



**Fermi National Accelerator Laboratory**

FN-466

**Calculation of Electron Polarization in High-Energy  
Storage Rings Including Nonlinear Spin-Orbit Coupling**

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October 1987



Operated by Universities Research Association Inc. under contract with the United States Department of Energy

Calculation of electron polarization in high-energy storage rings  
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An algorithm (and computer program) have previously been described by the author to calculate electron polarization in high-energy storage rings to arbitrary orders of spin resonances. The algorithm and program treat linear orbital dynamics and linear spin-orbit coupling. Here the algorithm is extended to include nonlinear spin-orbit coupling. Numerical results are presented.

## I. INTRODUCTION

In a recent paper,<sup>1</sup> I published an algorithm to calculate the equilibrium polarization of electron beams in high-energy storage rings. The algorithm was coded into a computer program, called SMILE, subject to the restriction of treating only linear orbital dynamics and linear spin-orbit coupling. I shall define these terms mathematically below. The concepts of nonlinear orbital dynamics and nonlinear spin-orbit coupling are in fact independent, and the SMILE algorithm can treat nonlinear spin-orbit coupling. This paper describes extensions to SMILE so as to include nonlinear spin-orbit coupling.<sup>2</sup> Numerical results are also presented.

## II. ALGORITHM

### A. Polarization formula

The formula for the equilibrium degree of polarization is<sup>3,4</sup>

$$P_{eq} = \frac{8}{5\sqrt{3}} \frac{\left\langle \frac{1}{|\rho|^3} \hat{b} \cdot \left( \hat{n} - \gamma \frac{\partial \hat{n}}{\partial \gamma} \right) \right\rangle}{\left\langle \frac{1}{|\rho|^3} \left( 1 - \frac{2}{9} (\hat{n} \cdot \hat{v})^2 + \frac{11}{18} \left| \gamma \frac{\partial \hat{n}}{\partial \gamma} \right|^2 \right) \right\rangle}, \quad (1)$$

where  $\rho$  is the local radius of curvature of the particle trajectory,  $\hat{v}$  is the direction of particle motion,  $\hat{b} = \vec{v} \times \dot{\vec{v}} / |\vec{v} \times \dot{\vec{v}}|$ ,  $\hat{n}$  is the spin quantization axis on the particle trajectory, and the angular brackets denote an ensemble average over the particle trajectories and accelerator azimuth (or arc-length). The principal aim of the algorithms of both Refs. 1 and 2 is to calculate  $\hat{n}$ . Given  $\hat{n}$ , the derivative  $\gamma(\partial \hat{n} / \partial \gamma)$  can be computed, and the relevant ensemble averages taken. The calculation of  $\gamma(\partial \hat{n} / \partial \gamma)$  from  $\hat{n}$  is explained in Ref. 1 and again below.

### B. SMILE formalism

Both Refs. 1 and 2 use perturbation theory to calculate  $\hat{n}$ ; no nonperturbative algo-

rithm is yet known. The perturbation expansion of Ref. 1 will be described here. First, the spin quantization axis  $\hat{n}$  is chosen so that spin states quantized along  $\hat{n}$  are eigenstates of the Hamiltonian in semiclassical approximation, i.e. to the leading order in  $\hbar$ . See Ref. 3 for details. The axis  $\hat{n}$  depends on the orbital trajectory:  $\hat{n} = \hat{n}(\vec{r}, \vec{p})$ , because particles following distinct orbital trajectories see different magnetic fields, and so their spin eigenstates are quantized in different directions. The perturbation expansions of Refs. 1 and 2 consist of solving for  $\hat{n}$  as a function of an orbital oscillation away from the equilibrium closed orbit. In Ref. 1, the orbital dynamics is assumed to be linear. The unperturbed solution is  $\hat{n}_0$ , the value of  $\hat{n}$  on the equilibrium closed orbit, and the expansion parameter is the amplitude of the orbital oscillation (strictly, the orbital action variable). There are actually three expansion parameters, because the orbital motion has three degrees of freedom, but for formal purposes they are all treated on the same footing. Numerically, my program<sup>1</sup> allows one to calculate to different orders in the various orbital modes of oscillation, thus one can calculate to high order in a mode that has a strong influence on the spin precession, and to low order in one that does not.

