



**Fermi National Accelerator Laboratory**

**FERMILAB-FN-617**

## **Adiabatic Trapping Due to Current Ripple**

King-Yuen Ng

*Fermi National Accelerator Laboratory  
P.O. Box 500, Batavia, Illinois 60510*

January 1994

Presented at the *6th Advanced ICFA Beam Dynamics Workshop*, Funchal, Portugal, October 25-30, 1993

## **.. Disclaimer**

*This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.*

# ADIABATIC TRAPPING DUE TO CURRENT RIPPLE

King-Yuen Ng

*Fermi National Accelerator Laboratory,\* P.O. Box 500, Batavia, IL 60510*

## I. INTRODUCTION

Current ripple in the dipoles will perturb the longitudinal phase space when the ripple frequency, usually  $f_m = 60$  Hz, is close to the synchrotron frequency  $f_{s0}$ , creating an island system of order one. The increase in the longitudinal emittance of the bunch depends very critically on how fast the synchrotron frequency is changing. If the synchrotron frequency changes very rapidly, the emittance growth may not be big. On the other extreme, the synchrotron frequency can ramp through 60 Hz so slowly that the beam particles will be trapped inside the outer island. As the synchrotron frequency is increased, the outer island moves farther and farther away from the origin of the phase space and the trapped particles are carried to the boundary of the rf bucket leading eventually to beam loss. This mechanism is called adiabatic trapping, and has been studied briefly in a previous paper. [1] Here, we shall derive the condition for adiabatic trapping and verify it via simulations.

## II. ISLAND STRUCTURE

The detailed derivation of the island structure of the modulated system has been given in Ref. 1. The following is just a brief recap. Let us start from the Hamiltonian of a driven pendulum

$$H = \frac{1}{2}\nu_{s0}\delta^2 + \nu_{s0}(1 - \cos\varphi) + a\delta\nu_{s0}\sin\nu_m\theta. \quad (2.1)$$

In the language of the longitudinal phase space,  $\varphi$  and  $\delta$  denote the phase and *normalized* fractional momentum spread of a particle in the rf bucket with unperturbed synchrotron tune  $\nu_{s0}$ . The driving force has a dimensionless amplitude  $a$  and tune  $\nu_m = f_m/f_0$ , where  $f_0$  is the revolution frequency. In terms of the action-angle variables  $J$ - $\psi$ , the Hamiltonian becomes

$$H = \int_0^J \nu_s(J)dJ + 2ak\nu_{s0}\text{cn}(u|k)\sin\nu_m\theta, \quad (2.2)$$

where  $u = \left(\psi + \frac{\pi}{2}\right)\nu_{s0}/\nu_s(J)$ ,  $k = \sin\frac{\varphi_0}{2}$ ,  $\varphi_0$  is the maximum amplitude of oscillation, and  $\text{cn}(u|k)$  is a Jacobian elliptic function. Expanding the synchrotron tune  $\nu_s(J)$  and the elliptic function for small  $J$  and retaining only the lowest-order resonance, the Hamiltonian becomes

$$H = \nu_{s0}\left(J - \frac{J^2}{16}\right) - \frac{1}{2}a\nu_{s0}(2J)^{1/2}\cos(\psi - \nu_m\theta). \quad (2.3)$$

Transforming to a rotating frame of frequency  $f_m$ , the Hamiltonian

$$H = \nu_{s0}\left(xJ - \frac{J^2}{16} - \frac{a}{2}\cos\psi\right), \quad (2.4)$$

---

\*Operated by the Universities Research Association, Inc., under contract with the U.S. Department of Energy.

becomes “time” independent. For the sake of convenience, we have used the same notations  $J$ - $\psi$  for the canonical variables in the rotating frame, and defined

$$x = 1 - \frac{\nu_m}{\nu_{s0}} \quad (2.5)$$

to take care of the driving frequency. Bifurcation occurs when

$$x_{bf} = \frac{3}{16}(4a)^{2/3} \quad \text{and} \quad g_{bf} = (4a)^{1/3}, \quad (2.6)$$

where the phase amplitude is defined as  $g = \sqrt{2J}$ . We *normalize* the variables with respect to their bifurcation values:

$$\frac{x}{x_{bf}} \rightarrow \hat{x} \quad \text{and} \quad \frac{g}{g_{bf}} \rightarrow \hat{g}, \quad (2.7)$$

thus scaling away the driving amplitude  $a$ , so that the equation for fixed points

$$\hat{g}^3 - 3\hat{x}\hat{g} + 2 = 0, \quad (2.8)$$

depends on the variable  $\hat{x}$  only. When the driving frequency is below bifurcation ( $\hat{x} > 1$ ), the solutions can be readily written as [2]

