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Injection Mismatch and Phase Space Dilution*

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M. J. Syphers

Introduction

Modern high energy proton accelerator facilities employ a series of machines of various intermediate energies. In the design of beam transport systems between accelerators, the primary concern is to match the betatron amplitude functions, dispersion functions, and of course the ideal beam trajectory coming from the first synchrotron to those of the second synchrotron. If a proper match is not provided, an increase in the transverse emittance, the area in transverse phase space occupied by the particles, will occur. The degree to which deviations from an ideal match affect the transverse phase space emittances is the subject of this article.

It is assumed throughout that the particles of the beam do not interact with one another, and that each obeys linear and uncoupled equations of motion in the two transverse degrees of freedom. In a real synchrotron, a mismatch will result in gradual filamentation in the transverse phase space as a consequence of field nonlinearities. Here, the time average distribution of the linear motion will be used to provide a model-independent account of the emittance dilution pursuant to filamentation.

Given a particle with an initial coordinate in phase space, the resulting time average distribution in the transverse coordinate may be obtained for that particle. Using this result an expression for the final distribution of many particles, given their initial distribution, may be found. The area in phase space which contains a certain fraction (95%, say) of an incoming distribution after dilution may be computed as well. Initial distributions generated by various forms of mismatch may then be inserted into these expressions to yield resulting time average distributions and emittance dilution factors for the particle beam. This process is performed for the three types of mismatch mentioned above and, through a set of simple statistical arguments, general expressions for the variances of these distributions are derived.

Time Average Distribution of a Single Particle

If the transverse motion in one degree of freedom of a single particle is observed at a particular longitudinal location s in a synchrotron, the solution to Hill's Equation yields a trajectory of the particle in transverse phase space given by

$$x^2 + (\beta(s)x' + \alpha(s)x)^2 = \beta(s)A^2$$

where A is a constant determined by the initial conditions x_o and x'_o. The quantity β is the characteristic betatron amplitude function of the ring and $\alpha \equiv -\beta'/2$. If now a new phase space variable $\varsigma \equiv \beta x' + \alpha x$ is defined, then the trajectory may be written as

$$x^2 + \varsigma^2 = a^2$$

which is the equation of a circle in $x \cdot \varsigma$ space with radius $a = (\beta)^{1/2}A$. The quantity a will be referred to as the amplitude of the oscillation and has units of length.

A particle enters the accelerator and upon its first passage through point s the particle has phase space coordinates (x_o, ς_o). Upon subsequent revolutions about the machine, the particle will reappear at point s with phase space coordinates (x, ς) which lie on a circle of radius $a = (x_o^2 + \varsigma_o^2)^{1/2}$. The exact location on the circle after each revolution will depend upon the phase advance of the betatron oscillation for one complete revolution. This phase advance may depend upon the amplitude a. (In fact, in a real synchrotron, the phase advance undoubtedly will depend upon amplitude due to nonlinear field imperfections within the machine, and the phase space trajectory will be distorted from that of a circle. Here, these distortions are assumed to be small enough to ignore.) Over a long period of time, the probability of finding the particle at a specific transverse displacement x may be computed. If the quantities x and ς are parameterized by the relationships

$$x = a\cos\omega t, \qquad \varsigma = a\sin\omega t,$$

then the phase space distribution of the particle will be given by

$$f(x,\varsigma,t)\ dx\ d\varsigma \ = \ \delta(x-a\cos\omega t)\ \delta(\varsigma-a\sin\omega t)\ dx\ d\varsigma$$

where $\delta(u)$ is the Dirac delta function. Integrating over ς yields

$$g(x,t)\ dx \ = \ \delta(x - a\cos\omega t)\ dx\ .$$

To find the time average distribution in x, g(x,t) is integrated over a cycle of period $\tau = 2\pi/\omega$ yielding

$$n_a(x)dx = dx\ \frac{2}{\tau}\int_0^{\tau/2}\delta(x-a\cos\omega t)dt = dx\ \frac{2}{\tau}\int_{-a}^{a}\delta(x-u)\frac{du}{a\omega[1-(u/a)^2]^{1/2}}$$

or,

$$n_a(x)\ dx \ = \ \frac{1}{\pi a}\ \frac{dx}{\sqrt{1 - (x/a)^2}}\ .$$

So, given the initial condition in transverse phase space (x_o,x'_o), over a long period of time the probability of finding the particle between x and x+dx is $n_a(x)dx$.

Time Average Distribution Given an Initial Distribution of Many Particles

Given an initial distribution of particles $n_o(x,\varsigma)dxd\varsigma$ within the synchrotron at location s, then the resulting time average distribution of the particles may be found. By converting x and ς to polar coordinates, the number of particles which are located within a circle of radius a is given by

$$f(a) = \int_0^{2\pi} \int_0^a n_o(r,\theta)\ r\ dr\ d\theta$$

and the number of particles between two circles of radii a and a+da is

$$\frac{\partial f(a)}{\partial a}\, da = da \int_0^{2\pi} n_o(a,\theta)\ a\ d\theta\ .$$

Thus, the contribution of a particular ring of radius a and thickness da to the resulting time average distribution in x is

$$n_a(x)\ dx = \frac{1}{\pi} \frac{dx\ da}{\sqrt{1 - (x/a)^2}} \int_0^{2\pi} n_o(a,\theta)\ d\theta\ .$$

Upon adding up all contributions due to all pertinent rings (i.e., $a \geq |x|$), the resulting time average distribution in x will be

$$n(x)\ dx = \frac{dx}{\pi} \int_{|x|}^{\infty} \int_0^{2\pi} \frac{n_o(a,\theta)}{\sqrt{1 - (x/a)^2}}\ d\theta\ da\ .$$

Using the above equation, the resulting time average distribution of particles in one transverse degree of freedom may be computed given the initial distribution of particles delivered by the beamline. A perfect match of the beamline to the synchrotron would produce a resulting time average distribution of $n(x) = \int n_o(x,\varsigma)d\varsigma$.

Emittance Dilution

Let the emittance ϵ be defined as the area in transverse $x \cdot x'$ phase space which contains 95% of the particles. The differential element of area in $x \cdot \varsigma$ phase space is simply

$$dx\ d\varsigma = \beta\ dx\ dx'.$$

Thus, the area in $x \cdot \varsigma$ phase space which contains 95% of the particles is $\beta\epsilon$.

This area may be found by determining the value of a_{95} which satisfies the equation

$$f(a_{95}) = \int_0^{2\pi} \int_0^{a_{95}} n_o(r,\theta)\ r\ dr\ d\theta = 0.95\ N$$

where N is the number of particles in the beam. If the incoming distribution is Gaussian in x with variance $\sigma_o{}^2$ and is perfectly matched to the synchrotron, the resulting distribution after many revolutions will also be Gaussian with variance $\sigma_o{}^2$. The area in $x \cdot \varsigma$ phase space which contains 95% of the particles will be $\pi a_{95}{}^2 = \beta\epsilon_o = 6\pi\sigma_o{}^2$. If the incoming distribution is not perfectly matched to the synchrotron, then the resulting time average distribution in x will be broader than the incoming distribution and the value of a_{95} will be greater than $(6\sigma_o{}^2)^{1/2}$. Thus, the effective area in phase space which the beam will occupy increases by a factor of

$$F = \frac{\beta\epsilon}{\beta\epsilon_o} = \frac{\pi a_{95}{}^2}{6\pi\sigma_o{}^2} = \frac{a_{95}{}^2}{6\sigma_o{}^2}$$

due to a mismatch. F will be referred to as the dilution factor.

Simplifying Assumptions and Units

Different sources of mismatch will produce different initial distributions n_o and hence will result in different final distributions and dilution factors. Before investigating three of these sources, some simplifying assumptions are made and units are defined which are used throughout the discussions.

All quantities associated with the beamline are denoted by a subscript "1" and those of the synchrotron are denoted by a subscript "2." For example, if s represents the injection point or some other point within the synchrotron "downstream" of the injection point, then $\beta_2(s)$ would be the value of the natural amplitude function of the machine at that point and $\beta_1(s)$ would be the value of the amplitude function at that point as delivered by the beamline. The distribution of particles at point s is considered to be Gaussian in $x_1 \cdot \varsigma_1$ space and may be written as

$$n(x_1,\varsigma_1)\ dx_1\ d\varsigma_1 = \frac{N\ e^{-\left[x_1^{\ 2} + \varsigma_1^{\ 2}\right]/2\sigma_1^{\ 2}}}{2\pi\sigma_1^{\ 2}} \quad dx_1\ d\varsigma_1\ .$$