The Hamiltonian for the orbital and spin motion can be written in the form

$$\mathcal{H} = \mathcal{H}_{\text{orb}}(\vec{r}, \vec{p}) + \vec{s} \cdot \vec{\Omega}(\vec{r}, \vec{p}) , \quad (2)$$

where  $\mathcal{H}_{\text{orb}}$  describes purely orbital motion,  $\vec{\Omega}$  is the spin precession vector, and  $\vec{s}$  is the spin operator. Expressions for  $\vec{\Omega}$  in various standard accelerator elements such as dipole and quadrupole magnets will be given below. Note that  $\vec{\Omega}$  depends on the orbital trajectory; this is the spin-orbit coupling. The spin equation of motion is

$$\frac{d\vec{s}}{d\theta} = \vec{\Omega} \times \vec{s} , \quad (3)$$

where  $\theta$  is the azimuth around the storage ring. The notation in Eq. (3), and below, follows Ref. 1. By definition,  $\hat{n}$  satisfies Eq. (3) on a given orbital trajectory. I now write

$$\vec{\Omega} = \vec{\Omega}_0 + \vec{\omega} , \quad (4)$$

where  $\bar{\Omega}_0$  is the value of  $\bar{\Omega}$  on the equilibrium closed orbit and  $\bar{\omega}$  is the additional part due to an orbital oscillation. I can now define the concepts of (non)linear spin-orbit coupling mathematically. In general,  $\bar{\omega}$  is a nonlinear function of the orbital trajectory. The term ‘linear spin-orbit coupling’ means that  $\bar{\omega}$  is expanded in a power series in  $\vec{r}$  and  $\vec{p}$  and only the terms proportional to the first power in  $\vec{r}$  and  $\vec{p}$  are kept. Nonlinear spin-orbit coupling arises when higher-order terms are retained in the expression for  $\bar{\omega}$ . It is important to note that the concept of (non)linearity does *not* depend on the linearity or otherwise of the orbital motion — it is an independent concept. Expressions for  $\bar{\omega}$  in various standard accelerator elements are given in Appendix A, up to second order in the trajectory.

Now, by definition,

$$\frac{d\hat{n}_0}{d\theta} = \bar{\Omega}_0 \times \hat{n}_0 , \quad (5a)$$

with the boundary condition

$$\hat{n}_0(\theta + 2\pi) = \hat{n}_0(\theta) . \quad (5b)$$

Two other solutions of Eq. (3), on the equilibrium closed orbit, are also introduced, called  $\hat{l}_0$  and  $\hat{m}_0$ . They are not periodic, but instead obey the relation

$$\begin{pmatrix} \hat{l}_0 \\ \hat{m}_0 \end{pmatrix}_{\theta+2\pi} = \begin{pmatrix} \cos(2\pi\nu) & -\sin(2\pi\nu) \\ \sin(2\pi\nu) & \cos(2\pi\nu) \end{pmatrix} \begin{pmatrix} \hat{l}_0 \\ \hat{m}_0 \end{pmatrix}_{\theta} . \quad (6)$$

The quantity  $\nu$  is the spin tune on the equilibrium closed orbit; it is the frequency, in units of revolution frequency around the ring, at which  $\hat{l}_0$  and  $\hat{m}_0$  precess around  $\hat{n}_0$ . Then  $\hat{n}$  is written in the form

$$\hat{n} \equiv V_0 \hat{n}_0 - V_1 \frac{\hat{l}_0 - i\hat{m}_0}{\sqrt{2}} + V_{-1} \frac{\hat{l}_0 + i\hat{m}_0}{\sqrt{2}} . \quad (7)$$

The use of a spherical basis simplifies the subsequent analysis, which is cumbersome in a Cartesian basis. The equation of motion for  $V_0$ , etc. is<sup>1</sup>

$$\frac{d}{d\theta} \begin{pmatrix} V_1 \\ V_0 \\ V_{-1} \end{pmatrix} = i(\bar{\omega} \cdot \vec{J})^T \begin{pmatrix} V_1 \\ V_0 \\ V_{-1} \end{pmatrix} , \quad (8)$$

where  $\vec{J}$  is a vector of spin-1 angular momentum matrices, with  $J_1$  along  $\hat{l}_0$ ,  $J_2$  along  $\hat{m}_0$  and  $J_3$  along  $\hat{n}_0$ . Using standard notation,

