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Coherent Betatron Oscillations and Emittance Growth Due to Random Kicks

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1 Introduction

Coherent betatron oscillations of the beam centroid and a slow but nonzero emittance growth rate have been observed in the Tevatron.[1,2] These are not transient effects, in the sense that their time-averaged values are not zero, over an interval of hours. The emittance growth is important because it can decrease the beam luminosity significantly during collider operations. In this note, we study the excitation of coherent betatron oscillations of the beam centroid and emittance growth due to random dipole kicks, i.e. all particles at a given azimuth receive the same kick. Expressions for the time-averaged beam centroid amplitude and emittance growth rate are derived. The latter result is compared with some data from the Tevatron,[2] and the agreement appears to be satisfactory.

2 Notation

First we shall describe the notation to be used below. The orbit is described by the column vector

$$X(\theta) = \begin{pmatrix} x \\ p \end{pmatrix} \equiv \begin{pmatrix} x \\ \alpha x + \beta x' \end{pmatrix} = \sqrt{2I\beta} \begin{pmatrix} \sin(\Psi + \psi - \nu\theta) \\ \cos(\Psi + \psi - \nu\theta) \end{pmatrix} . \tag{1}$$

Here I is the action, Ψ is the angle, ν is the linear tune and ψ is the Floquet phase. We make the approximation that the phase-space trajectories in $\{x, p\}$ space are circles with action-dependent tunes. The unnormalized emittance ϵ is given by

$$\epsilon \equiv \frac{\pi}{\beta}\langle X^T X\rangle = 2\pi\langle I\rangle \ , \tag{2}$$

assuming $\langle X\rangle = 0$. The angular brackets denote an average over the beam at fixed θ. The emittance growth rate is

$$r = 2\pi f\frac{d\epsilon}{d\theta} \ , \tag{3}$$

where f is the revolution frequency. Note that to calculate r it is not necessary that $\langle X\rangle = 0$; it is sufficient if $\langle X\rangle$ is bounded, because then $d\langle X\rangle/d\theta$ averages to zero.

The normalized emittance which encloses 95% of the beam is a factor $6\beta\gamma$ larger than the above value. (Here $\beta = v/c$; to avoid confusion we shall not use the $\beta\gamma$ factor until the end.)

3 Coherent betatron motion

3.1 Linear response theory

Let us now study the excitation of coherent betatron oscillations, i.e. the motion of the beam centroid. Suppose there is a random horizontal dipole kick at location θ, so that

$$\begin{aligned} x(\theta+\delta\theta) - x(\theta) &= 0 \\ p(\theta+\delta\theta) - p(\theta) &= N\delta\theta \ . \end{aligned} \tag{4}$$

We shall linearize the response of the beam with respect to the kicks, i.e. we calculate the changes to x and p to linear order in N only, and so the emittance growth rate will be of $O(N^2)$.

The changes to x and p at azimuth θ, due to a kick $N\delta\theta$ at azimuth θ', are

$$\begin{aligned} \delta\left(\frac{x}{\sqrt{\beta}}\right) &= \sin[\Phi(\theta) - \Phi(\theta')]\frac{N(\theta')}{\sqrt{\beta(\theta')}}\,\delta\theta \\ \delta\left(\frac{p}{\sqrt{\beta}}\right) &= \cos[\Phi(\theta) - \Phi(\theta')]\frac{N(\theta')}{\sqrt{\beta(\theta')}}\,\delta\theta \ . \end{aligned} \tag{5}$$

Here $\Phi = \Psi + \psi - \nu\theta$. The factors of $\sqrt{\beta}$ have been introduced because the quantity of real interest is $\langle x\rangle/\sqrt{\beta}$, because $\langle x\rangle$ itself is proportional to $\sqrt{\beta}$. To obtain the kick to the beam centroid $\langle x \rangle$, we must average over the beam. We shall assume that the kick N is the same for all the particles, i.e. $\langle Nx \rangle = N\langle x \rangle$, etc. This is a valid assumption for a dipole kick due to a displaced quadrupole. The change to the beam centroid is

$$\delta\left(\frac{\langle x \rangle}{\sqrt{\beta}}\right) = D(\theta,\theta')\frac{N(\theta')}{\sqrt{\beta(\theta')}}\sin[\Phi_0(\theta) - \Phi_0(\theta')]\,\delta\theta \ , \tag{6}$$

where Φ_0 is the linear phase advance, i.e. $\Phi_0 = \Psi_0 + \psi - \nu\theta$, where Ψ_0 is the linear angle variable. The function $D(\theta,\theta')$ is a decoherence factor. It appears because the individual particles have different tunes, and so even though the kick N is the same for all particles, they get out of phase as time goes by.