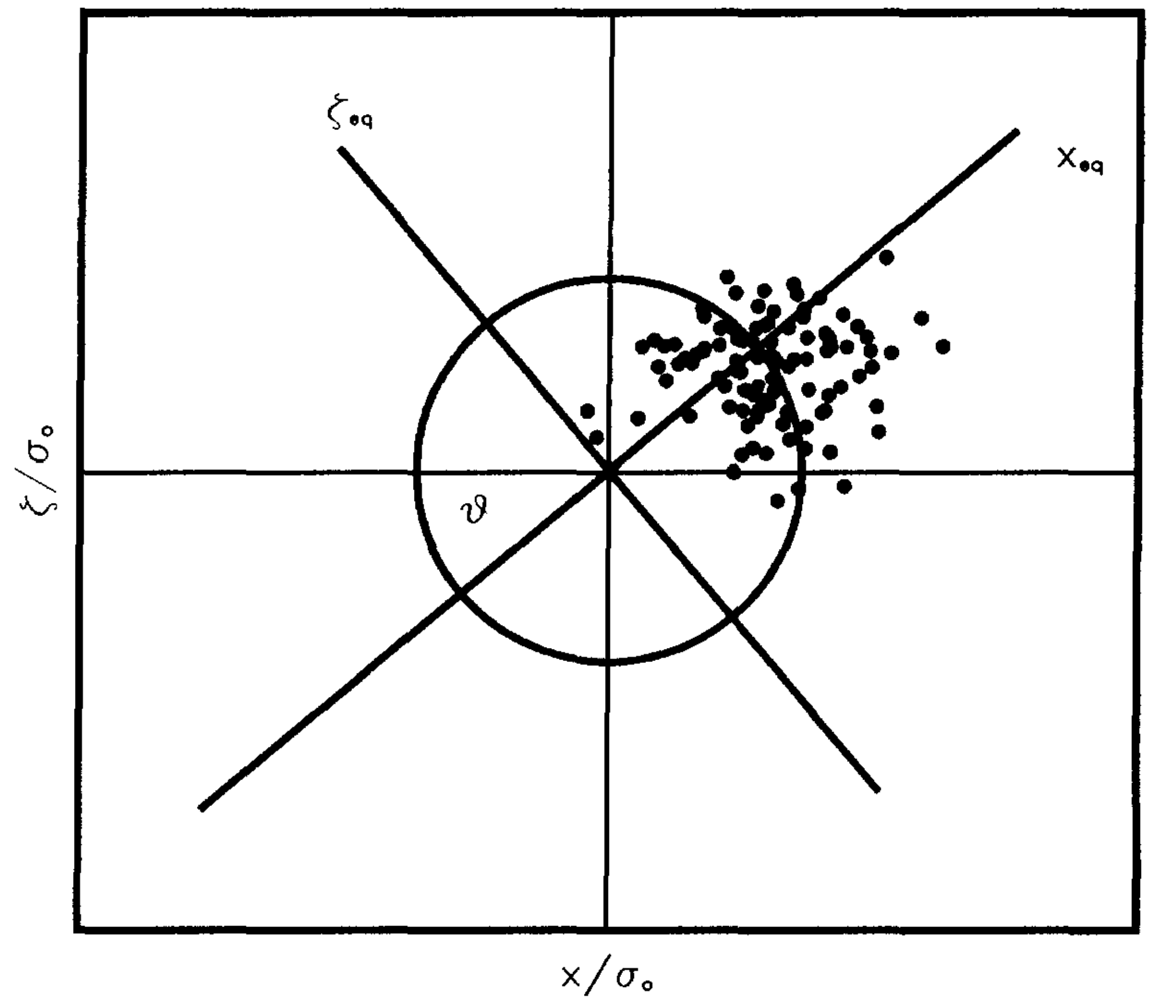
Amplitudes of particle motion in phase space are assumed not to vary with time so that the techniques described in the last two sections may be applied to determine the resulting particle distributions.[1]

Because the resulting trajectory of a particle in $x_2 \cdot \varsigma_2$ phase space is a circle, the resulting time average particle distribution in phase space will always be symmetric about the origin. For the cases of Gaussian beams which are studied, the $x_2 \cdot \varsigma_2$ axes may be rotated through an angle θ to make the original distribution $n_o(x_2,\varsigma_2)$ symmetric about the x_2 axis. For example, if both the position and slope of the incoming trajectory are mismatched by the amounts Δx and $\Delta x'$ respectively, the axes may be rotated and the problem treated as though only x_2 was mismatched by an amount Δx_{eq}. (See Figure 1.) The values of F computed with the axes rotated or not rotated will be the same.

To simplify the expressions, all lengths (x, ς, a) are written in units of the standard deviation of the perfectly matched beam. That is, $\sigma_{2o} = (\epsilon_o\beta_2/6\pi)^{1/2} \equiv \sigma_o \equiv 1$. The quantity ϵ_o is the transverse emittance delivered by the beamline. In addition, the particle distribution is normalized to one particle, i.e., N = 1.

Figure 1

Determination of Equivalent Mismatch
Through Rotation of Phase Space Axes



Injection Position Mismatch

For the case of a mismatch of the ideal trajectory, the amplitude functions and dispersion functions delivered by the beamline are assumed to be matched to the synchrotron: $\beta_1 = \beta_2$, $\alpha_1 = \alpha_2$, $D_1 = D_2$, and $D'_1 = D'_2$. If Δx and $\Delta x'$ are the errors in the incoming position and slope for one transverse degree of freedom of the beamline's ideal trajectory, then

$$x_2 = x_1 + \Delta x \qquad \text{and} \qquad x'_2 = x'_1 + \Delta x'$$

which implies

$$\varsigma_2 = \varsigma_1 + \Delta\varsigma \ .$$

The $x_2 \cdot \varsigma_2$ axes may be rotated through an angle θ given by

$$\tan\theta = \Delta\varsigma/\Delta x$$

so that the problem is equivalent to one in which the incoming distribution is displaced only in position by an amount

$$\Delta x_{eq} = (\Delta x^2 + (\beta\Delta x' + \alpha\Delta x)^2)^{1/2} \ .$$

From now on, Δx will be used to represent Δx_{eq}.

Since the amplitude functions are assumed to be matched, $\sigma_1 = \sigma_2 = \sigma_o = 1$ and the incoming distribution may be written as

$$n_o(x_2,\varsigma_2)\ dx_2\ d\varsigma_2 = \frac{e^{-\left[(x_2-\Delta x)^2 + \varsigma_2^2\right]/2}}{2\pi}\qquad dx_2\ d\varsigma_2 \ .$$

Using $n_x(x)dx$ to denote the final time average distribution due to a position mismatch, dropping the subscript "2" for the moment, and switching to polar coordinates,

$$n_x(x)\ dx \ = \frac{dx}{2\pi^2}\int_{|x|}^{+\infty}\int_0^{2\pi} \frac{e^{-\frac{1}{2}\left(a^2+\ \Delta x^2-\ 2a\Delta x\cos\theta\right)}}{\sqrt{1\ -(x/a)^2}}\ d\theta\ da$$

or, reducing the expression to a single integral,

$$n_x(x)\ dx = \frac{dx}{\pi} \int_{|x|}^{+\infty} e^{-(a-|\Delta x|)^2/2} \frac{e^{-|a\Delta x|} I_o(a\Delta x)}{\sqrt{1-(x/a)^2}}\ da$$

where $I_o(z)$ is the modified Bessel function of order zero.[2] The dilution factor F_x for a particular position error Δx is found from

$$F_x = a_{95}^2/6$$

where a_{95} satisfies

$$f_x(a_{95}) = \int_0^{2\pi} \int_0^{a_{95}} \frac{1}{2\pi} e^{-(r^2+\Delta x^2-2r\Delta x\cos\theta)/2}\ r\ dr\ d\theta$$

$$= \int_0^{a_{95}} e^{-(r-|\Delta x|)^2/2} \left(e^{-|r\Delta x|} I_o(r\Delta x) \right) r\ dr = 0.95 \quad .$$

As a check, notice that for $\Delta x = 0$, the expressions for $n_x(x)dx$ and f_x reduce to

$$n_x(x)\ dx = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\ dx$$

and,

$$f_x(a_{95}) = 1 - e^{-a_{95}^2/2} = 0.95$$

which implies

$$a_{95}^2 = 6 \text{ and thus } F_x = 1 \text{ , as expected.}$$

Numerical integration of the expressions for n_x and f_x yields the results displayed in Figures 2 and 3. If the injected beam is displaced by more than about twice the standard deviation of the initial particle distribution, the resulting time average distribution exhibits a "double hump" appearance. Up

