Fermi National Accelerator Laboratory

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Longitudinal Beam Motion Due to Ground Motion

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I. Introduction

The ground motions at the proposed SSC tunnel due to passing trains, quarry blasts, or ambient noise have frequencies lower than 10 Hz. On the other hand, the synchrotron frequency of the SSC is $f_s = 4$ Hz. Therefore, the quadrupole motion due to ground motions can be in resonance with the synchrotron oscillation leading to a continuous growth of the bunch area and thus limiting the lifetime of the beam storage. Here, we are going to emphasize perturbations with *long correlation time* or *nonrandom* and low frequencies. Some of the on-site ground-motion measurements[1] do show that the waveforms of ground displacement due to train crossing and quarry blasts are quite periodic with definite frequencies. The investigation in this note follows a preliminary study by Rossbach.[2]

II. The Kick and Equations of Motion

Assume the horizontal kick $A(f_m)$ on a quadrupole of focal length F for the j-th turn has the form

$$\theta_j(f_m) = \frac{A(f_m)}{F}\sin(2\pi\nu_m j + \xi) \ , \tag{2.1}$$

where $\nu_m = f_m/f_0$ and f_m and f_0 are, respectively, a frequency of the ground motion, and the revolution frequency of the storage ring. The phase ξ is a parameter of a particular ensemble. Since we want the quadrupole to receive a small incremental kick when the ground wave arrives, we need to put $\xi = 0$, otherwise the first kick the quadrupole experience will be proportional to $\sin\xi$.

The incremental kick from the $(j-1)$ to j-th turn is

$$\Delta\theta_j = \theta_j - \theta_{j-1} \ , \tag{2.2}$$

leading to a change of orbit length (derivation given in Appendix)

$$\Delta C_j = \Delta\theta_j D \ , \tag{2.3}$$

where D is the horizontal dispersion function at the quadrupole. As a result, the synchronous particle will arrive at the rf cavity late by the rf angle

$$\Delta\psi_j = 2\pi\frac{\Delta C_j}{\lambda_{rf}} = \frac{2\pi h D}{C}\Delta\theta_j \ , \tag{2.4}$$

where $\lambda_{rf} = C/h$ is the rf wavelength with C equal to the ideal orbit length and h the rf harmonic. In order to understand the mechanism clearly, we want to point out that even if the quadrupole experiences the same kick $\Delta\theta_j$ every turn (without any turn-by-turn increment), the synchronous particle will always have an orbit length longer than the ideal one by ΔC_j as given by Eq. (2.3), so that the rf phase lag $\Delta\psi_j$ given by Eq. (2.4) will accumulate turn by turn in such a way that the total rf phase lag will become $N\Delta\psi_j$ in N turns. However, as is shown in the next section, this constant kick will not throw the synchronous particle out of the rf bucket, as one might naively expect.

The turn-by-turn equations of motion for the rf phase ϕ_j and fractional energy offset δ_j at the j-th turn are given by

$$\begin{aligned} \frac{d\phi_j}{dn} &= 2\pi\eta h\delta_j + \psi_j \ , \\ \frac{d\delta_j}{dn} &= -\frac{eV}{E}\phi_j \ , \end{aligned} \tag{2.5}$$

where η is the phase-slip parameter, V the rf voltage (for a stationary bucket), and E the synchronous energy. In the above, the accumulated rf phase lag or mismatch due to the kick is

$$\psi_j = \sum_{i=1}^{j} \Delta\psi_j = \frac{2\pi hD}{C}\theta_j \ . \tag{2.6}$$

We want to point out that it is ψ_j and *not* $\Delta\psi_j$ that enters into the first equation of motion.

IV. Effect of a Constant Kick

In order to have a clear picture, we first study the situation of a constant turn-by-turn kick θ. This just implies a new closed orbit with an extra length $\Delta C = \theta D$, or for each turn there is an increase in rf phase lag given by

$$\psi = \frac{2\pi hD}{C}\theta \ . \tag{3.1}$$

For simplicity, we use time t as the continuous variable. Then the equations of motion transform into

$$\begin{aligned} \frac{d\phi}{dt} &= \eta h\omega_0\delta + \frac{\omega_0}{2\pi}\psi \ , \\ \frac{d\delta}{dt} &= -\frac{eV\omega_0}{2\pi E}\phi \ , \end{aligned} \tag{3.2}$$

where $\omega_0/2\pi$ is the revolution frequency. We want to study the trajectory of the synchronous particle. Its initial condition is $\phi = \psi$ and $\delta = 0$ at $t = 0$ just after seeing the first kick. The equations can be easily solved by a Laplace transform. The solution is