The response to a sequence of kicks, using the linear response approximation, is

$$\frac{\langle x \rangle}{\sqrt{\beta}} = D(\theta,\theta_0)\left(\frac{\langle x \rangle}{\sqrt{\beta}}\right)_{\theta_0} + \int_{\theta_0}^{\theta} D(\theta,\theta')\frac{N(\theta')}{\sqrt{\beta(\theta')}}\sin[\Phi_0(\theta) - \Phi_0(\theta')]\,\delta\theta' \ . \tag{7}$$

We shall use only the asymptotic solution, i.e. we shall neglect the contribution from θ_0 in Eq. (7). Mathematically, we take $\theta_0 \to -\infty$. This is a formal limit, and simply means that we are assuming that after sufficient time the contribution of the initial state of the beam centroid decoheres and becomes negligible in comparison to the effect of the kicks experienced by the beam after injection.

3.2 Decoherence factor

3.2.1 Sample models

The decoherence factor for a beam with a uniform octupole moment, i.e. $d\Psi/d\theta = \nu + \nu' I$, has been calculated in Ref. [3]. The result is (with $\Theta = \theta - \theta'$)

$$D(\theta,\theta') = D(\theta - \theta') = \frac{1}{1+(\Theta/\Theta_0)^2}\exp\left[-\frac{1}{2}\frac{(\Theta/\Theta_1)^2}{1+(\Theta/\Theta_0)^2}\right] \ , \tag{8}$$

where Θ_0 and Θ_1 are constants. Note that this is even under time-reversal, i.e. the beam decoheres even if $\theta < \theta'$, as expected for decoherence. From Ref. [3], the exponential factor may be ignored ($\Theta_1 \to \infty$) unless the beam centroid displacement is much greater than the beam size, and so

$$D(\theta,\theta') \simeq \frac{1}{1+(\Theta/\Theta_0)^2} \ . \tag{9}$$

Another model, which is motivated by radiation damping in synchrotron radiation theory, is to put

$$\begin{aligned} D(\theta,\theta') &= e^{-\alpha_d(\theta-\theta')} \qquad && \theta > \theta' \\ &= 0 && \theta < \theta' \ . \end{aligned} \tag{10}$$

This has the disadvantage of not being even under time-reversal. A more reasonable approximation in this context would be

$$D(\theta,\theta') = \frac{1}{2\cosh[\alpha_d(\theta-\theta')]} \ . \tag{11}$$

Since we do not know the detailed decoherence mechanism in general, the choice of model is somewhat arbitrary. The expression Eq. (10) is the simplest when evaluating Eq. (7), because we shall decompose N into Fourier harmonics below, and an exponential decoherence factor yields the simplest analytical solution. In particular, we shall only be interested in $\theta > \theta'$, and so we shall use Eq. (10). It must be understood that this is a phenomenological step.

3.2.2 Caveat

In all of the models of decoherence presented above, the beam never recoheres. Instead, if the beam were to suffer only one kick, say at $\theta = \theta_0$, the value of $\langle a \rangle$ would approach zero monotonically as θ increased. In practice, the beam would recohere and decohere again repeatedly. The use of the above expressions assumes that the recoherence time is much longer than the timescale of the phenomena being studied in this note. The matter will be discussed below, in the section on "approximations," in the context of other relevant timescales.

3.3 Solution of beam centroid motion

3.3.1 Fourier harmonics

It is useful at this point to introduce some Fourier harmonics. Define

$$\begin{aligned} N(\theta) &= \int \frac{d\omega}{2\pi}\, \widetilde{N}(\omega) e^{i(\omega\theta + i\phi_N(\omega,\theta))} \\ \frac{e^{i(\psi-\nu\theta)}}{\sqrt{\beta}} &= \sum_k L_k e^{ik\theta} \ . \end{aligned} \tag{12}$$

The latter function is periodic in θ, and so one gets a sum, not integral, of harmonics. The function ϕ_N is a random phase, whose value fluctuates equally over all values between 0 and 2π. It means that $N(\theta)$ consists of wave trains that shift phase randomly from time to time. It will be more convenient below to deal with the combined transform

$$\begin{aligned} \frac{N e^{i(\psi-\nu\theta)}}{\sqrt{\beta}} &= \int \frac{d\omega}{2\pi} \widetilde{M}(\omega) e^{i(\omega\theta+\phi(\omega,\theta))} \\ \widetilde{M}(\omega) e^{i\phi(\omega,\theta)} &= \sum_k \widetilde{N}(\omega-k) L_k e^{i\phi_N(\omega-k,\theta)} \ . \end{aligned} \tag{13}$$

The function ϕ is the random phase associated with $\widetilde{M}$. In practice, one must interpret the Fourier transform as a sum over (angular) "frequency bins" of size $\Delta\omega$, i.e.

$$\int F(\omega)\,d\omega = \Delta\omega \sum_k F(k\Delta\omega) \ , \tag{14}$$

where $\Delta\omega$ is determined by the experimental apparatus used to measure, say, the power spectrum of the noise.