$$J_3 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}, \quad J_+ = J_1 - iJ_2 = \begin{pmatrix} 0 & \sqrt{2} & \\ & 0 & \sqrt{2} \\ & & 0 \end{pmatrix} \quad (9)$$

and  $J_- = J_+^\dagger$ . Blanks denote zeros in the above matrices. Then the formal solution for  $\hat{n}$  is

$$\begin{pmatrix} V_1 \\ V_0 \\ V_{-1} \end{pmatrix} = \mathbb{T} \left\{ \exp \left( i \int_{-\infty}^{\theta} (\vec{\omega} \cdot \vec{J})^T d\theta' \right) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \quad (10)$$

where  $\mathbb{T}\{\dots\}$  denotes a  $\theta$ -ordered product. As explained in Ref. 1, a convergence factor is needed to make the integral well-defined at the lower limit. This led to certain subtleties in Ref. 1, concerning the so-called ‘‘periodic terms,’’ and one had to be careful in taking the limit as the convergence factor went to zero. Throughout this paper the use of a convergence factor will be understood, as well as proper handling of the limits involved.

The above solution is completely general, but unfortunately it is also merely formal. One now has to make sense of the  $\theta$ -ordered product. This is where the restrictions needed for the SMILE algorithm of Ref. 1 come in.

### C. SMILE perturbation theory

In the SMILE algorithm, described in Ref. 1, the r.h.s. of Eq. (10) is expanded in a power series and the terms are evaluated one by one: this is the perturbation expansion. The algorithm is recursive, because each new order is obtained by multiplying the previous order by  $i(\vec{\omega} \cdot \vec{J})^T$  and integrating. It was shown in Ref. 1 how to reduce the range of integration from  $-\infty$  to  $\theta$  to one circumference, from  $\theta$  to  $\theta + 2\pi$ . Symbolically,

$$\begin{pmatrix} V_1 \\ V_0 \\ V_{-1} \end{pmatrix}_N = i \int_{-\infty}^{\theta} d\theta' (\vec{\omega} \cdot \vec{J})^T \begin{pmatrix} V_1 \\ V_0 \\ V_{-1} \end{pmatrix}_{(N-1)}, \quad (11)$$

where the subscript  $N$  denotes the  $N^{\text{th}}$  order in the perturbation expansion. In SMILE, the orbital trajectory is written in the form

$$\begin{pmatrix} \vec{r}(\theta) \\ \vec{p}(\theta) \end{pmatrix} = \sum_{\lambda} a_{\lambda} E_{\lambda}(\theta), \quad (12)$$

where  $E_{\lambda}$  is a normal mode of the orbital motion, assumed linear. The  $a_{\lambda}$  are constants which specify a particular trajectory. The approximation in SMILE is that  $\vec{\omega}$  is first-order (linear) in the orbital trajectory, i.e.

$$\vec{\omega} = \sum_{\lambda} a_{\lambda} \vec{\omega}_{\lambda}, \quad (13)$$

where  $\vec{\omega}_{\lambda} \propto E_{\lambda}$ . Hence one can say that the  $N^{\text{th}}$  term in the power series expansion of the r.h.s. of Eq. (10) is also of  $N^{\text{th}}$  order in the perturbative solution for  $\hat{n}$ ; multiplying by  $\vec{\omega}$  in Eq. (11) adds one power of  $a_{\lambda}$  to the solution for  $\hat{n}$ .

#### D. Nonlinear spin-orbit coupling

Let us now write  $\vec{\omega}$ , still using linear orbital dynamics, as a series

$$\vec{\omega} = \sum_{\lambda} a_{\lambda} \vec{\omega}_{\lambda} + \sum_{\lambda, \lambda'} a_{\lambda} a_{\lambda'} \vec{\omega}_{\lambda\lambda'} + \dots \quad (14)$$

