distribution

$$f(\nu) = \begin{cases} \dfrac{1}{2\Delta\nu} & |\nu - \nu_0| < \Delta\nu \\ 0 & \text{otherwise}\,, \end{cases} \tag{5.9}$$

the accumulated particle displacement is bounded by

$$\bigl|x_c\bigr| \approx \sqrt{\beta_k\beta_c}\,(\Delta x')_k \frac{1}{8\Delta\nu}\;. \tag{5.10}$$

In the case of a Gaussian distribution with rms spread σ_ν, $f(\nu)$ is given by Eq. (3.1). The asymptotic bound is therefore

$$\bigl|x_c\bigr| \approx \frac{1}{4}\sqrt{\beta_k\beta_c}\,(\Delta x')_k \frac{1}{\sqrt{2\pi}\sigma_\nu} e^{-(\nu_\beta-\nu_0)^2/(2\sigma_\nu^2)} = \sqrt{\beta_k\beta_c}\,\varepsilon_k\, e^{-a^2}\;, \tag{5.11}$$

where the last result has been written in terms of the kicker strength ε_k defined in Eq. (3.7) and the normalized tune offset a defined in Eq. (3.5). We obtain similar conclusion as in the previous section. The presence of a tune spread in the kicker limits the accumulated growth of the particle displacement. For the simple case when $\nu_\beta = \nu_0$, the growth of the displacement is initially linear in turn number and becomes saturated when $N \sim 1/\Delta\nu$. Thus although large spread in the kicker tune will encompass a wider range of betatron tune in the particle bunch, however, it leads to early saturation and the saturated particle displacement will be small. As a result, the tune spread in the kicker should be chosen to be roughly the betatron tune spread in the particle bunch.

The result of Eq. (5.11) should be compared with Eq. (3.8), the second line of Eq. (3.9), and Eq. (3.13). We see that the correct asymptotic bound in Eq. (5.11) has been reproduced.

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