Figure 2

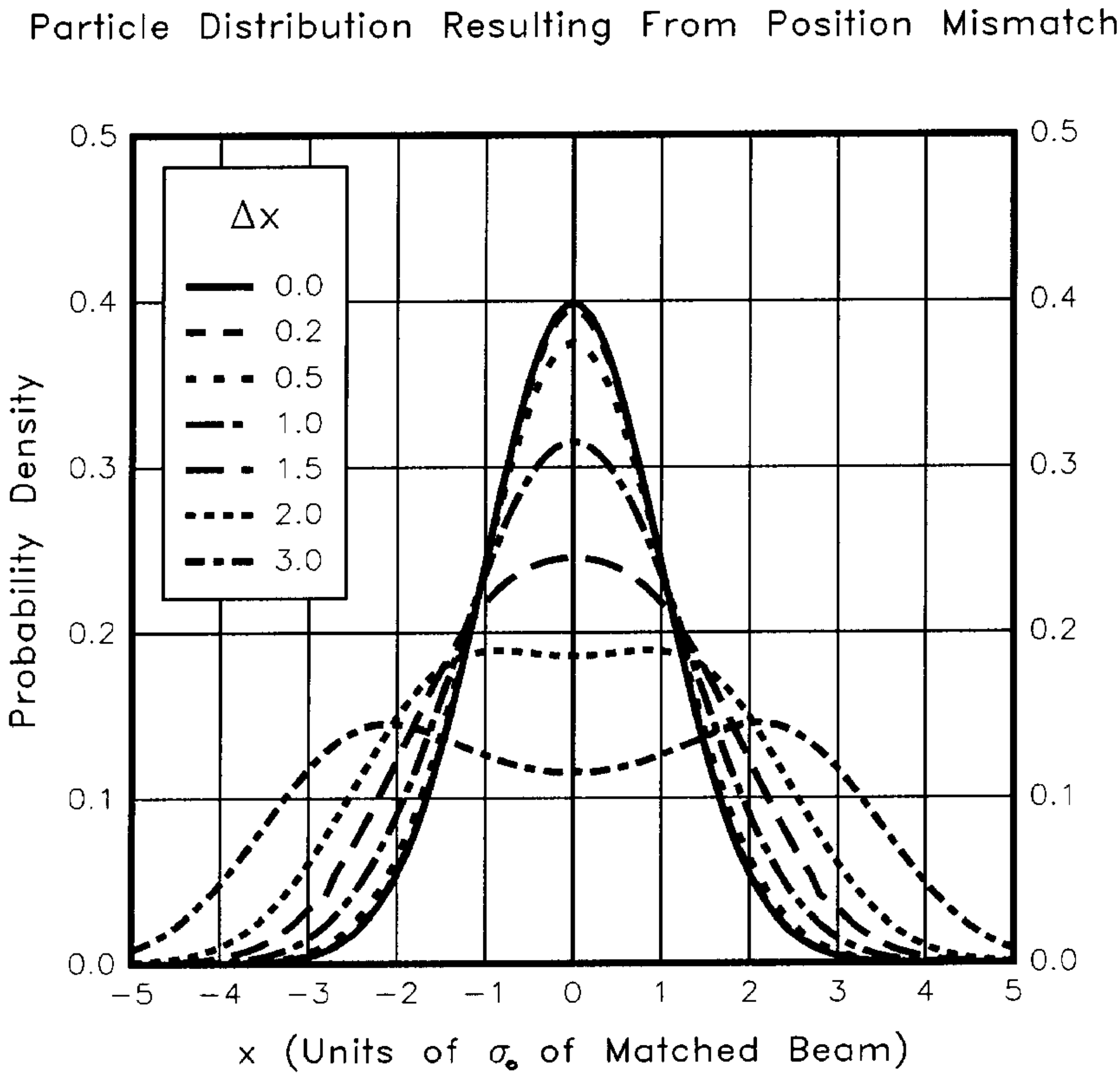
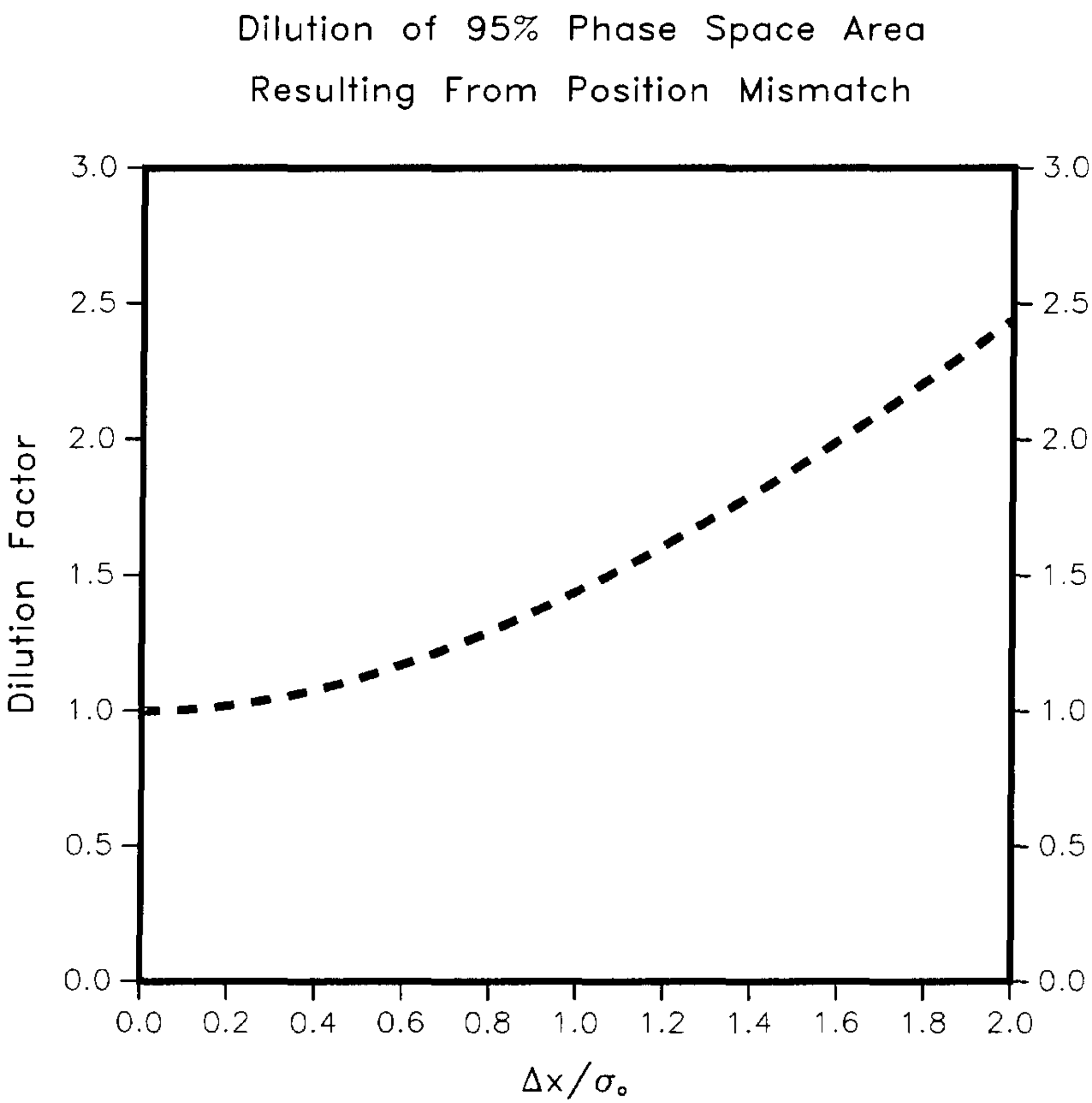

Particle Distribution Resulting From Position Mismatch
Δx
0.0
0.2
0.5
1.0
1.5
2.0
3.0
Probability Density
0.0
0.1
0.2
0.3
0.4
0.5
−5
−4
−3
−2
−1
0
1
2
3
4
5
x (Units of σ₀ of Matched Beam)

Figure 3


Dilution of 95% Phase Space Area
Resulting From Position Mismatch
Dilution Factor
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0
$\Delta x/\sigma_o$

to this point, the variance of the resulting distribution is equal to the dilution factor, to within a few percent. For Δx greater than about $2\sigma_o$, the variance is greater than F_x. An explicit relationship between the variance of the distribution and the position error Δx is derived later in this chapter.

Dispersion Function Mismatch

For this discussion the ideal trajectory and the amplitude functions of the beamline are assumed to be matched to those of the synchrotron: $\beta_1 = \beta_2$, $\alpha_1 = \alpha_2$, $x_2 = x_1$, and $x'_2 = x'_1$. For a particle of momentum $p+\Delta p$, where p is the ideal momentum, the equilibrium orbit within the synchrotron is given by $(x_2,x'_2) = (D\Delta p/p, D'\Delta p/p)$, where D is the dispersion function of the accelerator. The distribution of momenta $\Delta p/p$ is assumed to be Gaussian with a mean of zero and a variance σ_p^2. The quantity δ is defined by

$$\Delta p/p = \sigma_p \delta .$$

Thus, for a particular momentum, the ideal trajectory within the synchrotron is located at the phase space point

$$(x_2,\varsigma_2) = (D\sigma_p\delta, \Lambda\sigma_p\delta), \quad \text{where } \Lambda \equiv \beta D' + \alpha D .$$

The displacement of the trajectory of a particle of momentum $p+\sigma_p\delta$ about its respective ideal orbit may be written in terms of

$$\xi = x_2 - D\sigma_p\delta \quad \text{and} \quad \eta = \varsigma - \Lambda\sigma_p\delta .$$

Because a mismatch of the dispersion function may be thought of as a steering error for off-momentum particles, the results of the last section may be used. The dispersion function delivered by the beamline, and its slope, may be written as

$$D_2 = D_1 + \Delta D, \quad \text{and} \quad D'_2 = D'_1 + \Delta D' .$$

As before, an equivalent mismatch of the dispersion function alone may be defined, namely

$$\Delta D_{eq} \equiv (\Delta D^2 + (\beta\Delta D' + \alpha\Delta D)^2)^{1/2} .$$

From now on, ΔD is used to represent ΔD_{eq}. Using these definitions, the resulting time average distribution of particles of momentum $p+\sigma_p\delta$ due to a mismatch of the dispersion function by an amount ΔD is

$$n_\delta(\xi)\ d\xi = \frac{d\xi}{\pi} \int_{|\xi|}^{\infty} e^{-(a-|\Delta\xi|)^2/2} \frac{e^{-|a\Delta\xi|} I_o(a\Delta\xi)}{\sqrt{1-(\xi/a)^2}}\ da$$

where $\Delta\xi = \Delta D\sigma_p\delta$. In the above equation, the distribution is normalized to the number of particles with momentum $p+\sigma_p\delta$. By integrating over the entire momentum distribution, the resulting time average distribution in x is found to be

$$n_D(x)\ dx = dx \int_{-\infty}^{+\infty} n_\delta(x) \frac{1}{\sqrt{2\pi}} e^{-\delta^2/2}\ d\delta$$

$$= \frac{dx}{\sqrt{2\pi^3}} \int_{-\infty}^{+\infty} \int_{|x-D\sigma_p\delta|}^{+\infty} e^{-\delta^2/2}\ e^{-(a-|\Delta D\sigma_p\delta|)^2/2} \frac{e^{-|a\Delta D\sigma_p\delta|} I_o(a\Delta D\sigma_p\delta)}{\sqrt{1-\left(\frac{x-D\sigma_p\delta}{a}\right)^2}}\ da\ d\delta.$$

Here, $D = D_2$ is the dispersion function of the synchrotron at the point of observation.