$$\begin{aligned} \hat{g}_a(\hat{x}) &= -2\hat{x}^{1/2} \cos \frac{\xi}{3}, \\ \hat{g}_b(\hat{x}) &= 2\hat{x}^{1/2} \sin \left( \frac{\pi}{6} - \frac{\xi}{3} \right), \\ \hat{g}_c(\hat{x}) &= 2\hat{x}^{1/2} \sin \left( \frac{\pi}{6} + \frac{\xi}{3} \right), \end{aligned} \quad (2.9)$$

where  $\xi = \tan^{-1} \sqrt{\hat{x}^3 - 1}$ . Above bifurcation ( $\hat{x} < 1$ ), there is only one real solution:

$$\hat{g}_a(\hat{x}) = \begin{cases} -2\hat{x}^{1/2} \cosh \frac{\xi}{3} & 0 \leq \hat{x} < 1, \\ -2|\hat{x}|^{1/2} \cosh \frac{\xi}{3} & \hat{x} \leq 0, \end{cases} \quad (2.10)$$

where

$$\xi = \begin{cases} \tanh^{-1}(1 - \hat{x}^3)^{1/2} & 0 \leq \hat{x} < 1, \\ \tanh^{-1}(1 - \hat{x}^3)^{-1/2} & \hat{x} \leq 0. \end{cases} \quad (2.11)$$

The locations of the fixed points together with the separatrices in the longitudinal phase space are shown in Fig. 1 when the perturbing frequency is below and right at bifurcation.

### III. ADIABATIC TRAPPING

When the synchrotron tune  $\nu_{s0}$  is so small that the system is away from bifurcation ( $\hat{x} < 1$ ), particles revolve around the only stable fixed point  $a$  which is close to the center of the bucket. As  $\nu_{s0}$  increases ( $\hat{x}$  increases), this fixed point moves to the right as is indicated in Fig. 1. The normalized amplitude  $|\hat{g}_a|$  is plotted in Fig. 2. The rate at which  $|\hat{g}_a|$  increases is

$$\frac{d|\hat{g}_a|}{d\theta} = \frac{d|\hat{g}_a|}{d\hat{x}} \frac{d\hat{x}}{d\theta}. \quad (3.1)$$

We have from Eqs. (2.5) and (2.7),

$$\frac{d\hat{x}}{d\theta} = \frac{\nu_m}{x_{bf}\nu_{s0}^2} \frac{d\nu_{s0}}{d\theta}. \quad (3.2)$$

Figure 2 shows a plot of  $d|\hat{g}_a|/d\hat{x}$  having a maximum at  $\hat{x} = 0$  when  $\nu_{s0} = \nu_m$ . In fact, from Eqs. (2.9) and (2.10), one obtains near  $\hat{x} = 0$ ,

$$\frac{d|\hat{g}_a|}{d\hat{x}} \approx \frac{1}{2^{1/3}} \left( 1 - \frac{\hat{x}^2}{2^{4/3}} \right). \quad (3.3)$$

On the other hand, the rate of change of the amplitude  $\hat{g}$  of the particle due to the resonance can be obtained from Eq. (2.4) and has a maximum of

$$\frac{d|\hat{g}|}{d\theta} = \frac{1}{2} a \nu_{s0}, \quad (3.4)$$

when  $\sin \psi = \pm 1$ ; i.e., when the particle is moving in the direction midway between the stable fixed point  $a$  and the unstable fixed point  $c$ .

Adiabatic trapping will occur when the amplitude-change rate of the center of the outer island due to the slow variation of  $\nu_{s0}$  is less than the amplitude-change rate caused by the motion around the resonance, [3, 4] or, by combining Eqs. (3.1), (3.2), and (3.4),

$$\frac{\nu_m}{x_{bf}\nu_{s0}^2} \frac{d\nu_{s0}}{d\theta} \frac{d|\hat{g}_a|}{d\hat{x}} \Big|_{\max} < \frac{1}{2} a \nu_{s0}, \quad (3.5)$$

Putting in the maximum for  $d|\hat{g}_a|/d\hat{x}$  and letting  $\nu_{s0} = \nu_m$ , we arrive at the simple criterion

$$\frac{d\nu_{s0}}{d\theta} < \frac{3}{16} a^{4/3} \nu_m^2. \quad (3.6)$$

### IV. APPLICATION TO CURRENT RIPPLE

Let  $\epsilon$  be the fractional level of the 60 Hz ripple. This leads to a deflection of the beam by

$$\hat{\theta} = \frac{\hat{B}\ell}{B\rho} = 2\pi\epsilon, \quad (4.1)$$

for *all* the dipoles, where  $\hat{B}$  is the peak value of the ripple magnetic field and  $\ell$  is the *total* length of all the dipoles. For one turn, the beam has an rf phase lag

$$\Delta\varphi = \frac{2\pi h\Delta C}{C} = \frac{4\pi^2 h D \epsilon}{C}, \quad (4.2)$$

where  $h$  is the rf harmonic,  $D$  is the dispersion at the dipole, and  $C$  is the length of the ideal orbit. The dimensionless driving amplitude is then [5]

$$a = \frac{\Delta\varphi}{2\pi\nu_{s0}} = \frac{2\pi h D \epsilon}{\nu_{s0} C}. \quad (4.3)$$