We substitute the above expression into Eq. (6),

$$\begin{aligned} \frac{\langle x \rangle}{\sqrt{\beta}} &\simeq \mathrm{Re}\left\{ \frac{1}{2i} \int \frac{d\omega}{2\pi} \widetilde{M}(\omega) e^{i\phi(\omega,\theta)} \int_{-\infty}^{\theta} e^{-\alpha_d(\theta-\theta')} e^{i\omega\theta'} e^{-i\nu(\theta-\theta')}\,d\theta' \right\} \\ &= \frac{1}{2}\mathrm{Im}\left\{ \int \frac{d\omega}{2\pi} \frac{\widetilde{M}(\omega) e^{i\phi(\omega,\theta)} e^{i\omega\theta}}{\alpha_d + i(\omega-\nu)} \right\} \ . \end{aligned} \tag{15}$$

Let us try to visualize the beam centroid motion pictorially. Note that the integrand has a random phase. Hence if we average over the noise, $\langle x \rangle$ will average to zero. The physical picture is that of successive wave trains of coherent betatron motion, separated by random fluctuations in phase (Fig. 1).

3.3.2 Time averages

Since $\langle x \rangle$ averages to zero, it is more useful to calculate $\langle x \rangle^2/\beta$, averaged over time. Note that this is *not* the beam emittance, but the r.m.s. beam centroid amplitude squared and divided by β. Formally, the time average of any function F is given by

$$F_{avg} = \lim_{\theta\to\infty} \left[\frac{1}{\theta} \int_{-\theta/2}^{\theta/2} F(\theta')\,d\theta' \right] \ . \tag{16}$$

The limit $\theta \to \infty$ is again a formal limit, and does *not* mean that we are going back to values of θ when the beam was first injected into the machine. It simply means a long time,

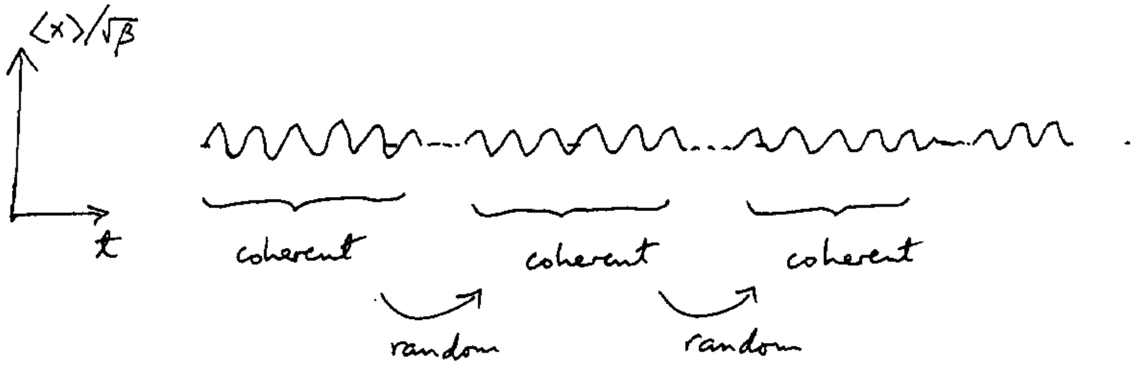

Figure 1: Coherent wave trains and random fluctuations in beam centroid motion

$\theta = 2\pi/\Delta\omega$, where $\Delta\omega$ is the "bin size" of (angular) frequencies in the Fourier transform. Beyond this time the Fourier harmonics are not well-defined. The approximations involved in the next few equations are also discussed in the section on approximations below.