The phase space dilution factor is given by $F_D = a_{95}^2/6$ where

$$\frac{1}{\sqrt{2\pi}} \int_0^{a_{95}} \int_{-\infty}^{+\infty} e^{-\delta^2/2}\ e^{-(r-|\Delta D\sigma_p\delta)^2/2} \left[e^{-|r\Delta D\sigma_p\delta|} I_o(r\Delta D\sigma_p\delta)\right] r\ d\delta\ dr = 0.95$$

Once again, for $\Delta D = 0$, $a_{95}^2 = 6$ and $F_D = 1$. Also, for $\Delta D = 0$, the distribution n_D becomes

$$n_D(x)\ dx \ = \ \frac{1}{\sqrt{2\pi(1+D^2\sigma_p{}^2)}} \quad e^{-x^2/2(1+D^2\sigma_p{}^2)} \ dx$$

which is a Gaussian distribution with variance $\sigma^2 = 1 + D^2\sigma_p{}^2$. (Note that $1 = \sigma_o{}^2 = \epsilon_o\beta_o/6\pi$.) The variance of the beam distribution at a location in the beamline or synchrotron where the dispersion function is nonzero is given by $\sigma^2 = \sigma_t{}^2 + D^2\sigma_p{}^2$ where σ_t is due to the transverse emittance alone. The result of a dispersion function mismatch is to increase the transverse emittance and hence increase σ_t. By studying the resulting time average distribution with D set to zero, the total variance of the distribution will be given by $D^2\sigma_p{}^2$ + the variance of $n_D(D=0)$.

Figure 4 shows the distribution $n_D(D=0)$ for several values of dispersion mismatch. The dilution factor due to a dispersion mismatch is displayed in Figure 5. The severity of the emittance dilution depends upon both ΔD and σ_p as it must. If all of the particles are of the exact same momentum, the beam size would not increase no matter how large a value for ΔD is obtained. Likewise, any small deviation from the ideal dispersion function significantly affects the emittance of a beam which has a large enough momentum spread. Even for values of $\Delta D\sigma_p$ as large as 1.5, the variance of the resulting time average distribution provides a good estimate of the dilution factor F_D.

To remind the reader, the quantities Δx and $\Delta D\sigma_p$ used in these discussions are in units of $\sigma_o = (\epsilon_o\beta_o/6\pi)^{1/2}$ of the ideally matched beam. Thus, for a beam that is inherently large upon entrance to the synchrotron, the effects of mismatches due to steering and dispersion function errors actually may be small.

Figure 4

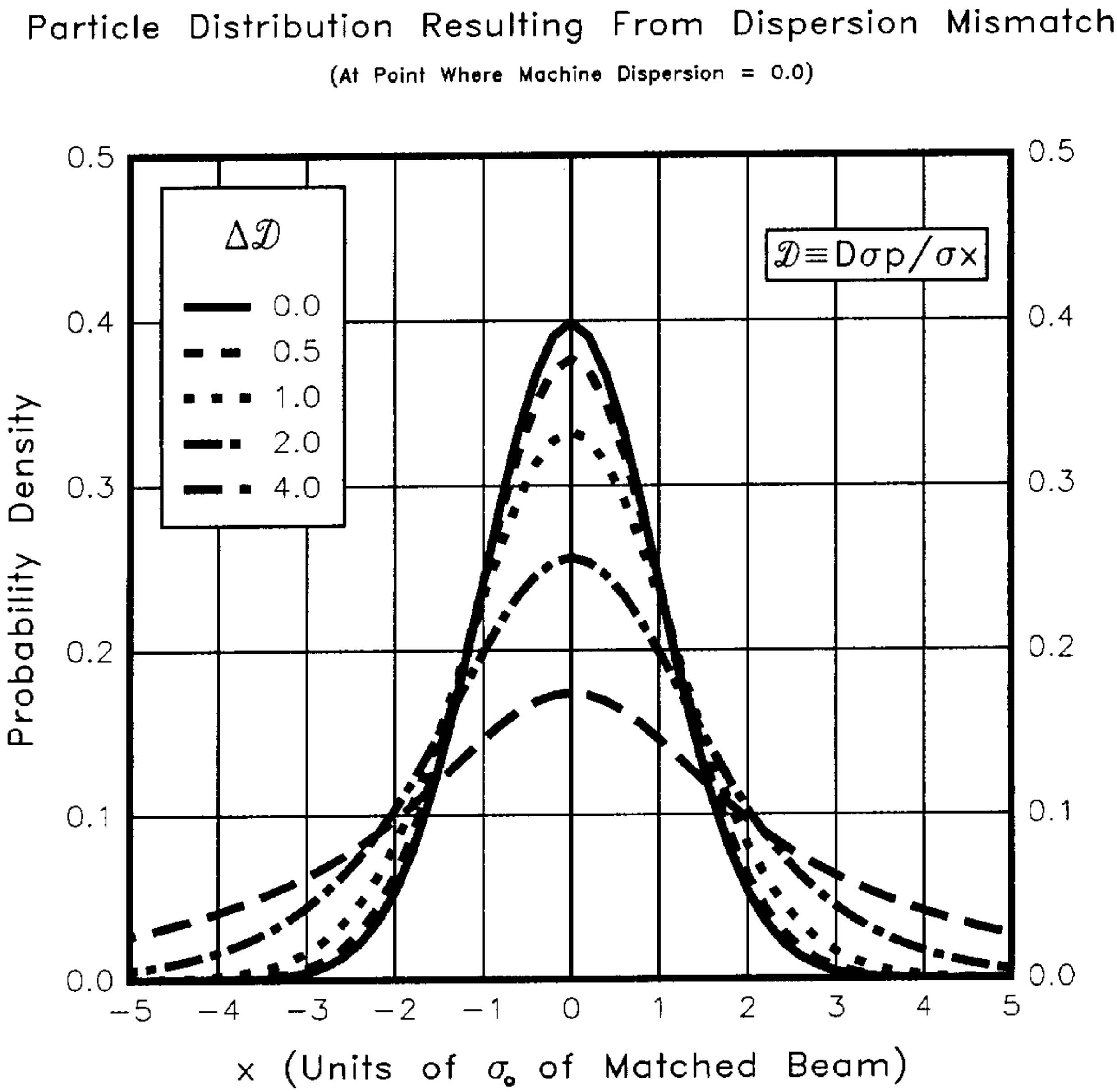
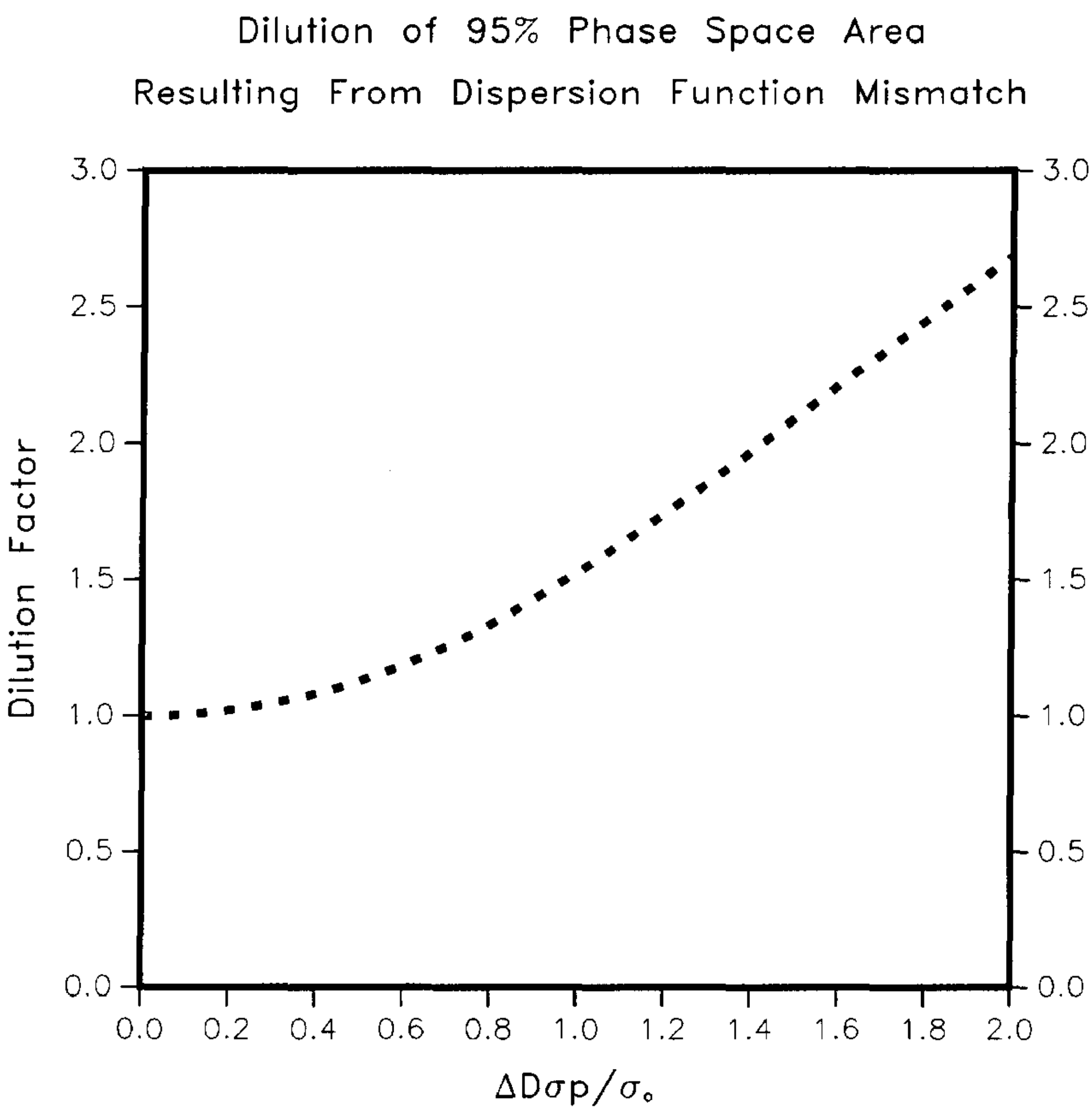