The circumferential length of the Relativistic Heavy Ion Collider (RHIC) is  $C = 3833.84$  m, and  $\nu_{s0}$  is changing at the rate of  $2.45 \times 10^{-9}$  per turn in the vicinity of  $f_{s0} = 60$  Hz or ( $\nu_{s0} = 7.67 \times 10^{-4}$ ). With  $h = 342$  and  $D \sim 1$  m, Eq. (3.6) says that there will be adiabatic trapping when the current ripple  $\epsilon > 2.0 \times 10^{-5}$ . A simulation was performed for a particle initially at the center of the rf bucket with its location plotted for every perturbation period (or  $1/\nu_m$  turns). The results in Fig. 3 show that there is adiabatic trapping when  $\epsilon \geq 1.68 \times 10^{-5}$ , which is not far from the theoretical value.

Adiabatic trapping can become very serious for RHIC due to its extremely slow ramping rate, for example,  $\dot{\gamma} = 1.6 \text{ sec}^{-1}$  at transition, although the trapping mechanism usually takes a very long time to develop. Markers 0, 1, 2, 3, and 4 in Fig. 3 denote the instant when  $\hat{x} = 0$ , 1, 5, 10, 15, and 20, respectively, or  $\nu_{s0}/\nu_m = 1, 1.0256, 1.144, 1.336$ , and  $1.607$ . From  $\hat{x} = 0$ , it takes 0.10, 0.56, 1.30, and 2.37 sec. to reach Marker 1, 2, 3, 4. For other machines, usually the ramping rate is much faster. However, we may still expect a rather large emittance blowup when the adiabatic trapping criterion is met.

The synchrotron frequency as a function of momentum or energy during ramping is shown in Fig. 4 for the Fermilab Main Ring and the Tevatron. For both machines, the synchrotron frequency is ramped up extremely fast at injection and therefore adiabatic trapping will not be a problem.

For the Tevatron, the synchrotron frequency drops through 60 Hz around 360 GeV. We would like to ask the question: Will adiabatic trapping occur so that the bunch emittance becomes smaller eventually? The answer is no. Figure 3(a) is a plot of the position of a particle for every perturbation period starting from some particular turn or in a frame rotating with the frequency of 60 Hz. We discover that the tracked particle moves to the right side of the bucket during adiabatic trapping. In other words, only some particles in a bunch will be trapped in the outer island and carried outward in some direction only, while some particles will remain untrapped. However, the bunch emittance will be increased. In the same way, when the synchrotron frequency is reduced slowly, only some particles will be trapped and move inward towards the center of the bucket and some particles will not be trapped. For this reason, there will not be a reduction of bunch emittance.

## V. CRITERION VERIFICATION

We would like to verify the adiabatic trapping criterion of Eq. (3.6) by simulation. The situation of RHIC was used. We first test the dependency of  $d\nu_{s0}/d\theta$  on  $\nu_m$  by holding the

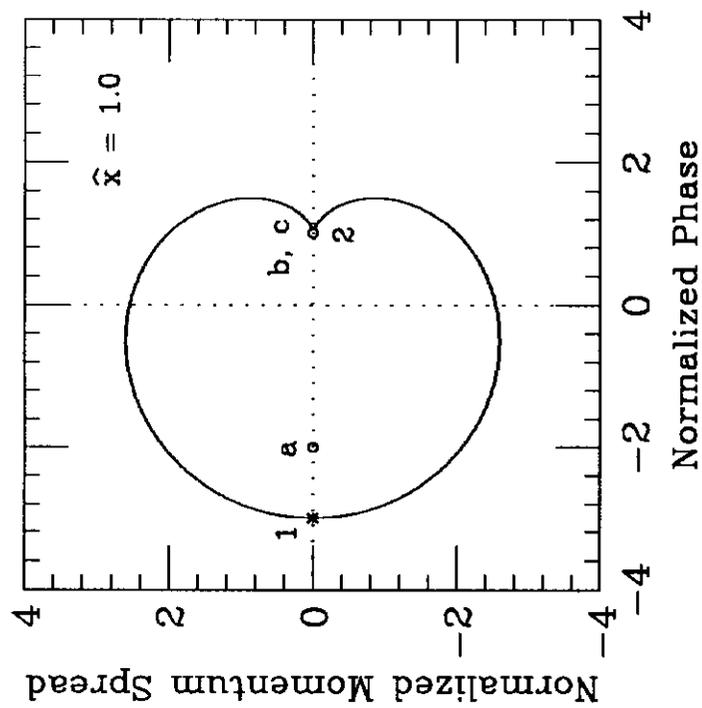
amplitude of the current ripple fixed at  $\epsilon = 1.468 \times 10^{-5}$ . The simulation result is plotted in Fig. 5, along with the theoretical values. It is clear that the  $\nu_m^2$  dependency is nicely verified, although the simulated  $d\nu_{s0}/d\theta$  points are slightly larger than the theoretical values.