Thus we actually evaluate

$$\begin{aligned}\left[\frac{\langle x\rangle^2}{\beta}\right]_{avg} &\simeq \frac{1}{2}\frac{\Delta\omega}{2\pi}\int_{-\pi/\Delta\omega}^{\pi/\Delta\omega}\frac{\langle x\rangle^2}{\beta}\,d\theta' \\ &= \frac{\Delta\omega}{4\pi}\int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} d\theta' \int\!\!\int \frac{d\omega d\omega'}{(2\pi)^2}\,\frac{\widetilde{M}^*(\omega)\widetilde{M}(\omega')e^{i(\omega'-\omega)\theta'}e^{i(\phi(\omega',\theta)-\phi(\omega,\theta))}}{[\alpha_d - i(\omega-\nu)][\alpha_d + i(\omega'-\nu)]}\ . \qquad (17)\end{aligned}$$

We now need to interchange the orders of integration, which involves further assumptions about the uniformity of the convergence of the integrals involved. Doing so, we obtain

$$\begin{aligned}\left[\frac{\langle x\rangle^2}{\beta}\right]_{avg} &\simeq \frac{\Delta\omega}{4\pi}\int\!\!\int \frac{d\omega d\omega'}{(2\pi)^2}\,\frac{\widetilde{M}^*(\omega)\widetilde{M}(\omega')}{[\alpha_d - i(\omega-\nu)][\alpha_d + i(\omega'-\nu)]} \\ &\qquad \times \int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} e^{i(\phi(\omega',\theta')-\phi(\omega,\theta'))}e^{i(\omega'-\omega)\theta'}\,d\theta'\ . \qquad (18)\end{aligned}$$

The integral over θ' contains two factors both of which average to zero unless $\omega = \omega'$. More precisely, if we recall that the integration over frequencies consists of sums over bins of size $\Delta\omega$, then the integrals over ω' and θ' yield

$$\int \frac{d\omega'}{2\pi}F(\omega')\int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} e^{i(\phi(\omega',\theta')-\phi(\omega,\theta'))}e^{i(\omega'-\omega)\theta'}\,d\theta' \;\simeq\; \frac{\Delta\omega}{2\pi}\sum_{\omega'} F(\omega')\frac{2\pi}{\Delta\omega}\delta_{\omega,\omega'}$$

$$\simeq \quad F(\omega' = \omega) \; . \tag{19}$$

If we were to take $\Delta\omega \to 0$ in the above integrals, *then* we would get true δ-functions

$$\begin{aligned} \int \frac{d\omega'}{2\pi} F(\omega') \int_{-\infty}^{\infty} e^{i(\phi(\omega',\theta')-\phi(\omega,\theta'))} e^{i(\omega'-\omega)\theta'} \, d\theta' &= \int \frac{d\omega'}{2\pi} F(\omega') \, 2\pi\delta(\omega' - \omega) \\ &= F(\omega' = \omega) \; , \end{aligned} \tag{20}$$

but we have a global factor of $\Delta\omega$ in Eqs. (17) and (18), and so we must keep $\Delta\omega \neq 0$.

Using the above results in Eq. (18),

$$\left[\frac{\langle x\rangle^2}{\beta}\right]_{avg} = \frac{\Delta\omega}{4\pi} \int \frac{d\omega}{2\pi} \frac{|\widetilde{M}(\omega)|^2}{\alpha_d^2 + (\omega - \nu)^2} \; . \tag{21}$$

The integrand is a Lorentzian with a maximum at $\omega = \nu$ and width α_d. Thus the conclusion is that the harmonics of the noise which contribute significantly to the coherent betatron motion are those in a range $\pm\alpha_d$ around the betatron tune ν. Since, usually, $\alpha_d \ll 1$ in practice, and $\widetilde{M}(\omega)$ is slowly varying in the interval $|\omega - \nu| < \alpha_d$, we can set $M(\omega) \simeq M(\nu)$ and pull it out of the Lorentzian integral, whose value is π/α_d. Then

$$\left[\frac{\langle x\rangle^2}{\beta}\right]_{avg} \simeq \frac{1}{8\pi\alpha_d} |\widetilde{M}(\nu)|^2 \Delta\omega \; . \tag{22}$$

This result agrees with Siemann's finding[1] that $\langle x\rangle \propto \sqrt{\tau_d}$, where $\tau_d = \alpha_d^{-1}$ is the decoherence time. The quantity $|\widetilde{M}(\nu)|^2\Delta\omega$ is the power (times various lattice functions) in the noise source in the angular frequency bin at $\omega = \nu$. Using Eq. (13), this means that the original noise source N must have a nonzero harmonic at some $\nu - k$, i.e., the betatron frequency plus some multiple of the revolution frequency.