Particle Distribution Resulting From Dispersion Mismatch
(At Point Where Machine Dispersion = 0.0)
ΔD
0.0
0.5
1.0
2.0
4.0
D≡Dσp/σx
Probability Density
0.0
0.1
0.2
0.3
0.4
0.5
−5
−4
−3
−2
−1
0
1
2
3
4
5
x (Units of σ₀ of Matched Beam)

Figure 5


Dilution of 95% Phase Space Area
Resulting From Dispersion Function Mismatch
Dilution Factor
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
ΔDσp/σ₀

Amplitude Function Mismatch

For this case, the ideal trajectory as well as the dispersion functions of the beamline are assumed to be matched to the synchrotron: $x_1 = x_2 = x$, $x'_1 = x'_2 = x'$, $D_1 = D_2$, and $D'_1 = D'_2$. The incoming particle distribution may be written in the form

$$n(x_1,\varsigma_1)\, dx_1\, d\varsigma_1 = \frac{e^{-\left[x_1^2 + \varsigma_1^2\right]/2\sigma_1^2}}{2\pi\sigma_1^2} \quad dx_1\, d\varsigma_1$$

Since $\varsigma_1 = \alpha_1 x + \beta_1 x'$ and $\varsigma_2 = \alpha_2 x + \beta_2 x'$, then x_1 and ς_1 may be written in terms of x_2 and ς_2:

$$x_1 = x_2 \quad ,$$

$$\varsigma_1 = (\alpha_1 - \beta_1\alpha_2/\beta_2)x_2 + \beta_1\varsigma_2/\beta_2 \ .$$

Let the emittance being delivered by the beamline be denoted by ϵ_o. If the beamline were perfectly matched, then the beam size in the second synchrotron after injection would be given by $\sigma_o^2 = \epsilon_o\beta_2/6\pi$. This same emittance is assumed to be delivered through the mismatched beamline giving $\sigma_1^2 = \epsilon_o\beta_1/6\pi$, so that $\beta_1/\sigma_1^2 = \beta_2/\sigma_o^2$. Noting also that $dx_1\, d\varsigma_1 = \beta_1\, dx\, dx'$ and $dx_2\, d\varsigma_2 = \beta_2\, dx\, dx'$, the initial distribution in x_2·ς_2 phase space is (with the subscript 2's suppressed on x,ς's)

$$n_o(x,\varsigma)dx\, d\varsigma = \frac{e^{-\left[x^2 + (\alpha_1 x + \beta_1(\varsigma - \alpha_2 x)/\beta_2)^2\right]\beta_2/2\beta_1}}{2\pi} \quad dx\, d\varsigma.$$

noting that $\sigma_o^2 = 1$. The bracket in the exponent may be rewritten as

$$[\ ] = a\, x^2 + 2b\, x\varsigma + c\, \varsigma^2$$

where

$a = 1 + (\alpha_1 - \alpha_2\beta_1/\beta_2)^2$, $b = (\beta_1/\beta_2)(\alpha_1 - \alpha_2\beta_1/\beta_2)$, $c = (\beta_1/\beta_2)^2$.

By performing a rotation of coordinate axes,

$$\xi = x\cos\theta + \varsigma\sin\theta$$

$$\eta = -x\sin\theta + \varsigma\cos\theta$$

and choosing θ such that the $\xi\cdot\eta$ term is zero, i.e., $\tan 2\theta = 2b/(a-c)$, the exponent may be expressed as

$$[\ ] = A\,\xi^2 + B\,\eta^2$$

where

$$A = (1/2)[(a+c) + (a-c)\cos 2\theta + 2b\sin 2\theta],$$

$$B = (1/2)[(a+c) - (a-c)\cos 2\theta - 2b\sin 2\theta].$$

After substituting the expressions for a, b, and c above, the coefficient A reduces to

$$A = (\beta 1/\beta 2)[\ D + (D^2 - 1)^{1/2}]$$

where

$$D \equiv (\beta 2\gamma 1 + \beta 1\gamma 2 - 2\alpha 1\alpha 2)/2.$$

Similarly, B is given by

$$B = (\beta 1/\beta 2)/[\ D + (D^2 - 1)^{1/2}].$$

Hence, the final expression for the incoming particle distribution due to a mismatch of the amplitude function may be written as

$$n_o(\xi,\eta)\,dx\,d\varsigma = \frac{e^{-\left[\beta_{eq}\xi^2 + \eta^2/\beta_{eq}\right]/2}}{2\pi}\,d\xi\,d\eta.$$

where

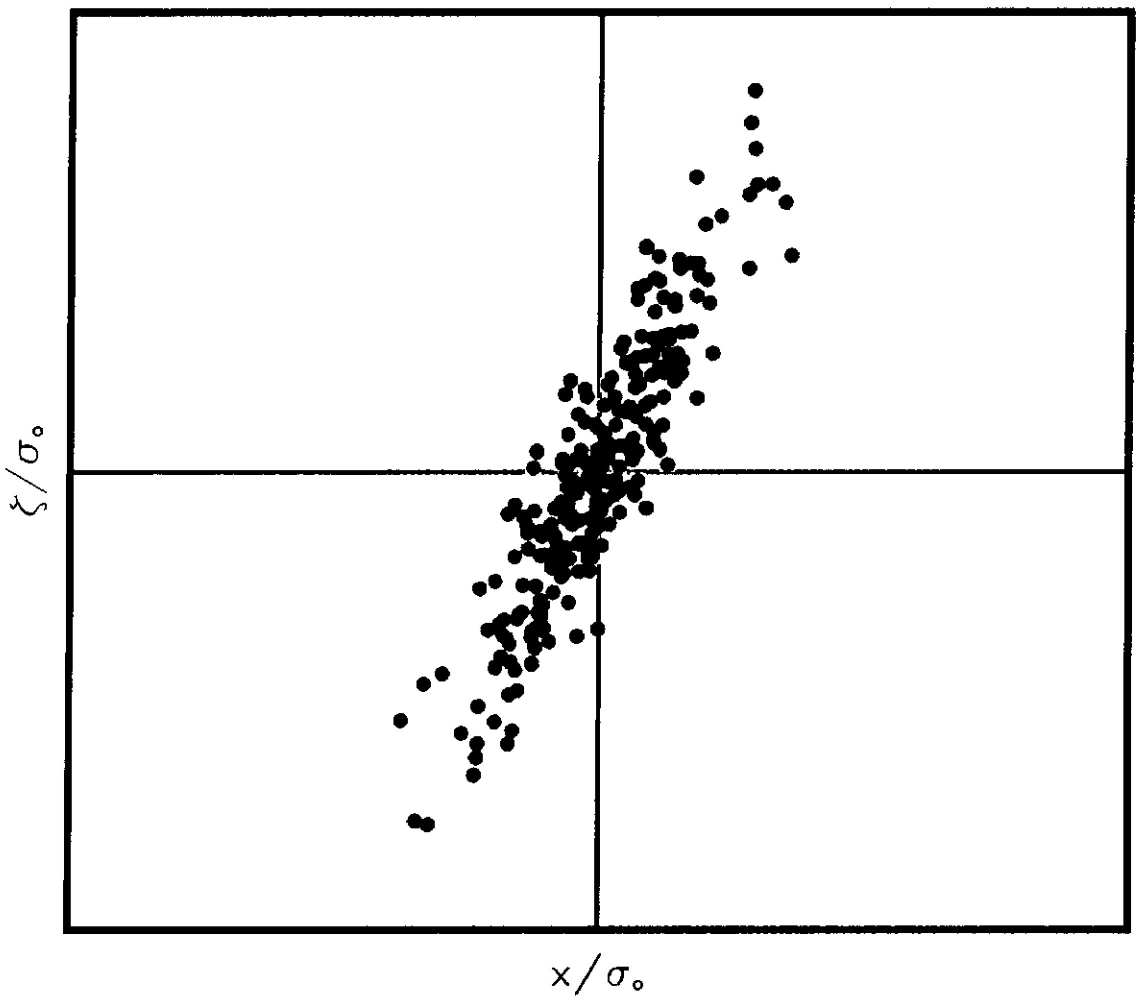
$$\beta_{eq} \equiv D + \sqrt{D^2 - 1}$$

$$\text{with}\quad D \equiv \frac{1}{2}\left[\beta_1\gamma_2 + \beta_2\gamma_1 - 2\alpha_1\alpha_2\right].$$

The incoming particle distribution resulting from a mismatch of the amplitude function and its slope will have the appearance depicted in Figure 6. Note that for the special case were $\alpha 1 = \alpha 2 = 0$, $\beta 1/\beta 2 \neq 1$, the above expressions yield $\theta = 0$, $D = [(\beta 1/\beta 2) + (\beta 2/\beta 1)]/2$, and $\beta_{eq} = \beta 1/\beta 2 = (\beta 2 + \Delta\beta)/\beta 2 = 1 + \Delta\beta/\beta$. That is, β_{eq} represents a pure amplitude function mismatch.