We next hold the tune of the current ripple fixed at  $\nu_m = 7.67 \times 10^{-4}$  and vary the amplitude  $\epsilon$ . The simulation results in Fig. 6 are close to the theoretical. But it appears that  $d\nu_{s0}/d\theta$  depends on  $\epsilon^{1.41}$  instead of  $\epsilon^{4/3}$  as predicted by Eq. (3.6).

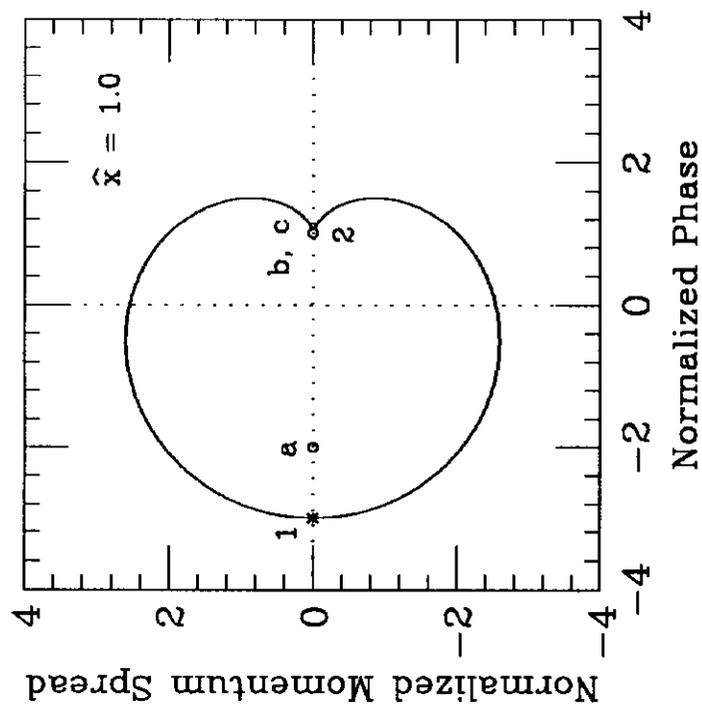
These tests suggest that the criterion given by Eq. (3.6) is qualitatively correct. At least it will tell us the correct order of magnitude.

## References

- [1] K.Y. Ng, *Longitudinal Emittance Increase due to Sinusoidal Perturbation*, Fermilab Internal Report FN-604, 1993.
- [2] There are some errors in the analytical expressions of the fixed points in Ref. 1.
- [3] A.W. Chao and M. Month, *Particle Trapping during Passage through a High-order Non-linear Resonance*, Nucl. Inst. Meth. **121**, 129 (1974)
- [4] L.R. Evans, *The Beam-beam Interaction*, CERN Accelerator School, Geneva 1984, CERN 84-15 (1984).
- [5] The dimensionless driving amplitude in Eq. (2.1) is defined as  $a = hD\hat{\theta}/(\nu_{s0}C)$ . See Ref. 1.



(a)



(b)

Figure 1: Fixed points and separatrices in the normalized longitudinal phase space when the driving frequency is (a) below bifurcation and (b) right at bifurcation.