3.3.3 Zero bandwidth

Note that if we were able to measure frequencies to infinite precision, then $\Delta\omega = 0$, and we would be led to conclude that $[\langle x\rangle^2/\beta]_{avg} = 0$. This is a correct conclusion for this model, and has the following interpretation. To measure a Fourier harmonic with a bin size $\Delta\omega \to 0$, it would truly take an infinite number of turns, and on *this* time scale even the most slowly varying harmonics in $\langle x\rangle^2/\beta$ would average to zero. In practice, we cannot observe the beam over such a long time, and so any harmonic that does not average to zero rapidly over the interval of experimental measurement $2\pi/\Delta\omega$ survives the average over θ in Eq. (17).

4 Emittance growth

Next, let us calculate the rate of emittance growth. For this we need to study $x^2 + p^2$. Now

$$[x^2 + p^2]_{\theta+\delta\theta} = [x^2 + (p + N\delta\theta)^2]_\theta \simeq [x^2 + p^2]_\theta + 2pN\delta\theta \tag{23}$$

and so

$$\beta\,\delta I = 2\pi(N\delta\theta)\sqrt{\beta}\int_{-\infty}^{\theta}\frac{N(\theta')}{\sqrt{\beta(\theta')}}\cos[\Phi(\theta) - \Phi(\theta')]\,d\theta' \;. \tag{24}$$

Averaging over the beam,

$$\frac{d\epsilon}{d\theta} = 2\pi\frac{N}{\sqrt{\beta}}\int_{-\infty}^{\theta} D(\theta,\theta')\frac{N(\theta')}{\sqrt{\beta(\theta')}}\cos[\Phi_0(\theta) - \Phi_0(\theta')]\,d\theta' \;. \tag{25}$$

We want the time average of this function, because $2\pi f d\epsilon/d\theta$ is the rate of emittance growth (f is the revolution frequency). The function ϵ itself grows indefinitely, and does not have a finite time average.

The approximations used here are the same as those used above in analyzing the behavior of $[\langle x\rangle^2/\beta]_{avg}$. Further, as noted in the previous section, since the beam centroid motion is bounded, we can get the emittance growth rate directly from the above equation. The time-averaged growth rate is

$$\begin{aligned}
r = 2\pi f\left[\frac{d\epsilon}{d\theta}\right]_{avg} &\simeq 4\pi^2 f\frac{\Delta\omega}{2\pi}\int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} d\theta\,\frac{N(\theta)}{\sqrt{\beta(\theta)}}\int_{-\infty}^{\theta} D(\theta,\theta')\frac{N(\theta')}{\sqrt{\beta(\theta')}}\cos[\Phi_0(\theta) - \Phi_0(\theta')]\,d\theta' \;. \\
&\simeq 2\pi f\Delta\omega\,\mathrm{Re}\left\{\int\!\!\int\frac{d\omega d\omega'}{(2\pi)^2}\,\widetilde{M}^*(\omega)\widetilde{M}(\omega')\frac{e^{i(\omega'-\omega)\theta'}e^{i(\phi(\omega',\theta')-\phi(\omega,\theta'))}}{\alpha_d - i(\omega' - \nu)}\right\} \\
&\simeq 2\pi f\Delta\omega\int\frac{d\omega}{2\pi}\,|\widetilde{M}(\omega)|^2\frac{\alpha_d}{\alpha_d^2 + (\omega - \nu)^2} \\
&\simeq \pi f|\widetilde{M}(\nu)|^2\Delta\omega \;.
\end{aligned} \tag{26}$$

Note that this result is independent of the decoherence rate. One can think of this as a manifestation of energy balance:[4] energy is fed into the beam at a rate $|\widetilde{M}(\nu)|^2\Delta\omega$, and is absorbed by the emittance growth — increasing the amplitudes (energy) of the betatron oscillations of the particles. The nonlinearities responsible for decoherence do not dissipate energy, but merely cause dephasing of the particles.

We see that the integrand above is also a Lorentzian with a maximum at $\omega = \nu$ and a width α_d, and so the emittance growth is driven by harmonics of the noise in a range $\pm\alpha_d$ centered on the betatron tune ν.

One can also write the above result in the form

$$r = 8\pi^2 f \alpha_d \left[\frac{\langle x \rangle^2}{\beta} \right]_{avg} . \tag{27}$$

Many of the approximations made in deriving the individual expressions for the time averages of $\langle x \rangle^2/\beta$ and $d\epsilon/d\theta$ cancel out in Eq. (27). Also, the above result also does not depend on the detailed form of the noise spectrum.