Figure 6

Example of Initial Particle Distribution
Resulting From Amplitude Function Mismatch



For the purpose of studying the time averaged resulting distribution due to an amplitude function mismatch, the initial distribution may be written as (with subscripts suppressed)

$$n_o(x,\varsigma)\ dx\ d\varsigma \ = \left[\frac{e^{-x^2/2\beta}}{\sqrt{2\pi\beta}}\right]\left[\frac{e^{-\beta\varsigma^2/2}}{\sqrt{2\pi/\beta}}\right] dx\ d\varsigma\ .$$

Here, β is used to represent β_{eq}. Upon transforming to polar coordinates, the integral for the resulting time average distribution due to an amplitude function mismatch becomes

$$n_\beta(x)\ dx \ = \frac{dx}{2\pi^2}\int_{|x|}^{\infty}\int_0^{2\pi}\frac{e^{-\frac{a^2}{2\beta}\left[\cos^2\theta\ +\ \beta^2\sin^2\theta\right]}}{\sqrt{1-(x/a)^2}}\ d\theta\ da$$

or,

$$n_\beta(x)dx = \frac{dx}{\pi}\int_{|x|}^{+\infty} e^{-a^2/2\beta}\ \frac{e^{-\left(\frac{a^2}{4}\ \frac{\beta^2-1}{\beta}\right)}\ I_o\left[\frac{a^2}{4}\ \frac{\beta^2-1}{\beta}\right]}{\sqrt{1-(x/a)^2}}\ da.$$

To obtain the dilution factor, $F_\beta = a_{95}^2/6$, a_{95} must satisfy

$$\int_0^{a_{95}} e^{-\frac{r^2}{2\beta}}\ e^{-\left(\frac{r^2}{4}\ \frac{\beta^2-1}{\beta}\right)}\ I_o\left[\frac{r^2}{4}\ \frac{\beta^2-1}{\beta}\right]\ r\ dr \ = \ 0.95\ .$$

Figures 7 and 8 show the dilution factor F_β and the resulting distribution n_β respectively. To more closely resemble the corresponding figures for position and dispersion mismatches, Figures 7 and 8 use the variable $\Delta\beta/\beta \equiv \beta_{eq} - 1$. As can be seen from the second figure, the amplitude function must be greatly mismatched to produce a significant increase in the variance of the distribution.

Figure 7

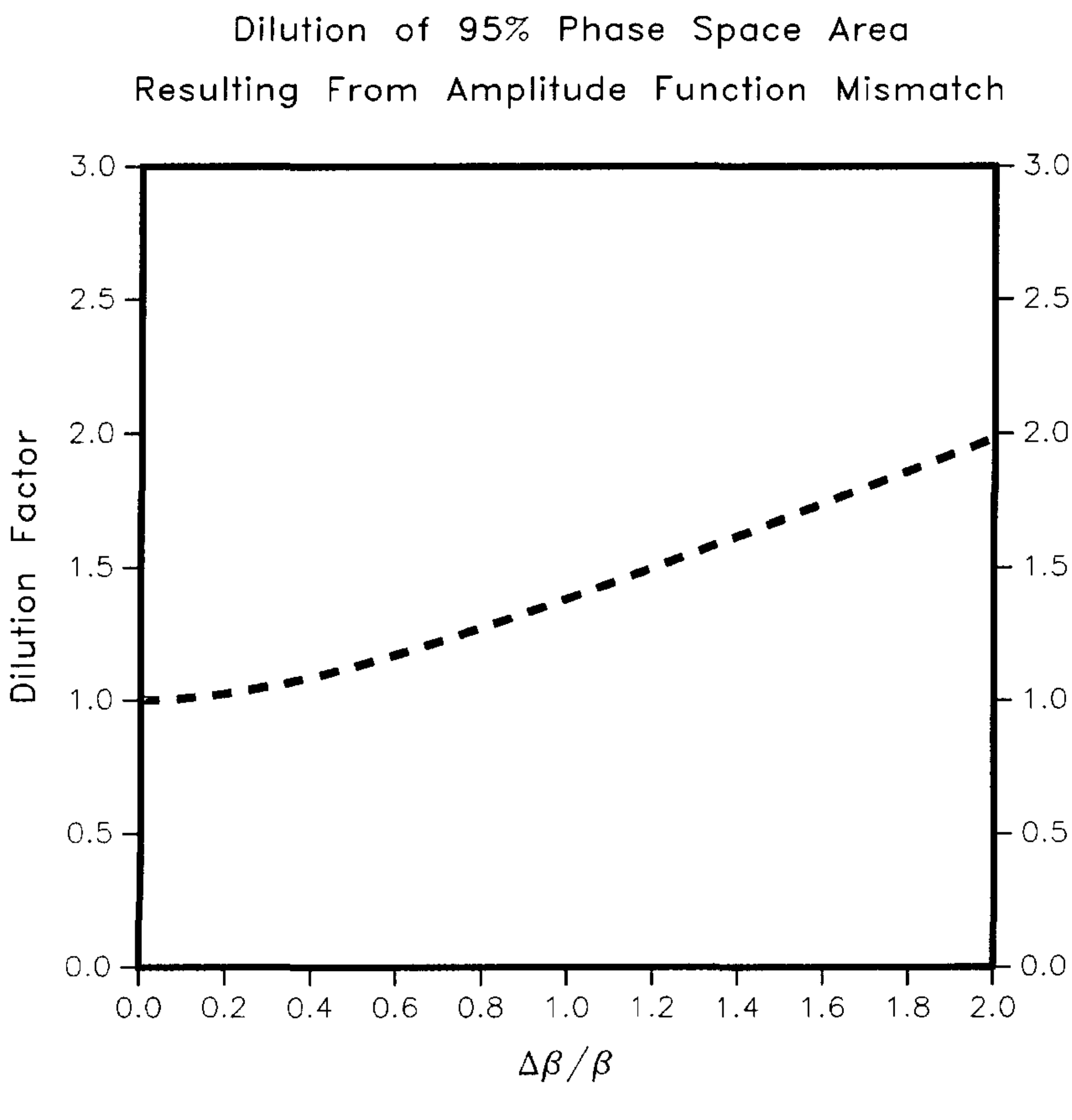
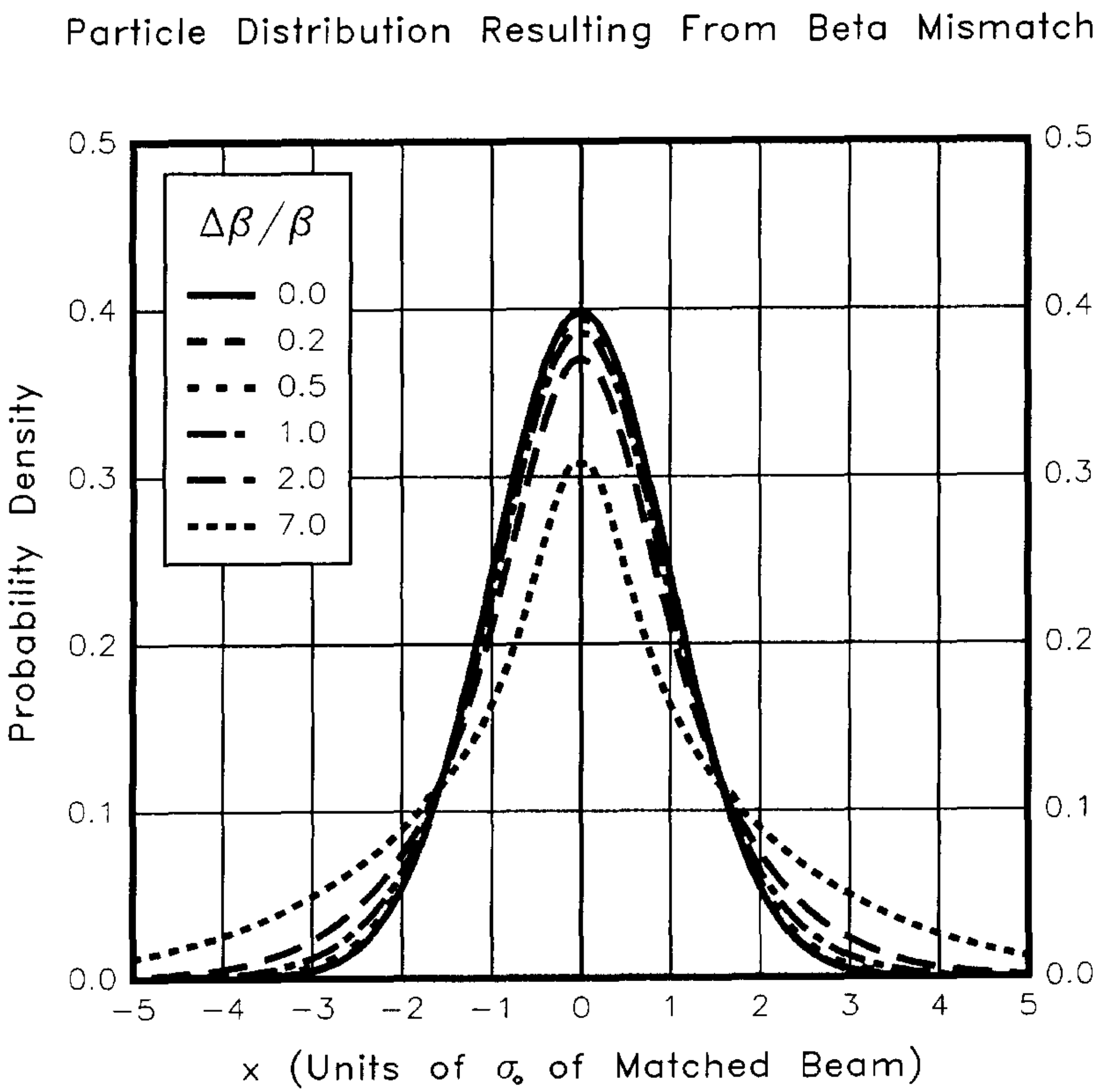