The vector  $\vec{\omega}_{\lambda}$  can be expressed as a  $3 \times 6$  matrix multiplying a six-component column vector  $E_{\lambda}$ , and  $\vec{\omega}_{\lambda\lambda'}$  can be expressed as a  $3 \times 6 \times 6$  tensor multiplying  $E_{\lambda}$  and  $E_{\lambda'}$ , etc. Then Eq. (11) becomes

$$\begin{aligned} \begin{pmatrix} V_1 \\ V_0 \\ V_{-1} \end{pmatrix}_N &= i \sum_{\lambda} a_{\lambda} \int_{-\infty}^{\theta} d\theta' (\vec{\omega}_{\lambda} \cdot \vec{J})^T \begin{pmatrix} V_1 \\ V_0 \\ V_{-1} \end{pmatrix}_{(N-1)} \\ &+ i \sum_{\lambda, \lambda'} a_{\lambda} a_{\lambda'} \int_{-\infty}^{\theta} d\theta' (\vec{\omega}_{\lambda\lambda'} \cdot \vec{J})^T \begin{pmatrix} V_1 \\ V_0 \\ V_{-1} \end{pmatrix}_{(N-2)} + \dots, \end{aligned} \quad (15)$$

where the dots indicate the obvious extension to higher orders in nonlinearity. Note that the algorithm is still recursive, as long as the vectors  $\vec{\omega}_\lambda$ ,  $\vec{\omega}_{\lambda\lambda'}$ , etc. are known. Symbolically, the expansion of the  $\theta$ -ordered product using only linear spin-orbit coupling is (repeated indices are summed over)

$$1 + \int a_\lambda \omega_\lambda + \int a_\lambda \omega_\lambda \int a_{\lambda'} \omega_{\lambda'} + \dots ,$$

whereas now it is

$$1 + \int (a_\lambda \omega_\lambda + a_\lambda a_{\lambda'} \omega_{\lambda\lambda'} + \dots) \\ + \int (a_\lambda \omega_\lambda + a_\lambda a_{\lambda'} \omega_{\lambda\lambda'} + \dots) \int (a_\lambda \omega_\lambda + a_\lambda a_{\lambda'} \omega_{\lambda\lambda'} + \dots) + \dots ,$$

which is regrouped into the form

$$1 + \int a_\lambda \omega_\lambda + \int a_\lambda a_{\lambda'} (\omega_{\lambda\lambda'} + \omega_\lambda \int \omega_{\lambda'}) + \dots .$$

Further, note that the relationship between the  $N^{\text{th}}$  and  $(N-2)^{\text{th}}$  orders in Eq. (15) is almost the same as that between the  $N^{\text{th}}$  and  $(N-1)^{\text{th}}$  orders; basically one still multiplies by  $i(\vec{\omega} \cdot \vec{J})^T$  and integrates. One just has to sum over more indices  $\lambda, \lambda', \dots$ . This means that the extension to nonlinear spin-orbit coupling can be naturally absorbed into SMILE. The major task is to get  $\vec{\omega}_{\lambda\lambda'}$ , etc.

#### E. Calculation of $\gamma(\partial\hat{n}/\partial\gamma)$

Before beginning the next Section, I shall close this one with a description of the derivative  $\gamma(\partial\hat{n}/\partial\gamma)$  which appears in Eq. (1). I showed in Ref. 1 that  $\gamma(\partial\hat{n}/\partial\gamma)$  could be obtained easily from  $\hat{n}$ , if one knew the expansion of  $\hat{n}$  as a power series in  $a_\lambda$ . One makes the substitution

$$a_\lambda \rightarrow \gamma \frac{\partial a_\lambda}{\partial \gamma} = \mp i E_{5\lambda}^* \quad (16)$$

for one of the  $a_\lambda$  in each term in the series for  $\hat{n}$ , and sums over all substitutions. Here  $E_{5\lambda}$  is the component of  $E_\lambda$  which describes the longitudinal offset of the normal mode  $E_\lambda$  from the equilibrium closed orbit. In the usual accelerator physics coordinate system  $\{x, p_x, y, p_y, \delta s, \delta E/E_0\}$  for describing horizontal, vertical and longitudinal oscillations (with associated conjugate momenta), this is the fifth component of  $E_\lambda$ . More precisely, if one writes

$$\hat{n} = \hat{n}_0 + \sum_{\lambda} a_\lambda \bar{n}_\lambda + \sum_{\lambda, \lambda'} a_\lambda a_{\lambda'} \bar{n}_{\lambda\lambda'} + \dots, \quad (17)$$

where the  $\bar{n}_{\lambda\dots}$  are defined via Eq. (17), then

$$\gamma \frac{\partial \hat{n}}{\partial \gamma} = \sum_{\lambda} j_\lambda E_{5\lambda}^* \bar{n}_\lambda + \sum_{\lambda, \lambda'} (j_\lambda E_{5\lambda}^* a_{\lambda'} + j_{\lambda'} E_{5\lambda'}^* a_\lambda) \bar{n}_{\lambda\lambda'} + \dots, \quad (18)$$

where  $j_\lambda = -i$  if  $\lambda > 0$  and  $j_\lambda = i$  if  $\lambda < 0$ . (The conventional numbering scheme is  $\lambda = \pm 1, \pm 2, \pm 3$ , with  $E_{-\lambda} = E_\lambda^*$ .) See Ref. 1 for details.