5 Approximations

In this section we recapitulate and discuss the approximations made in the above calculations. To begin with, we are only interested in the behavior of the beam long after injection, so that the initial state of the beam does not matter. We are assuming that a time interval exists where the contribution of the "transient" (initial value of $\langle x \rangle$ in Eq. (7)) is negligible. Now the "damping" of the beam centroid is really a decoherence of individual particle orbits, not true damping of $\langle x \rangle$, and, given time, the beam will recohere, and the magnitude of $\langle x \rangle$ will increase. Thus for the subsequent calculations to be valid, it is necessary that the recoherence time be much longer than the timescale of the entire phenomenon of excitation of coherent betatron oscillations and emittance growth. Specifically, we need to take various averages over time to obtain the average values for the beam centroid amplitude (Eqs. (17) - (22)) and emittance growth rate (Eq. (26)). These averages are supposed to extend over a sufficiently long time interval so that the oscillatory components in the observed values of $\langle x \rangle^2/\beta$ and r cancel out. The timescale is $2\pi/\Delta\omega$. The value of $\Delta\omega$ must therefore be much less than α_d in order for the Lorentzian integrals in Eqs. (21) and (26) to be meaningful, i.e. in order to justify the approximations made to the integrands in the integrations over ω' and ω. The interchange of the orders of integration between Eqs. (17) and (18) relies on the uniqueness of Fourier transforms, i.e. on the relations

$$\int_{-\infty}^{\infty} d\theta \, e^{-i\omega\theta} = 2\pi\delta(\omega) , \qquad \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, e^{i\omega\theta} = \delta(\theta) . \tag{28}$$

Since we only integrate to $\theta = \pm\pi/\Delta\omega$, we actually have

$$\begin{aligned} \int_{-\pi/\Delta\omega}^{\pi/\Delta\omega} d\theta \, e^{-i\omega\theta} &\simeq \frac{2\pi}{\Delta\omega} && |\omega| < \Delta\omega \\ &\simeq 0 && \text{otherwise.} \end{aligned} \tag{29}$$

Thus the interchange of the orders of integration involves errors of $O(\Delta\omega/2\pi)$ because the δ-functions are not "pointlike."

Hence the global picture is: we first wait a time much longer than $2\pi/\alpha_d$, so that the initial beam conditions "damp out," then define Fourier transforms over a time interval $2\pi/\Delta\omega$ which must exceed $2\pi/\alpha_d$, and the recoherence time must exceed the time interval of the whole process. The use of ∞ or $-\infty$ as a limit of integration simply means that the contribution from that end of the range of integration is being neglected. It does not imply tracking of a particle, or the beam, for an infinite number of turns.

6 Experimental results

6.1 Graphs

Measurements of coherent betatron oscillatons and emittance growth have been made in the Tevatron, and some of the results are shown in Figs. 2 and 3.[2] In both cases the ordinate is proportional to $|\widetilde{M}(\nu)|^2\Delta\omega$. The abscissa is the growth rate of the *normalized* emittance of 95% of the beam, which is a factor $6\gamma v/c \simeq 6\gamma$ larger than the unnormalized emittance used in all of the above calculations. Absolute values for $[\langle x\rangle^2/\beta]_{avg}$ are not yet available.

6.2 Numerical fit

The more useful graph is Fig. 3, because the values of all the relevant parameters are known, and so we can fit it quantitatively. The power in the kicks is

$$
\begin{aligned}
|\widetilde{N}|^2\Delta\omega &= \left(\eta\frac{eV_0}{E_0}\right)^2\theta_{rf}^2 \\
|\widetilde{M}|^2\Delta\omega &= \frac{|\widetilde{N}|^2\Delta\omega}{\beta_0}\,.
\end{aligned}
\tag{30}
$$

Here β_0 and η are the beta function and dispersion at the RF cavities, respectively, eV_0 is the RF energy gain per turn, E_0 is the beam energy and θ_{rf}^2 is the ordinate in Fig. 3. Recall that $N(\theta)$ is the kick given to the transverse particle momentum $p = \alpha x + \beta x'$. Since the kick is localized at one point, where $\beta = \beta_0$, we can write $|\widetilde{M}|^2 = |\widetilde{N}|^2/\beta_0$. The numerical

$$\boxed{\frac{d\epsilon_H}{dt} = (1.167 \pm 0.045)\, V_{RMS}^2 = (2.92 \pm 0.11)\, I_{RMS}^2 \, (mA)}$$

