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**Comment on the exact evaluation
of symplectic maps**

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Comment on the exact evaluation of symplectic maps

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Gjaja [1] has recently reported closed form expressions for $\exp[a : x^n p^m :]x$ and $\exp[a : x^n p^m :]p$. We present here a simpler derivation of these results using the fact that the vector field associated with a Hamiltonian H is $X_H = - : H :$. (See, for example, [2].) This means that $\exp[t : x^n p^m :]$ is the time-evolution operator associated with the Hamiltonian, $H = -x^n p^m$, and the desired expressions can be found by solving Hamilton's equations of motion.

Starting from the Hamiltonian, $H = -x^n p^m$, the vector field components for x and p are written,

$$\begin{aligned}\dot{x} &= -m x^n p^{m-1} = mH/p & (1) \\ \dot{p} &= n x^{n-1} p^m = -nH/x & (2)\end{aligned}$$

From these, and the fact that H is autonomous, we obtain immediately that xp is a linear function of time.

$$\begin{aligned}\frac{d}{dt}(xp) &= \dot{x}p + x\dot{p} = (m-n)H \\ xp &= x_0 p_0 + (m-n)Ht\end{aligned}$$

We use this in conjunction with Eq.(1) to obtain the solution for $x(t)$.

$$\begin{aligned}\dot{x} &= mH/p = mHx/xp \\ &= mHx/(x_0 p_0 + (m-n)Ht) \\ \frac{dx}{x} &= \frac{mH dt}{x_0 p_0 + (m-n)Ht}\end{aligned}$$

The case $m = n$ is simplest:

$$m = n \Rightarrow x = x_0 \exp[mHt/x_0 p_0] = x_0 \exp[-m x_0^{m-1} p_0^{m-1} t]$$

If $m \neq n$, then the integral is only slightly more complicated.

$$x = C (x_0 p_0 + (m-n)Ht)^{\frac{m}{m-n}}$$

Here, C is a constant, which will very quickly be replaced by a function of x_0 and p_0 . We substitute $H = -x_0^n p_0^m$ and perform a little factorization to get the final result.

$$\begin{aligned}
x &= C(x_0 p_0 + (n-m)x_0^n p_0^m t)^{\frac{m}{m-n}} \\
&= C(x_0 p_0)^{\frac{m}{m-n}} (1 + (n-m)x_0^{n-1} p_0^{m-1} t)^{\frac{m}{m-n}} \\
&= x_0 (1 + (n-m)x_0^{n-1} p_0^{m-1} t)^{\frac{m}{m-n}}
\end{aligned} \tag{3}$$

A completely analogous calculation, starting from Eq.(2), provides the corresponding result for p .

$$p = \begin{cases} p_0 \exp[mx_0^{m-1} p_0^{m-1} t], & n = m \\ p_0 (1 + (n-m)x_0^{n-1} p_0^{m-1} t)^{\frac{n}{n-m}}, & n \neq m \end{cases}$$

These results correspond to Equations 10 through 14 of Gjaja [1].

An even simpler development gives the same result when $m \neq n$, for we then can use the invariance of H itself to find the solutions.

$$\begin{aligned}
H = -x^n p^m &= -x^{n-m} (xp)^m \\
&= -x^{n-m} (x_0 p_0 + (n-m)x_0^n p_0^m t)^m \\
x^{n-m} &= x_0^n p_0^m (x_0 p_0 + (n-m)x_0^n p_0^m t)^{-m} \\
&= x_0^{n-m} (1 + (n-m)x_0^{n-1} p_0^{m-1} t)^{-m}
\end{aligned}$$

Taking the $(n-m)^{\text{th}}$ root reproduces Eq.(3).

We leave as a little exercise for the reader to determine what happens in the vicinity of $t \approx ((m-n)x_0^{n-1} p_0^{m-1})^{-1}$, and correspondingly to find the error in the solution as presented.

References

- [1] I. Gjaja. Exact evaluation of arbitrary symplectic maps. Preprint from University of Maryland. College Park, MD., January 1992.
- [2] Leo Michelotti. From vector fields to normal forms. Presented at the Fifth Advanced ICFA Beam Dynamics Workshop. Corpus Christi, Texas. October 3-8, 1991.