There is *no* modification to this scheme if the spin-orbit coupling is nonlinear. A more complicated scheme is required only when the orbital dynamics is nonlinear. Thus there is no change to this part of the formalism in Ref. 1.

### III. NUMERICAL RESULTS

Fig. 1 shows the result of calculation of the polarization, up to second order, for a simple model accelerator. The resonances are identified in the figure. On the horizontal axis we plot the value of  $a\gamma = E(\text{GeV})/.440652$ , where  $a = (g - 2)/2$ . The model is a ten-fold symmetric ring (a FODO lattice) with a circumference of 1000 m, one rf cavity and one vertical kicker. The r.m.s. vertical closed orbit distortion is 1.1 mm, and no orbit correction or spin-matching techniques were used. The solid curve in Fig. 1 is the polarization using only linear spin-orbit coupling. The dashed curve includes nonlinear coupling up to quadratic order (see Appendix A). There is no observable difference between

the curves on this scale, in fact. To get a better idea of the effect of nonlinearities, one can study a different model, with stronger resonances.

Fig. 2 is a similar plot to Fig. 1, but the model is different. It is a twenty-fold symmetric ring, with a circumference of 2000 m. The tunes are  $Q_x = 17.426$ ,  $Q_y = 17.371$  and  $Q_s = 0.058$ . The r.m.s. vertical closed orbit distortion is 0.5 mm. The energy is much higher than in Fig. 1, approximately that of PETRA (in Fig. 1 the energy was roughly that of SPEAR). From the expressions for  $\vec{\omega}$  in Appendix A, we see that the magnitude of  $\vec{\omega}$  increases roughly in proportion to  $a\gamma$ , hence the resonances are stronger because of the higher energy. Note also that, because the higher-order resonance strengths are proportional to powers of the beam emittances (which grow with energy), the *relative* strength of the higher-order resonances, compared to the first-order ones, is larger than in Fig. 1. Again the two curves are almost identical, except in a narrow region near  $a\gamma = 35.2$ , which corresponds to the spin resonance  $\nu = 70 - Q_x - Q_y$ . Examining this region more closely, in Fig. 3, we see that there *is* a difference between the curves, but it is small.

#### IV. CONCLUSION

It has been shown how to include nonlinear spin-orbit coupling into the SMILE algorithm, and numerical results have been displayed taking into account nonlinear coupling quadratic in the trajectory. The nonlinearity arises because the spin precession vector is a nonlinear function of the orbital trajectory. This can happen even if the orbital dynamics is linear. It is straightforward to include higher powers of nonlinearity into SMILE. It should be noted that these extra terms do not change the spectrum of spin resonances; they merely change the strengths of the resonances.

Overall, no instances were found where the nonlinear spin-orbit coupling makes a large difference to the polarization. The above models were examined at other energy points,

but the conclusions remain unchanged. It should be noted, however, that nonlinearities in the *orbital motion* have not been taken into account here. Such nonlinearities will cause tune spreads in the orbital motion, which have not been taken into account in the above calculations. Yokoya's algorithm<sup>2</sup> will include such effects.

The aim of the above calculations was to obtain a quantitative estimate of the magnitude of the nonlinearities in the spin-orbit coupling. The above results are not necessarily disappointing; in fact, they may indicate that we are now in a position to justifiably neglect higher orders in nonlinearity of the spin-orbit coupling. This would help to lend confidence to any perturbative calculation of the polarization. Once again, however, it should be noted that nonlinearities in the orbital motion have not been treated. Further work is required to establish the contribution of such terms.

#### ACKNOWLEDGEMENTS

This work was supported by the Universities Research Association Inc., under Contract DE-AC02-76CH03000 from the Department of Energy.