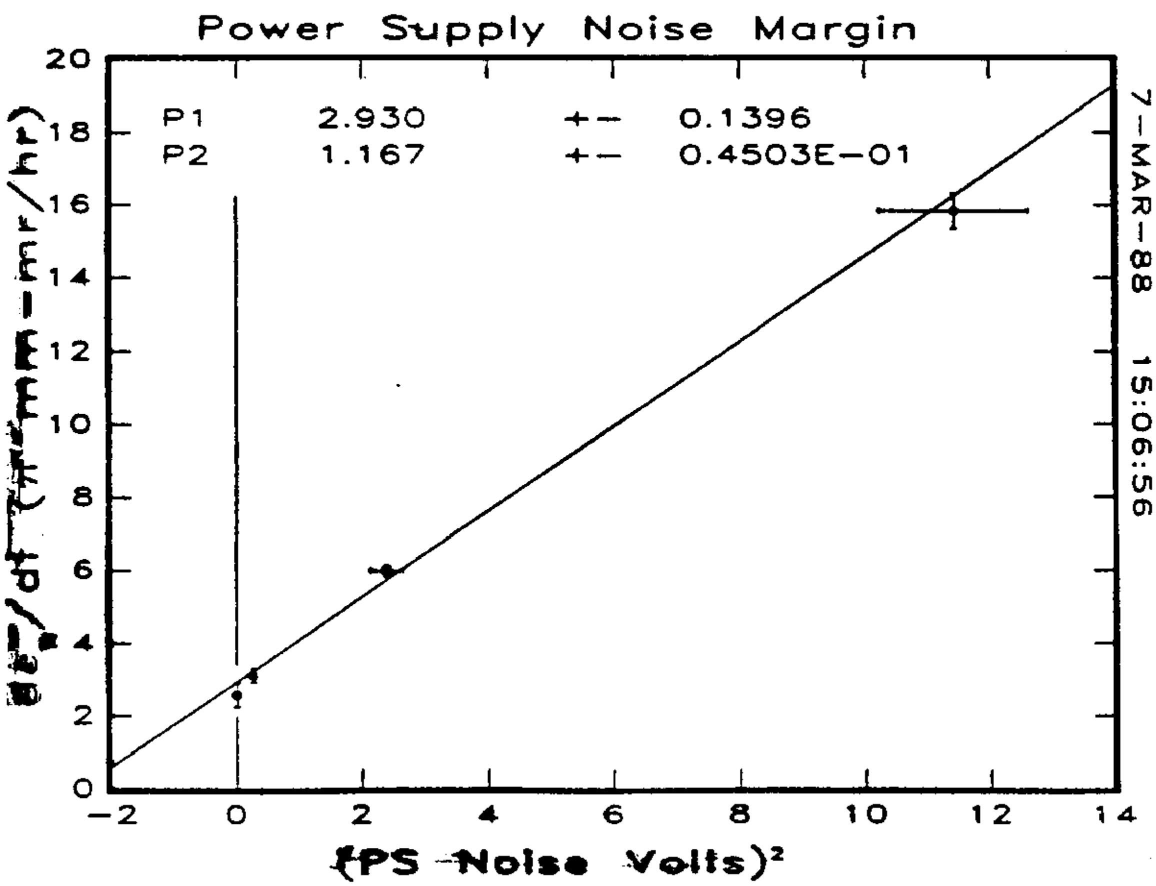

Figure 2: Emittance growth rate vs. noise in power supply (from Ref. [2])

$$\boxed{\frac{d\epsilon_H}{dt} = (71.6 \pm 8.2)\ \theta^2_{RMS}\ (\text{DEGREES})}$$