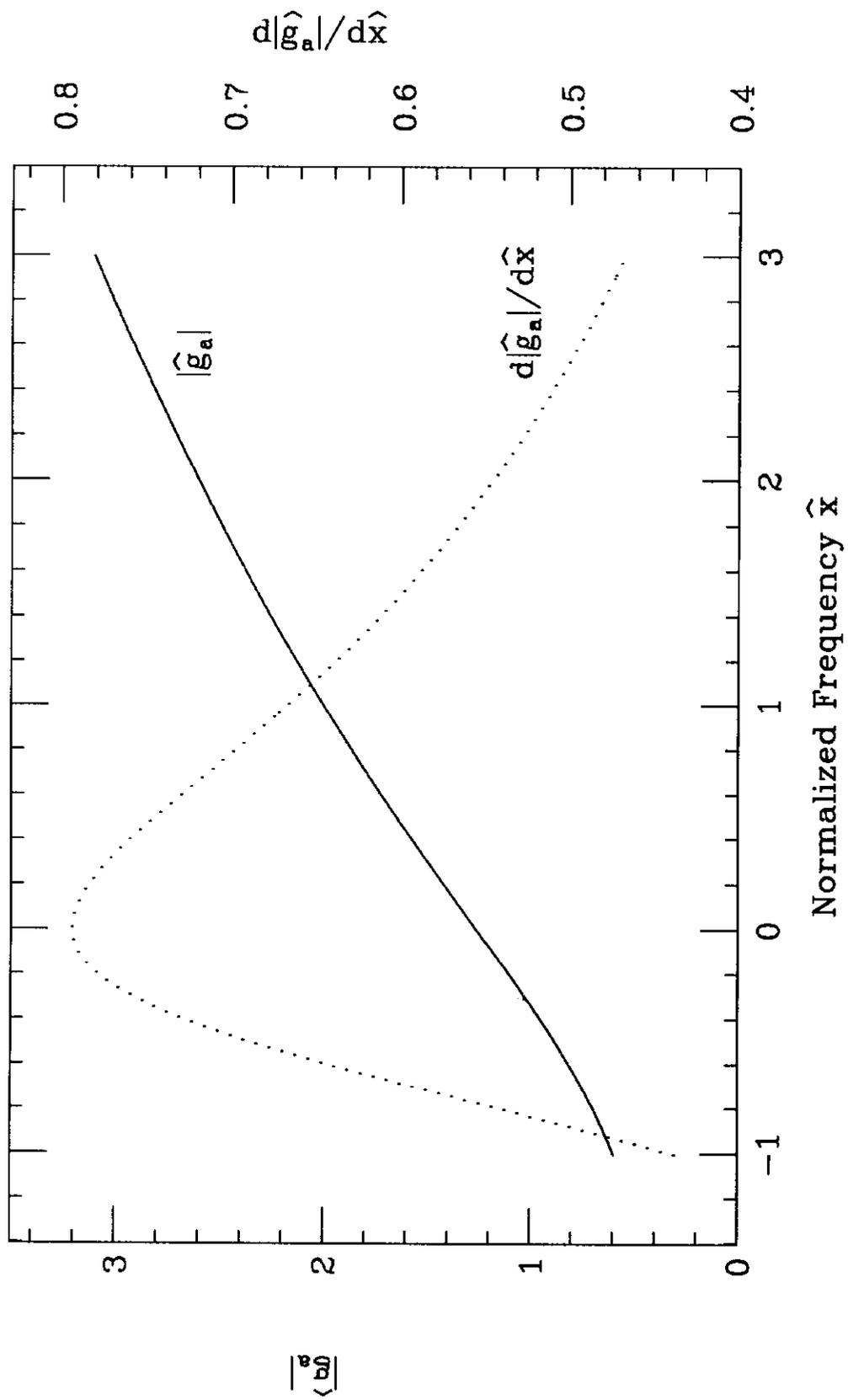


Figure 2. Plot of rate of movement of stable fixed point vs normalized frequency.