## APPENDIX A: SPIN PRECESSION VECTOR

This Appendix contains expressions for the integrated strength of the spin precession vector  $\vec{\omega}$  for horizontal bending magnets, quadrupole magnets and rf cavities, up to second-order in the orbital oscillation. The quantity actually listed is therefore  $\int \vec{\omega} d\theta$ , the integral of  $\vec{\omega}$  over the length of an element. A thin-lens approximation is used. The corresponding expressions for vertical bends, skew quadrupoles and dipole kickers can be easily obtained from the results below. In all cases the coordinate system is  $\{x, x', y, y', \delta s, \delta E/E_0\}$ , where  $x$  is horizontal,  $y$  is vertical, and  $\delta s$  is the longitudinal offset from the equilibrium closed orbit. Primes denote differentiation with respect to arc-length  $s$  along the equilibrium closed orbit,  $E_0$  is the average electron energy and  $\delta E$  is the energy deviation from  $E_0$ . Other important quantities are  $\gamma_0 = E_0/mc^2$ , where  $m$  is the electron mass,  $\delta\gamma/\gamma_0 = \delta E/E_0$ , and  $a = (g - 2)/2$ , the anomalous part of the electron  $g$  factor. It will also be assumed that  $\gamma_0 \gg 1$ . Specific parameters to characterize the strengths of particular magnets will be given below.

For a horizontal dipole, the magnet is specified by its length  $L$  and bending angle  $\theta_b$ , the angle through which the equilibrium closed orbit bends. It is assumed that there is no field gradient ( $\partial B/\partial x = 0$ ). A positive bend is to the right, hence  $\hat{x}$  points outwards and  $\hat{y}$  points upwards. Then  $\int \vec{\Omega}_0 = a\gamma_0\theta_b\hat{y}$ , where “ $\int \dots$ ” denotes the integral over the length of the magnet, and

$$\begin{aligned} \int \vec{\omega} = & a\gamma_0 y' \theta_b \hat{z} + \frac{\delta\gamma}{\gamma_0} \theta_b \hat{y} - (1 + a\gamma_0) \frac{\theta_b^2}{L} x \hat{y} \\ & - \left(\frac{\delta\gamma}{\gamma_0}\right)^2 \theta_b \hat{y} + a\gamma_0 \theta_b y' (x' \hat{x} + y' \hat{y}) + x \frac{\delta\gamma}{\gamma_0} \frac{\theta_b^2}{L} \hat{y}. \end{aligned} \quad (A1)$$

The effect of edge focussing (the term in  $\theta_b^2/L$ ) has been included above. It requires that  $L \neq 0$ , which is a little against the spirit of the thin-lens approximation  $L \rightarrow 0$ , but otherwise some needlessly complicated scheme would have to be used to get this term. For large accelerators, which typically have gentle bends ( $\theta_b \ll 1$ ), this term is not important.

For a quadrupole, the magnet is specified by its focussing strength  $k$ , the inverse of its focal length. Then  $\int \vec{\Omega} = 0$  and

$$\int \vec{\omega} = -(a\gamma_0 + 1)k(x\hat{y} + y\hat{x}) + \frac{\delta\gamma}{\gamma_0}k(x\hat{y} + y\hat{x}) + a\gamma_0k(xy' + yx')\hat{z} . \quad (A2)$$

For an rf cavity, the relevant parameter is  $r = (eV \sin \phi_s)/E_0$ , where  $e$  is the magnitude of the electron charge,  $V$  is the peak voltage and  $\phi_s$  is the phase of the synchronous orbit (the equilibrium closed orbit). Then  $\int \vec{\Omega} = 0$  and

$$\int \vec{\omega} = r(a\gamma_0 + 1)(y'\hat{x} - x'\hat{y}) - r\frac{\delta\gamma}{\gamma_0}(y'\hat{x} - x'\hat{y}) . \quad (A3)$$

These expressions have been coded into SMILE, together with expressions for  $\int \vec{\omega}$  for vertical dipoles, skew quadrupoles, dipole kickers, and an equilibrium closed orbit displaced from the “ideal equilibrium closed orbit” assumed above.