$$\begin{aligned} \phi(t) &= \frac{\psi}{2\pi\nu_s}\sqrt{1+(2\pi\nu_s)^2}\sin(\omega_s t+\chi) \ , \\ \delta(t) &= \frac{\psi}{2\pi\eta h}+\frac{\psi}{2\pi\eta h}\sqrt{1+(2\pi\nu_s)^2}\cos(\omega_s t+\chi) \ , \end{aligned} \tag{3.3}$$

where the synchrotron tune is given by

$$\nu_s = \sqrt{\frac{\eta h e V}{2\pi E}} \ , \tag{3.4}$$

$\omega_s = \nu_s \omega_0$, and

$$\tan\chi = 2\pi\nu_s \ . \tag{3.5}$$

The trajectory is therefore an ellipse passing through the point $(\phi,\delta) = (\psi, 0)$ with center O' at

$$\phi_c = 0 \qquad \text{and} \qquad \delta_c = -\frac{\psi}{2\pi\eta h} \ , \tag{3.6}$$

as shown in Figure 1.

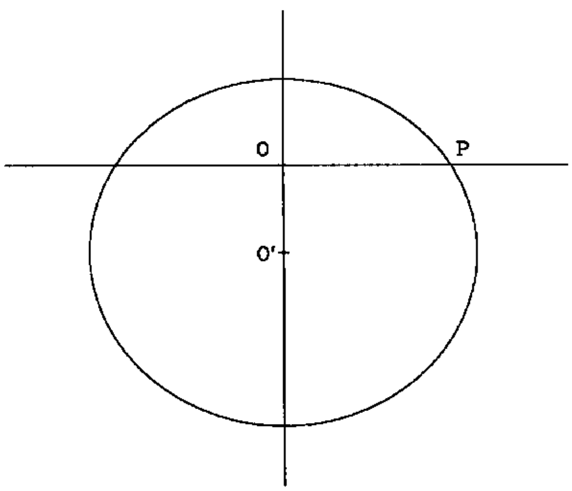

Fig. 1. Trajectory of a particle in an rf bucket, under a constant turn-by-turn longitudinal kick.

In other words, we have a new synchronous center O', which is shifted to a lower energy. This shift is required so that this lower-energy new synchronous particle has a shorter orbit length (without the kick) plus the lengthening of the orbit (due to the kick), which totals up to the original designed ideal orbit length in order that it will be exactly in phase with the rf voltage turn by turn. In fact, to decrease the orbit length by

$$\Delta C = \theta D \ , \tag{3.7}$$

the energy required to be lowered, $\Delta\delta$, is given by

$$\eta C \Delta\delta = \Delta C \ . \tag{3.8}$$

Combining Eqs. (3.1), (3.7), and (3.8), we get

$$\Delta\delta = \frac{\theta D}{\eta C} = \frac{\psi}{2\pi\eta h} \ , \tag{3.9}$$

which agrees with Eq. (3.6).

This simple model shows that a cumulative rf phase mismatch will not send the particle out of the bucket. The bucket chooses a new synchronous center instead. If we have a bunch at the *original* synchronous center O, the constant turn-by-turn kick will send the bunch to revolve around a new center O'. The bunch area will only increase through smearing due to synchronous tune dependence on amplitude.

IV. The Periodic Kick

We now want to solve

$$\begin{aligned} \frac{d\phi}{dt} &= \eta h \omega_0 \delta + \frac{\omega_0 h D A}{CF} \sin(\omega_m t) \ , \\ \frac{d\delta}{dt} &= -\frac{eV\omega_0}{2\pi E}\phi \ , \end{aligned} \tag{4.1}$$

with the initial condition $\phi = \phi_0$ and $\delta = \delta_0$ at $t = 0$. We introduce the new variable

$$\bar{\phi} = \frac{eV}{2\pi E \nu_s}\phi = \frac{\nu_s}{\eta h}\phi \ , \tag{4.2}$$

so that the elliptic trajectory becomes circular. Using Laplace transform, we obtain the solution

$$\begin{aligned} \bar{\phi}(t) &= \bar{\phi}_0 \cos\omega_s t + \delta_0 \sin\omega_s t + f(t) \ , \\ \delta(t) &= -\bar{\phi}_0 \sin\omega_s t + \delta_0 \cos\omega_s t + g(t) \ , \end{aligned} \tag{4.3}$$

where the particular solution is given by

$$\begin{aligned} f(t) &= \frac{\Omega^2}{\omega_m^2 - \omega_s^2}\frac{\omega_m}{\omega_s}\Big(\cos\omega_s t - \cos\omega_m t\Big) \ , \\ g(t) &= -\frac{\Omega^2}{\omega_m^2 - \omega_s^2}\left(\frac{\omega_m}{\omega_s}\sin\omega_s t - \sin\omega_m t\right) \ , \end{aligned} \tag{4.4}$$

with

$$\Omega^2 = \frac{eVhDA\omega_0^2}{ECF} = \frac{2\pi DA\omega_s^2}{\eta CF} \ . \tag{4.5}$$