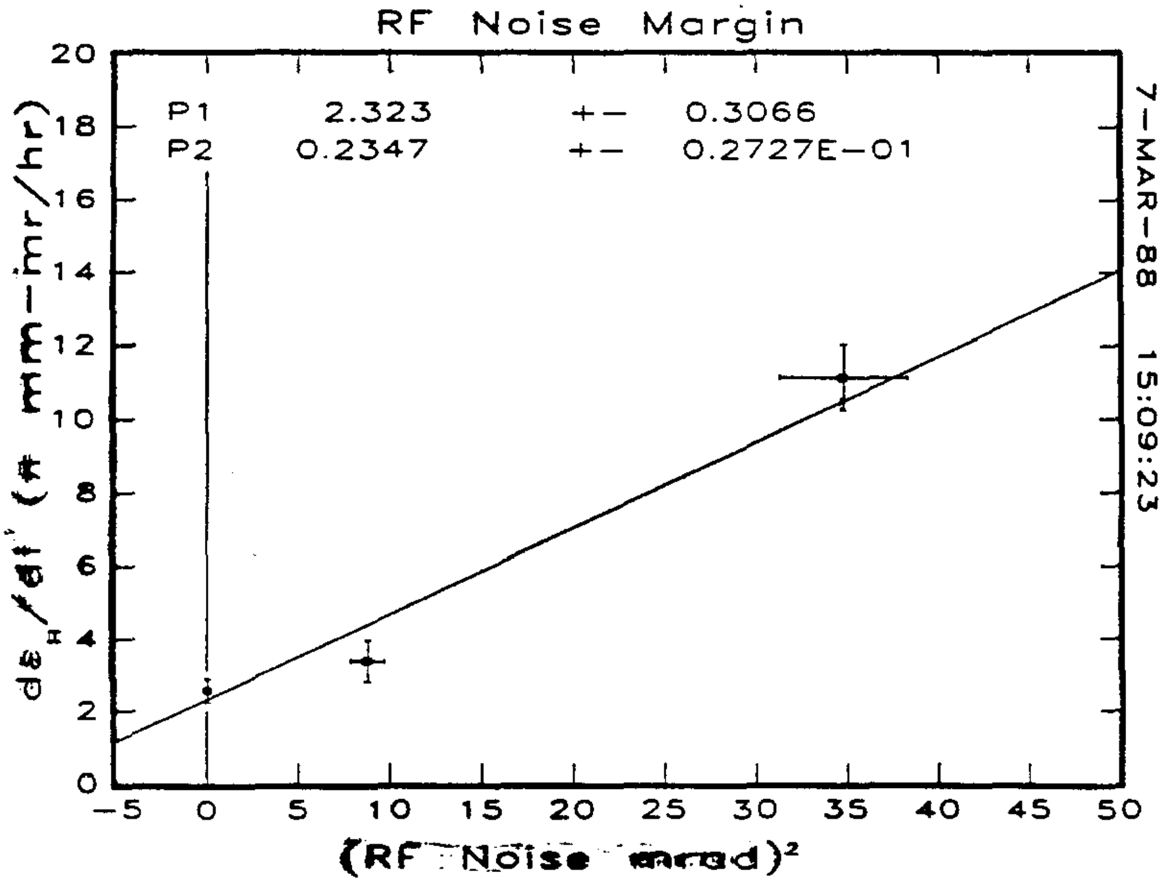

Figure 3: Emittance growth rate vs. noise in RF system (from Ref. [2])

values of the relevant parameters are

$$
\begin{aligned}
\eta &= 2.43 && \text{m} \\
\beta_0 &= 65.5 && \text{m} \\
eV_0 &= 1.16 && \text{MV/turn} \\
E_0 &= 900 && \text{GeV} \\
f &= 47.7 && \text{kHz}
\end{aligned}
\tag{31}
$$

From Fig. 3,

$$
\frac{d\epsilon_{norm}}{dt}\ (\pi\text{mm-mrad/hr}) = (71.6 \pm 8.2)\,\theta_{rf}^2\ (\text{degrees}) \tag{32}
$$

and so the experimental relation between r and θ_{rf}^2 is

$$
\begin{aligned}
r\ (\text{m-rad/sec}) &\simeq (71.6 \pm 8.2) \times \left[\frac{1}{6\gamma}\frac{\pi \times 10^{-6}}{3600}\left(\frac{180}{\pi}\right)^2\right]\theta_{rf}^2\ (\text{rad}) \\
&\simeq (3.6 \pm 0.4) \times 10^{-8}\,\theta_{rf}^2\ .
\end{aligned}
\tag{33}
$$

The theoretical value, using Eqs. (26) and (30), is

$$
\begin{aligned}
r &= \frac{\pi f}{\beta_0}\left(\eta\frac{eV_0}{E_0}\right)^2 \theta_{rf}^2 \\
&\simeq 2.2 \times 10^{-8}\,\theta_{rf}^2\ .
\end{aligned}
\tag{34}
$$

The agreement between the experimental and theoretical values is satisfactory. The discrepancy can be attributed to the uncertainty in the slope of the fit in Fig. 3 (about 10%), and the values of η and β_0 (also about 10% each).

7 Conclusion

We have derived expressions for the time-averaged beam centroid amplitude and emittance growth rate, assuming that all particles at a given azimuth receive the same kick. A number of additional approximations are required to derive the above results, however, and these have been listed in the section on "approximations." The results agree with those in Ref. [1], where some of these effects have previously been calculated. A comparison with data from Ref. [2] appears to be satisfactory.

Acknowledgements

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