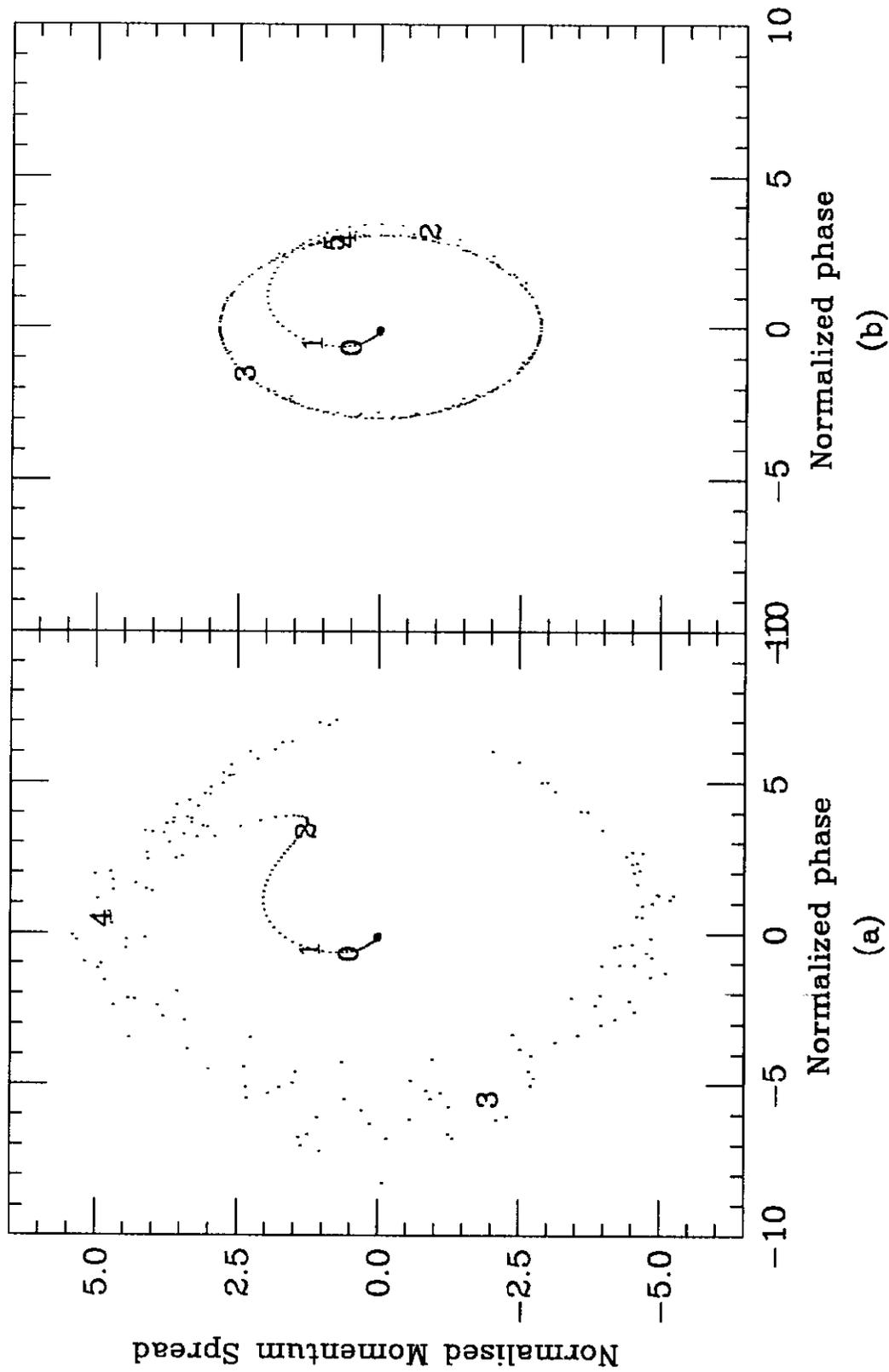


Figure 3. Adiabatic trapping occurs when (a) current ripple amplitude  $\epsilon = 1.68 \times 10^{-5}$  