V. Discrete Ground-Wave Frequency

We first consider the situation when the ground-wave frequency f_m is discrete and is very near to but different from the synchrotron frequency f_s. The particular solution of Eq. (4.4) can be approximately rewritten as a function of turn number N as:

$$\begin{aligned} f(N) &\approx \frac{\nu_\Omega^2}{\nu_s}\frac{\sin\pi(\nu_m - \nu_s)N}{\nu_m - \nu_s}\sin(2\pi\nu_s N) \ , \\ g(N) &\approx \frac{\nu_\Omega^2}{\nu_s}\frac{\sin\pi(\nu_m - \nu_s)N}{\nu_m - \nu_s}\cos(2\pi\nu_s N) - \frac{\nu_\Omega^2}{\nu_s}\frac{\cos\pi(\nu_m - \nu_s)N}{2\nu_s}\sin(2\pi\nu_s N) \ , \end{aligned} \tag{5.1}$$

where $\nu_\Omega = \Omega/\omega_0$. In complex notation, this is

$$(\bar{\phi}, \delta) = a_0 e^{-i2\pi\nu_s N} + \frac{i\nu_\Omega^2}{\nu_s}\frac{\sin\pi(\nu_m - \nu_s)N}{\nu_m - \nu_s}e^{-i2\pi\nu_s N} - i\frac{\nu_\Omega^2}{\nu_s}\frac{\cos\pi(\nu_m - \nu_s)N}{2\nu_s}\sin(2\pi\nu_s N) \ . \tag{5.2}$$

The a_0 in the first term represents the initial position of the particle (or particles forming a bunch). The second term represents the bunch being shifted up and down the energy axis by the amount

$$\Delta\delta = \frac{i\nu_\Omega^2}{\nu_s}\left|\frac{\sin\pi(\nu_m - \nu_s)N}{\nu_m - \nu_s}\right| \ . \tag{5.3}$$

With this term, the whole bunch is unchanged in shape and rotating at the rate ν_s about the center of the phase space. The third term shifts the center of rotation upward and downward, which is equivalent to the shift of synchronous center in the case of a constant kick discussed in Sect. 3. This term is small compared with the second term, which can become very big depending on how close ν_m and ν_s are. Since $D \approx \eta C$, we have from Eqs. (4.5) and (5.3)

$$\Delta\delta \approx \frac{2\pi A}{F} \frac{f_s}{|f_m - f_s|} = 1.56 \times 10^{-7} \ , \tag{5.4}$$

if we take $A = 0.05\ \mu$, $F = 80.8$ m, $f_s = 4$ Hz, and $|f_m - f_s| = 0.1$ Hz. The corresponding shift in rf phase is, from Eq. (4.2),

$$\Delta\phi = \frac{\eta h}{\nu_s} \Delta\delta = 1.54 \times 10^{-3} \text{ rad} \ , \tag{5.5}$$

where we have taken $\eta = 1.1 \times 10^{-4}$, and $h = 1.05 \times 10^5$. For $n = 1000$ quadrupoles kicking independently by the ground wave, we need to multiply the results of Eqs. (5.4) and (5.5) by $\sqrt{n}$. This gives $\Delta\phi = 0.049$ rad, which is still very small. We see that the periodic kicking just introduces more complicated motion inside the bucket: besides moving the center of rotation, the whole bunch is shifted upwards and downwards. However, coherent motion will not throw that particle outside the bucket. It requires synchrotron tune dependence on amplitude to smear out the phase space and lead to eventual filling up the whole bucket.

Of course, if the driving discrete frequency is right at $\omega_m = \omega_s$, the shifting of the particle in Eqs. (5.1) and (5.2) will be directly proportional to the turn number N, implying that the particle will be thrown out of the rf bucket in a matter of time. However, this situation is very unlikely, because the general ground motion has a continuous spectrum. Even if there is a discrete driving force, it is hard to imagine how its frequency will be locked on to the synchrotron frequency.