- <sup>1</sup> S.R. Mane, *Phys. Rev. A* **36**, 120 (1987).
- <sup>2</sup> K. Yokoya, *Nucl. Instrum. Meth.*, **A258**, 149 (1987), has published an algorithm, using Lie algebra, to include nonlinear orbital dynamics and nonlinear spin-orbit coupling, but a computer program is not yet available.
- <sup>3</sup> Ya.S. Derbenev and A.M. Kondratenko, *Zh. Eksp. Teor. Fiz.* **64**, 1918 (1973) [ *Sov. Phys. JETP* **37**, 968 (1973) ]. A rederivation is given in S.R. Mane, *Phys. Rev. A* **36**, 105 (1987).
- <sup>4</sup> Eq. (1) contains takes into account only the longitudinal recoil (energy loss) of an electron due to photon emission. There are also effects due to transverse momentum recoil, because of the nonzero (though small) angle of photon emission relative to the electron velocity. These lead to extra terms in the polarization formula. See J.S. Bell and J.M. Leinaas, *Nucl. Phys.* **B284**, 488 (1987), D.P. Barber and S.R. Mane, DESY 87-049 (1987) (unpublished) and *Phys. Rev. A*, in press, and S.R. Mane, FN-465 (1987) (unpublished).

Fig. 1

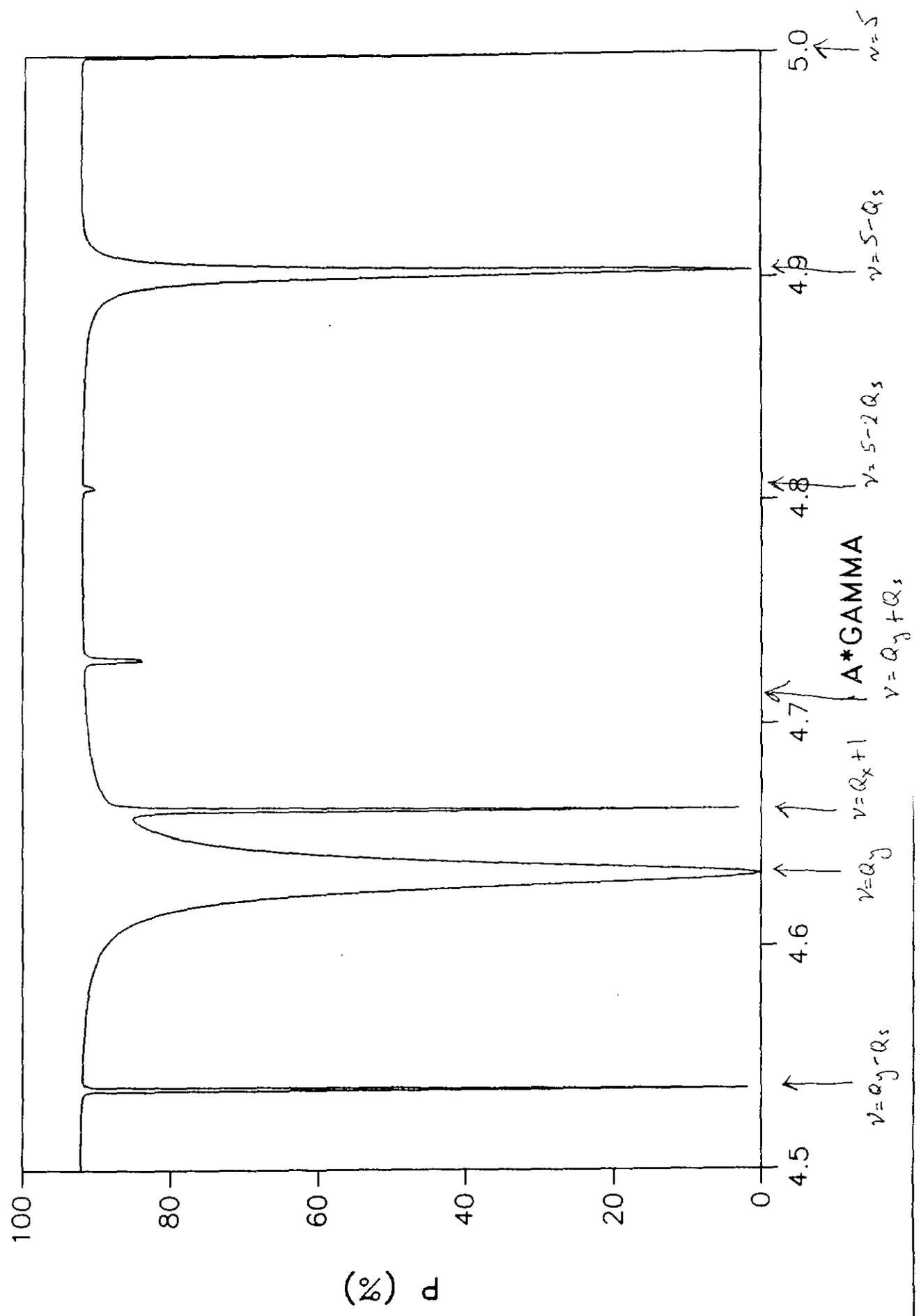


Fig. 2

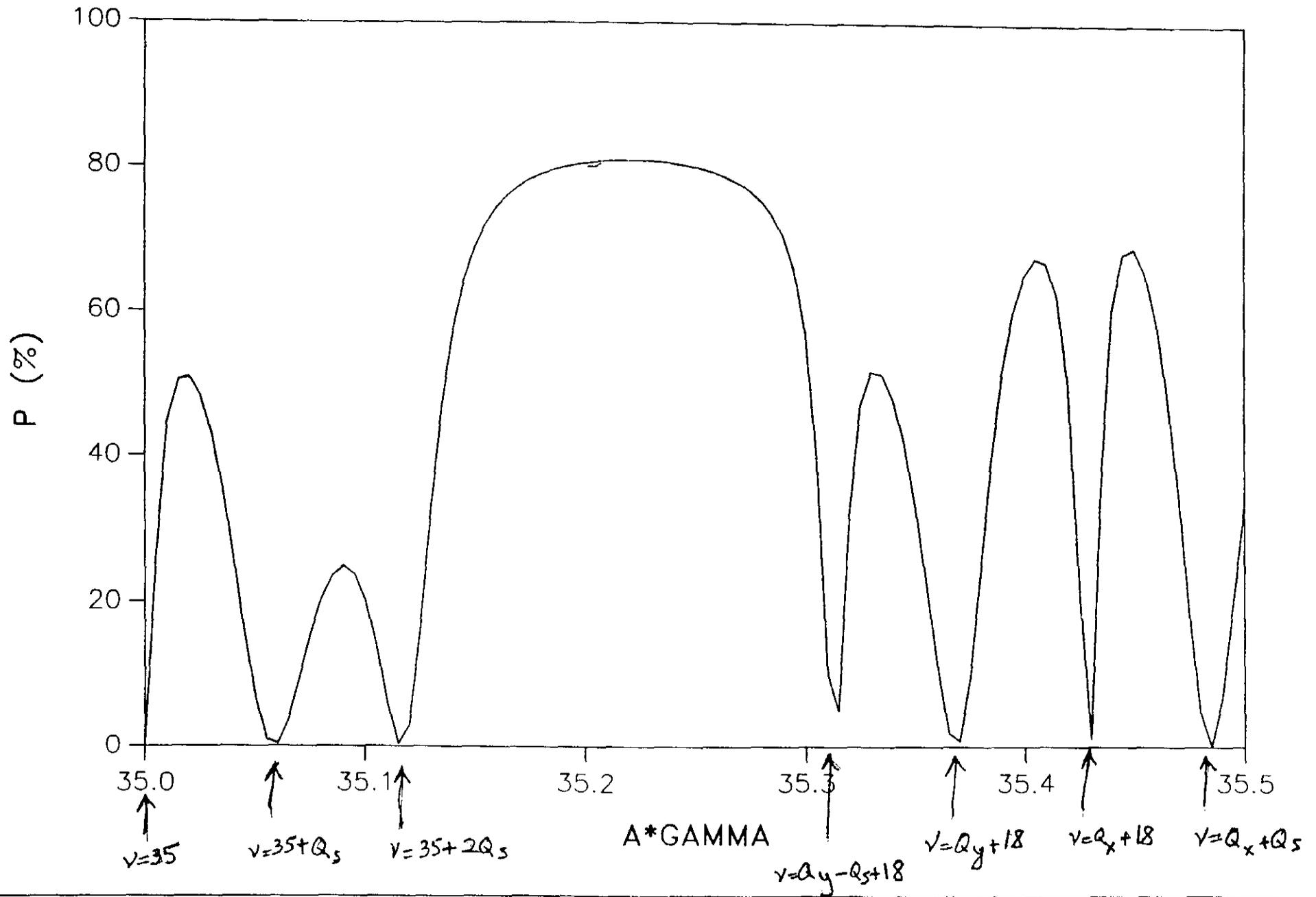


Fig. 3