and no trapping when (b)  $\epsilon = 1.67 \times 10^{-5}$ .

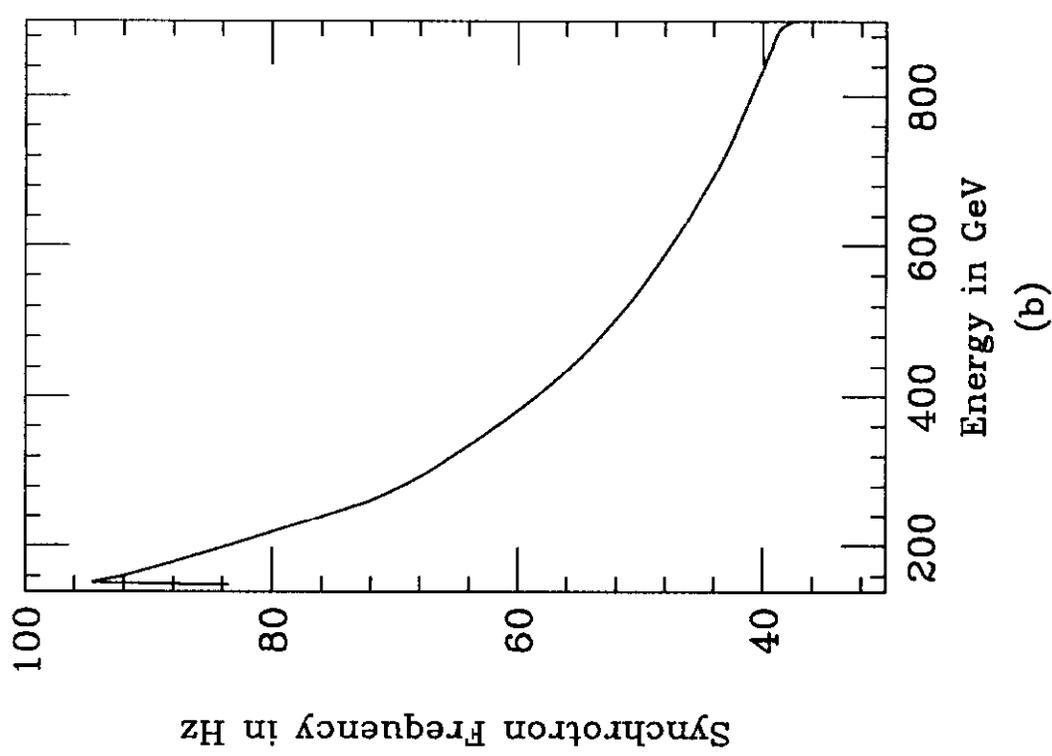
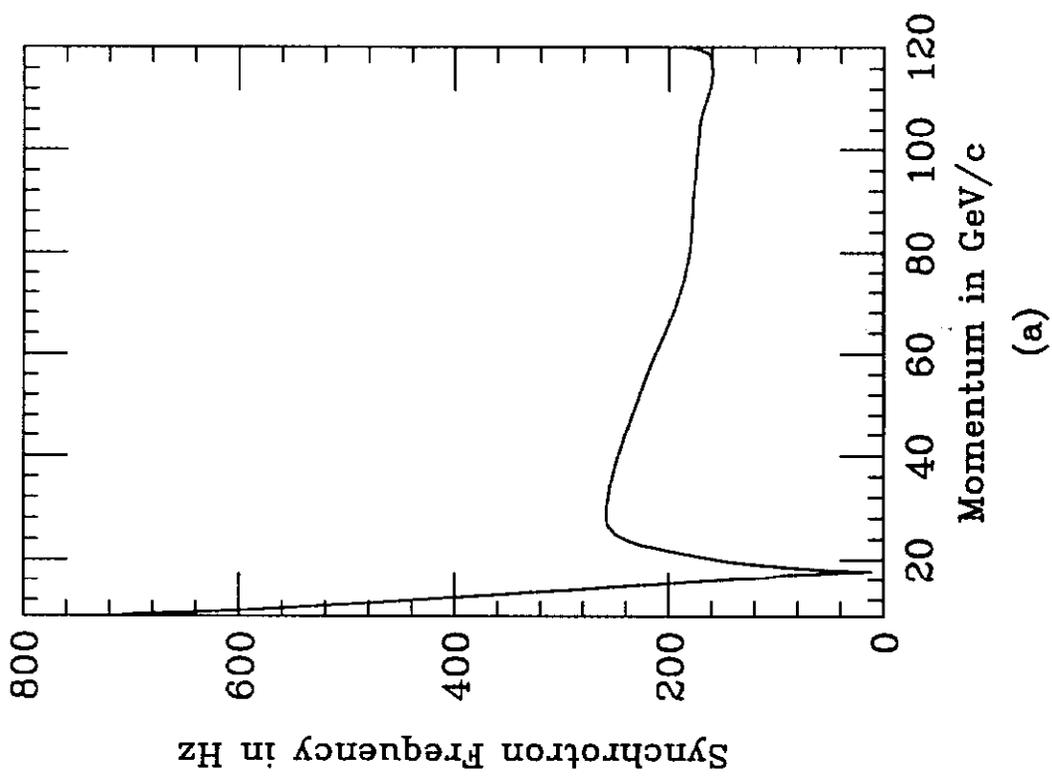