VI. Continuous Ground-Wave Frequency

The ground waves that were measured have a continuous spectrum of narrow width but covering the synchrotron frequency of $f_s = 4$ Hz. If we denote the spectral distribution by $\mu(\omega_m)$ which is normalized to

$$\int_0^\infty \mu(\omega_m) d\omega_m = 1 \ , \tag{6.1}$$

the convolutions with $f(\omega_m)$ and $g(\omega_m)$ give

$$
\begin{aligned}
f(N) &\approx \frac{\pi\Omega^2}{2\omega_s}\mu(\omega_s)\sin 2\pi\nu_s N \ , \\
g(N) &\approx \frac{\pi\Omega^2}{2\omega_s}\mu(\omega_s)\cos 2\pi\nu_s N \ ,
\end{aligned}
\tag{6.2}
$$

or in complex notation,

$$
(f(N), g(N)) \approx i\frac{\pi\Omega^2}{2\omega_s}\mu(\omega_s)e^{-i2\pi\nu_s N} \ . \tag{6.3}
$$

In the above, the kick amplitude $A(\omega_m)$ at one discrete frequency in the definition of Ω^2 should be replaced by the total kick amplitude integrated over the normalized spectral density $\mu(\omega_m)$. Also we have assumed that the poles of $\mu(\omega_m)$ in the complex ω_m-plane are far away from the poles at $\pm\omega_s$, so that their contributions can be neglected. Equation (6.3) is very similar to the second term in Eq. (5.2). However, there is no more periodic shifting motion, because the driving force now comprises many frequencies which average out to a fixed shift. There is also no more small denominator even if $\mu(\omega_m)$ peaks at ω_s, because for a continuous spectrum the amount of driving force right at resonance is of measure zero.

Unfortunately, only the power spectra of the ground waves had been measured. However, if we assume $\mu(\omega_m)$ to be Gaussian-like, peaked at ω_s with a spread $\sigma_\omega \approx 2\pi \times 0.1$ Hz, we obtain the shift in, respectively, fractional energy and rf phase for the effect of one quadrupole,

$$
\begin{aligned}
\Delta\delta &\approx \frac{\pi^{3/2}}{2^{1/2}}\frac{A}{F}\frac{\omega_s}{\sigma_\omega} = 3.90 \times 10^{-7} \ , \\
\Delta\phi &\approx \frac{\eta h}{\nu_s}\Delta\delta = 3.88 \times 10^{-3} \text{ rad} \ .
\end{aligned}
\tag{6.4}
$$

The total integrated kick is assumed to be $A = 0.05\ \mu$. These values are to be compared with those in Eqs. (5.4) and (5.5). (The actual spectrum due to a quarry blast peaks at 3 Hz with a width ~ 1 Hz.)

VII. Conclusion

We have studied the effect of ground waves on the longitudinal motion of a bunch assuming that the synchrotron frequency does not have any spread. We showed that the rf phase mismatch due to ground-wave kick on the quadrupole does not add up so as to throw the particles or bunch outside the rf bucket, except in the unlikely case of a discrete-frequency ground-wave kick right at exactly the synchrotron frequency. The shifting of the bunch inside the bucket was found to be very small. Any increase in bunch area has to come from the spread of synchrotron frequency.

The above treatment does not apply to the random motion of the ground. In that situation, the random kicks can be in resonance with synchrotron oscillation.

The synchrotron frequency is not unique. It decreases as the oscillation amplitude increases. As a result, the response due to a kick of any frequency will not lead to infinite growth in amplitude. If the particle is originally at the center of the rf bucket, a kick at the base synchrotron frequency (or at any frequency nearby) will only drive the particle into oscillation with a finite amplitude. This problem will be discussed elsewhere.

Appendix

The closed-orbit offset $x(\psi)$ at Floquet phase advance ψ (which runs from 0 to 2π around the ring) due to a kick $\Delta x'$ at phase advance ψ_0 is given by

$$x(\psi) = \frac{\Delta x'}{2\sin\pi\nu_\beta}\sqrt{\beta(\psi)\beta(\psi_0)}\cos(\pi - |\psi - \psi_0|)\nu_\beta ,$$

where ν_β is the betatron tune. The change in orbit length is therefore

$$\Delta C = \int_0^{2\pi} \frac{x(s)}{\rho(s)} ds = \int \frac{x(\psi)\beta(\psi)}{\rho(\psi)} \nu_\beta d\psi ,$$

or

$$\Delta C = \Delta x' \frac{\nu_\beta\sqrt{\beta(\psi_0)}}{2\sin\pi\nu_\beta} \int_0^{2\pi} d\psi \frac{\beta^{3/2}(\psi)}{\rho(\psi)} \cos(\pi - |\psi - \psi_0|)\nu_\beta \equiv \Delta x' D(\psi_0) ,$$

where $D(\psi_0)$ is just the periodic solution of the equation

$$\frac{d^2}{d\psi^2}\left(\frac{D}{\sqrt{\beta}}\right) + \nu_\beta^2\left(\frac{D}{\sqrt{\beta}}\right) = \frac{\nu_\beta^2\beta^{3/2}}{\rho}$$

and is therefore the dispersion function at phase advance ψ_0.

REFERENCES

1. K. Henon and D. Henon, SSCL Report SSC-SR-1043, *Field Measurements and Analysis of Underground Vibrations at the SSC Site,* Dec 1989.

2. J. Rossbach, private communication.