Dilution of 95% Phase Space Area
Resulting From Amplitude Function Mismatch
Dilution Factor
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6 1.8 2.0
Δβ/β

Figure 8


Particle Distribution Resulting From Beta Mismatch
$\Delta\beta/\beta$
0.0
0.2
0.5
1.0
2.0
7.0
Probability Density
0.0
0.1
0.2
0.3
0.4
0.5
−5
−4
−3
−2
−1
0
1
2
3
4
5
x (Units of σ_o of Matched Beam)

The significance of the quantity D may be seen by computing the determinant of the matrix ΔJ where J contains the Courant-Snyder parameters:

$$J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

Then,

$$\begin{aligned} \det(\Delta J) &= -(\alpha_2 - \alpha_1)^2 + (\gamma_2 - \gamma_1)(\beta_2 - \beta_1) \\ &= -[\alpha_2^2 - 2\alpha_1\alpha_2 + \alpha_1^2 - \gamma_2\beta_2 + \gamma_1\beta_2 + \gamma_2\beta_1 - \gamma_1\beta_1] \\ &= -[\ -1\ -2\alpha_1\alpha_2\ -1 + \gamma_1\beta_2 + \gamma_2\beta_1] \\ &= -2D + 2\ , \end{aligned}$$

or,

$$D = 1 - \det(\Delta J)/2\ .$$

The quantity $-\det(\Delta J)$ is positive definite and may be reduced to the form

$$-\det(\Delta J) = \frac{\left(\frac{\Delta\beta}{\beta}\right)^2 + \left(\Delta\alpha - \alpha\frac{\Delta\beta}{\beta}\right)^2}{1 + \frac{\Delta\beta}{\beta}} \geq 0\ .$$

This quantity is also an invariant. Suppose M is the 2 x 2 transport matrix from point s_a to point s_b. Then, $J_b = MJ_aM^{-1}$. If now J_a is altered by an amount ΔJ_a, then

$$J'_b = M(J_a + \Delta J_a)M^{-1} = MJ_aM^{-1} + M\Delta J_aM^{-1}$$

which implies

$$\Delta J_b = J'_b - J_b = M\Delta J_aM^{-1}$$

and so $\det(\Delta J_b) = \det(M)\ \det(\Delta J_a)\ \det(M^{-1}) = \det(\Delta J_a)$. Thus $\det(\Delta J)$ is invariant with s. This implies that the degree of mismatch due to amplitude functions may be measured at any point within the synchrotron, not just at the "injection point."

General Expressions for the Variances of Particle Distributions After Dilution

Using a simple statistical argument the resulting variance after dilution of an initially mismatched particle beam may be derived. The variances obtained with this method agree with the variances of the distributions arrived at by the straightforward method of the previous sections and allows one to write down simple formulas for the variances of the resulting time average distributions.

Given a particle with phase space amplitude a determined by the initial phase space coordinates x_o, ς_o, the subsequent motion in phase space is defined by the circle $x^2 + \varsigma^2 = a^2$. A distribution of particles of the form $g(x,\varsigma) = f(x)f(\varsigma)$ will have $\langle x^2\rangle = \langle\varsigma^2\rangle$, where $\langle\ \rangle$ denotes an average over time. Since $\langle x^2\rangle + \langle\varsigma^2\rangle = a^2$, then $\langle x^2\rangle = \langle\varsigma^2\rangle = a^2/2$. Therefore, a single particle with initial coordinates corresponding to amplitude a will provide a contribution of $a^2/2$ to the variance of the resulting time average distribution in x.

An initial distribution having a variance of $\sigma_o{}^2$ and which has rotational symmetry about the point $(\Delta x,0)$ is of the form

$$n_o(\rho,\phi) = f(\rho)/2\pi$$

where the distance from the point $(\Delta x,0)$ is denoted by ρ and the angle ϕ is the angle between the x axis and the line segment of length ρ, as shown in Figure 9. The distance a from the origin of a particle with coordinates ρ and ϕ is thus given by

$$a^2 = \rho^2 + \Delta x^2 - 2\rho\Delta x\cos\phi \ .$$

The final variance of the distribution in x averaged over a long period of time will then be

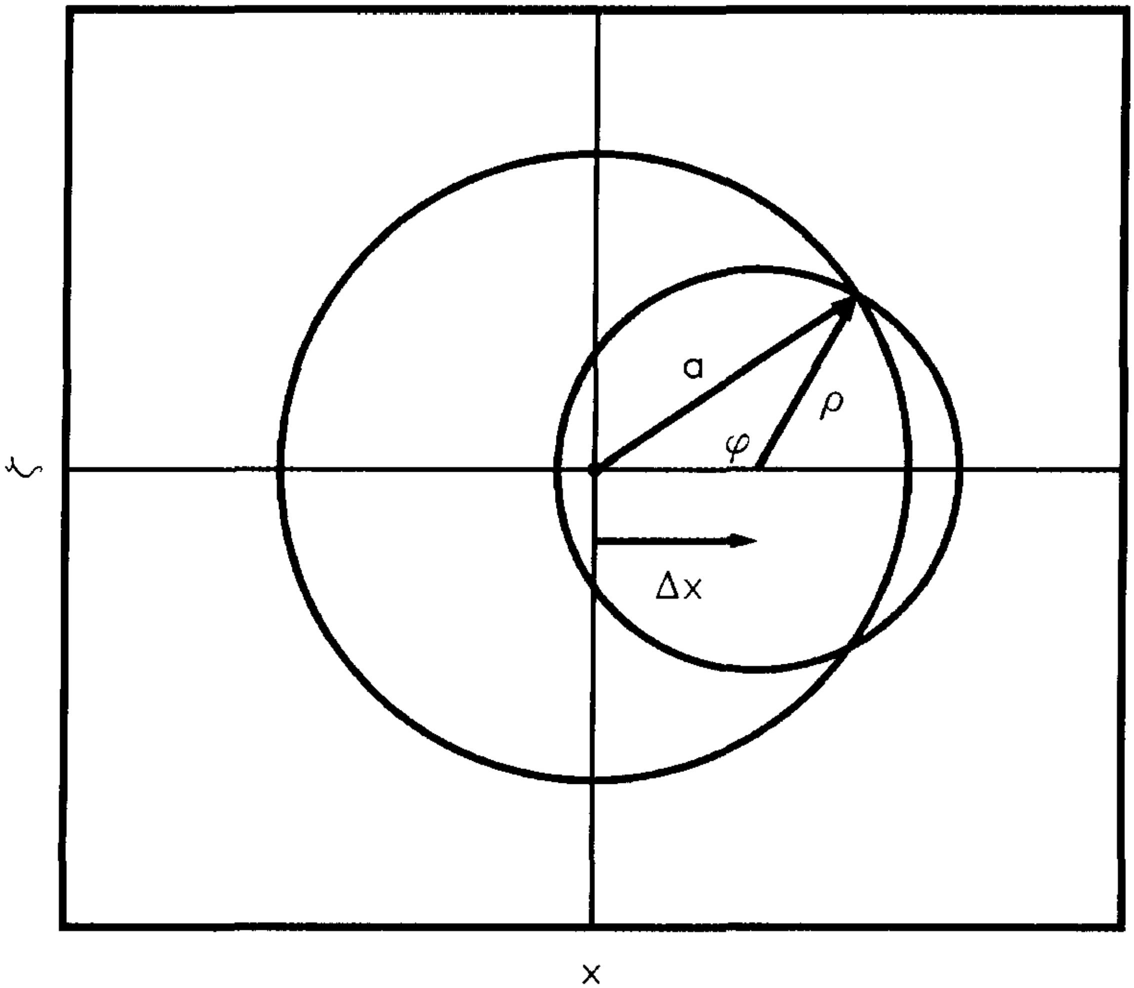
$$\sigma^2 \equiv \langle x^2\rangle = \int(a^2/2)n_o d\Sigma = \int(\rho^2/2)n_o d\Sigma + \int(\Delta x^2/2)n_o d\Sigma - \int\rho\Delta x\cos\phi d\Sigma$$

where $d\Sigma$ is the differential element of area in $x\cdot\varsigma$ phase space.