Figure 4. Synchrotron frequency as a function momentum or energy for the ramp of (a) Fermilab Main Ring and (b) Tevatron.

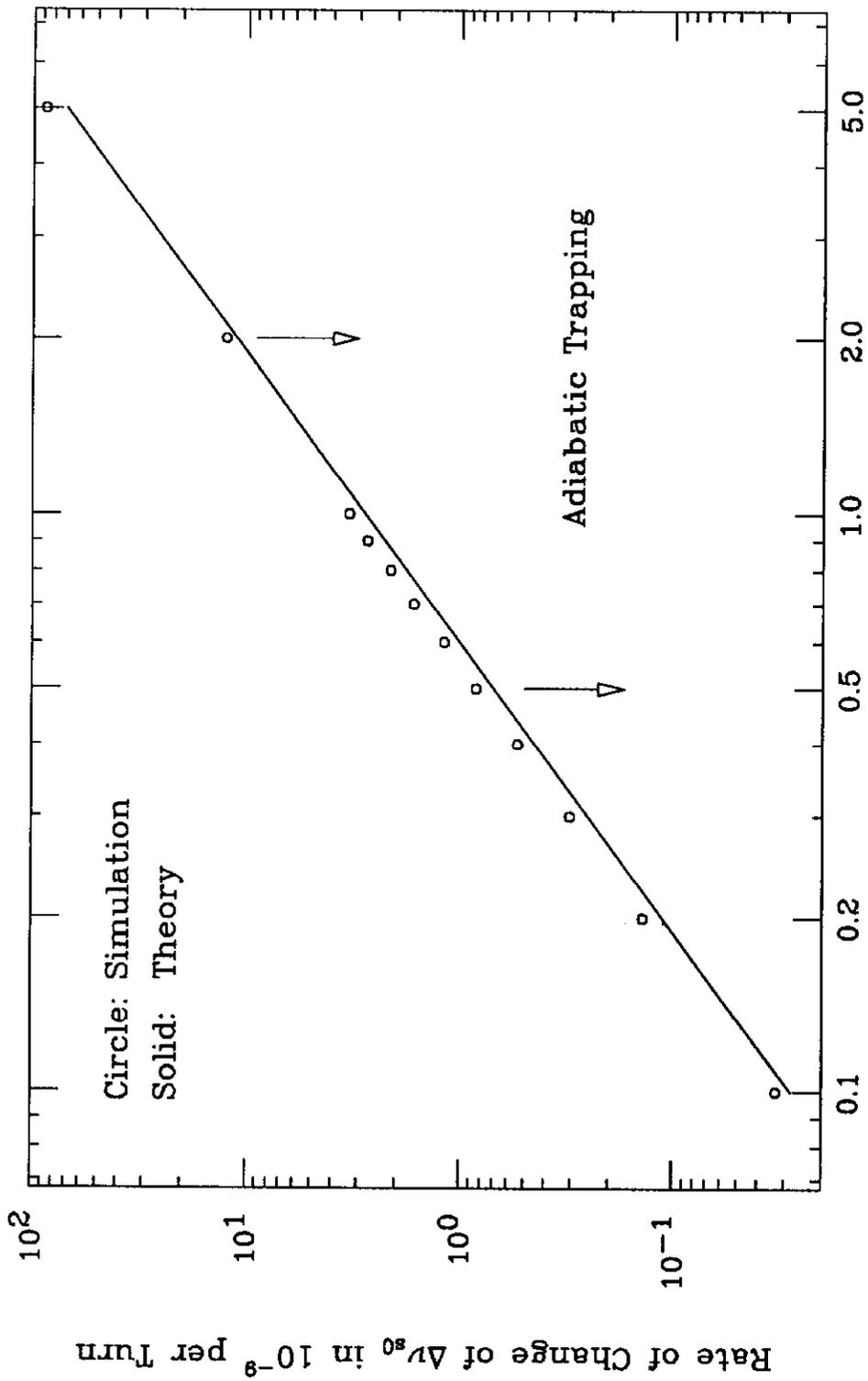


Figure 5. Rate of increase in synchrotron tune as a function of current-ripple tune, below which adiabatic trapping occurs.

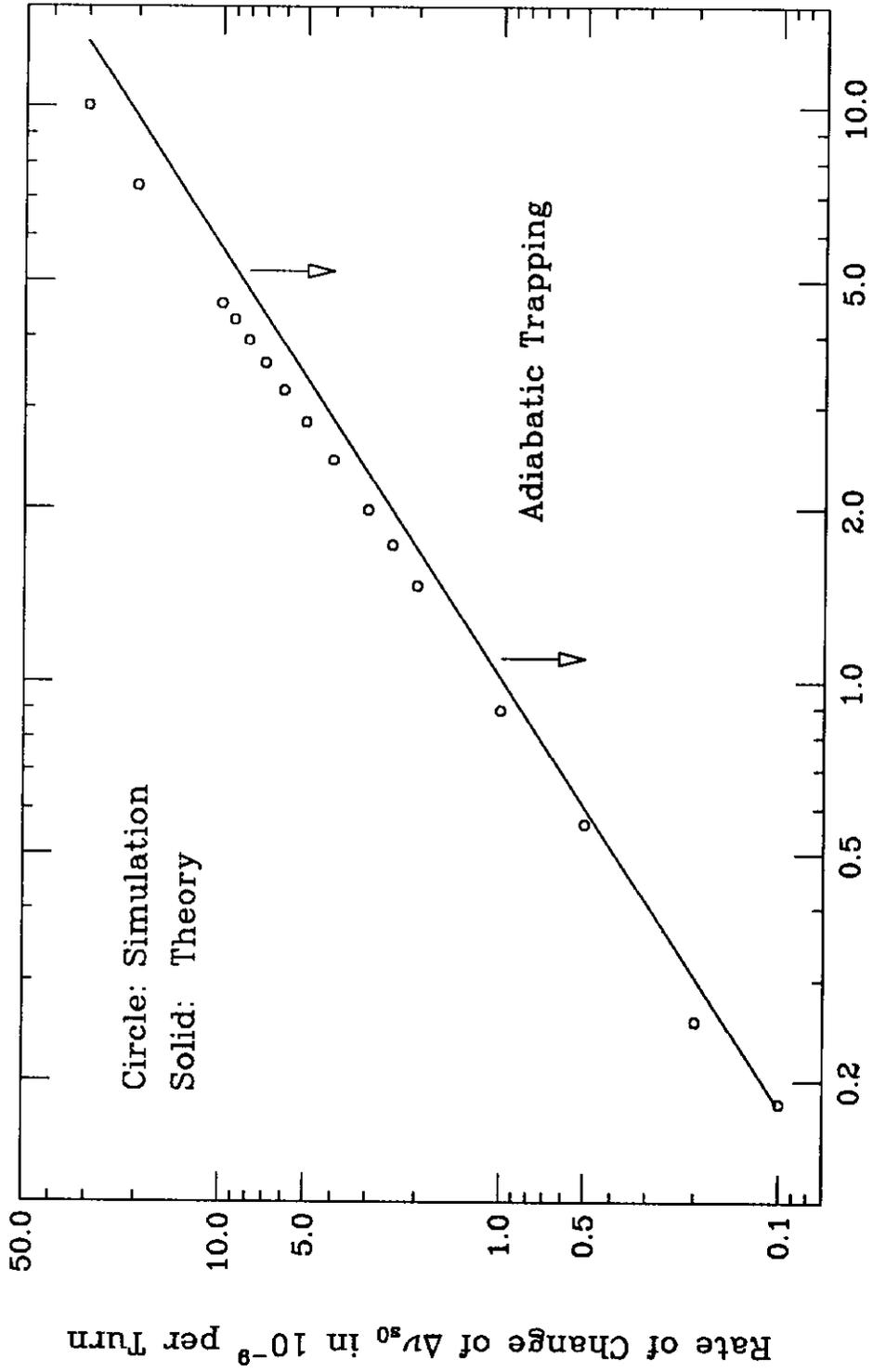


Figure 6. Rate of increase in synchrotron tune as a function of current-ripple amplitude, below which adiabatic trapping occurs.