Figure 9

Definition of ρ, φ

For Position Mismatch



Therefore,

$$\sigma^2 = (2\sigma_o^2)/2 + \Delta x^2/2 - (\Delta x/2\pi)\int\rho^2 f(\rho)d\rho\int\cos\phi d\phi$$
$$= \sigma_o^2 + \Delta x^2/2$$

or,

$$\sigma^2 = 1 + \Delta x^2/2 \ ,$$

where σ and Δx are in units of σ_o. This is the expression for the variance of the distribution resulting from a position mismatch of the ideal trajectory and is in agreement with the distributions displayed in Figure 2.

In exact analogy the resulting variance due to a mismatch of the dispersion function may be obtained. Since ΔD corresponds to an initial position error of $\Delta D\sigma_p\delta$ for a particle of momentum $p+\sigma_p\delta$, $\langle\xi^2\rangle = 1 + (\Delta D\sigma_p\delta)^2/2$, using the expression from the preceding paragraph. The position relative to the origin for a particle of momentum $p+\sigma_p\delta$ is given by $x_\delta = \xi + D\sigma_p\delta$ and thus

$$x_\delta^2 = \xi^2 + 2D\sigma_p\delta\xi + D^2\sigma_p^2\delta^2,$$

which yields

$$\langle x_\delta^2\rangle = \langle\xi^2\rangle + D^2\sigma_p^2\delta^2 = 1 + (\Delta D\sigma_p)^2\delta^2/2 + D^2\sigma_p^2\delta^2$$

since $\langle\xi\rangle = 0$. Therefore, upon integrating over momentum,

$$\langle x^2\rangle \equiv \sigma^2 = 1 + (\Delta D\sigma_p)^2/2 + D^2\sigma_p^2 \ .$$

Again, σ and $D\sigma_p$ are in units of σ_o. This expression again agrees with the distributions obtained earlier and shown in Figure 4.

For an amplitude function mismatch the initial distribution was written as

$$n_o(x,\varsigma)\ dx\ d\varsigma = \left(\frac{e^{-x^2/2\beta}}{\sqrt{2\pi\beta}}\right)\left(\frac{e^{-\beta\varsigma^2/2}}{\sqrt{2\pi/\beta}}\right) dx\ d\varsigma \ .$$

from which the variances $\langle x^2\rangle = \beta$ and $\langle\varsigma^2\rangle = 1/\beta$ are immediately evident. Given this initial distribution, the variance of the resulting time average distribution in x will be

$$\begin{aligned}\sigma^2 &= \int (a^2/2) n_o d\Sigma = \int (x^2/2) n_o d\Sigma + \int (\varsigma^2/2) n_o d\Sigma \\ &= \langle x^2\rangle/2 + \langle\varsigma^2\rangle/2 = \beta/2 + 1/2\beta = (\beta^2+1)/2\beta \\ &= ([D+(D^2-1)^{1/2}]^2 + 1)/2[D+(D^2-1)^{1/2}] \\ &= D \\ &= 1 + |\det(\Delta J)|/2\end{aligned}$$

where here, $D = (\beta_2\gamma_1 + \beta_1\gamma_2 - 2\,\alpha_1\alpha_2)/2$. Written in terms of $\Delta\beta/\beta = \beta - 1$, the form of this final variance more closely resembles the form of the two variances found for position and dispersion mismatches, namely

$$\sigma^2 = 1 + \frac{1}{2}\left[\frac{\Delta\beta/\beta}{\sqrt{1 + \Delta\beta/\beta}}\right]^2 .$$

The variance σ^2 is in units of the variance of the matched beam, σ_o^2. Once again, this formula for the resulting variance agrees with the results of the earlier distribution calculations.

As an example, suppose a "thin lens" gradient error of strength $\Delta q = \Delta B'L/(B\rho)$ presents itself within a beamline leading into a synchrotron. ($\Delta B'$ = gradient error, L = length of the error, and $B\rho$ is the magnetic rigidity.) At the location of the error, the slope of the amplitude function will be altered by the amount $\Delta\alpha = -\Delta(\beta'/2) = \Delta q\beta$ while the value of the amplitude function β will remain unchanged. Hence,

$$-\det(\Delta J) = -(-\Delta\alpha^2 + \Delta\beta\Delta\gamma) = \Delta\alpha^2 = \beta^2\Delta q^2$$

and thus, the variance of the particle distribution resulting from this mismatch will be given by

$$\sigma^2/\sigma_o^2 = D = 1 - \det(\Delta J)/2 = 1 + (\Delta q\beta)^2/2.$$

If the gradient error is produced by a mistuned quadrupole in the beamline, i.e., $\Delta q = \Delta B'L/B\rho = (\Delta B'/B')(B'L/B\rho) = (\Delta I/I_o)(1/F)$, where F = focal length of the quadrupole and I_o is the nominal current for the device, then

$$\sigma^2/\sigma_o^2 = 1 + (\beta^2/2F^2)(\Delta I/I_o)^2.$$

As the original definition of the phase space emittance was $\epsilon = 6\pi\sigma^2/\beta$ for a Gaussian distribution, a dilution factor could naively be defined as

$$\epsilon/\epsilon_o = \sigma^2/\sigma_o^2$$

so long as the resulting distribution after dilution resembles a Gaussian. Therefore, the expressions for the variances presented above are compared with the functions F_x, F_D, and F_β derived earlier for small degrees of mismatch, as shown in Figure 10. As can be seen, σ^2/σ_o^2 agrees with F reasonably well for small values of Δx, ΔD, and $\Delta\beta/\beta$. For large values of Δx, the variance is larger than the dilution factor due to the double hump appearance which shows up in the distribution. For large values of $\Delta\beta$, the dilution factor is larger than the variance due to the increased number of particles which reside in the tails of the distribution.

Figure 10

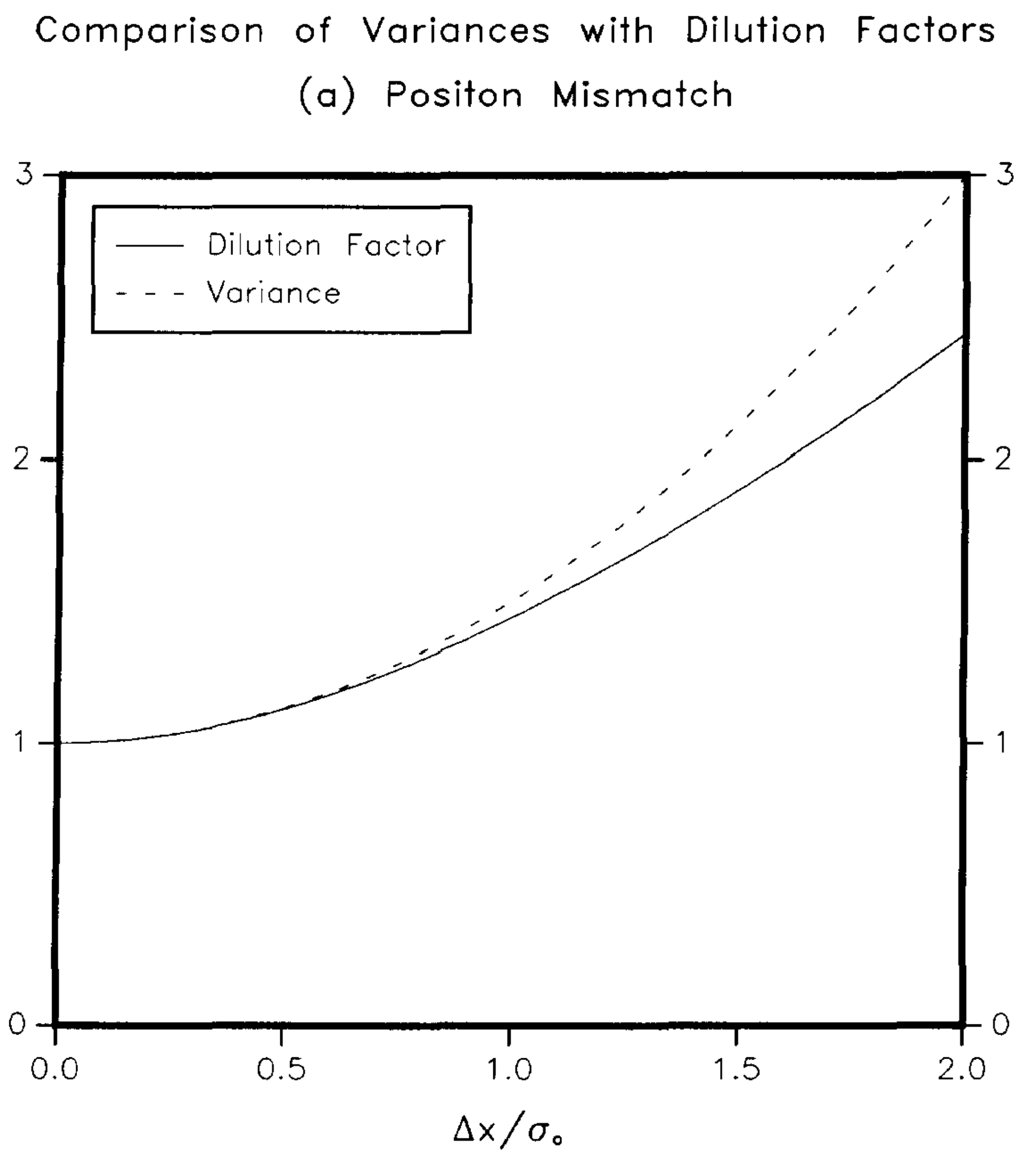

Comparison of Variances with Dilution Factors
(a) Positon Mismatch
Dilution Factor
Variance
0
1
2
3
0.0
0.5
1.0
1.5
2.0
Δx/σ₀

Figure 10 (Continued)

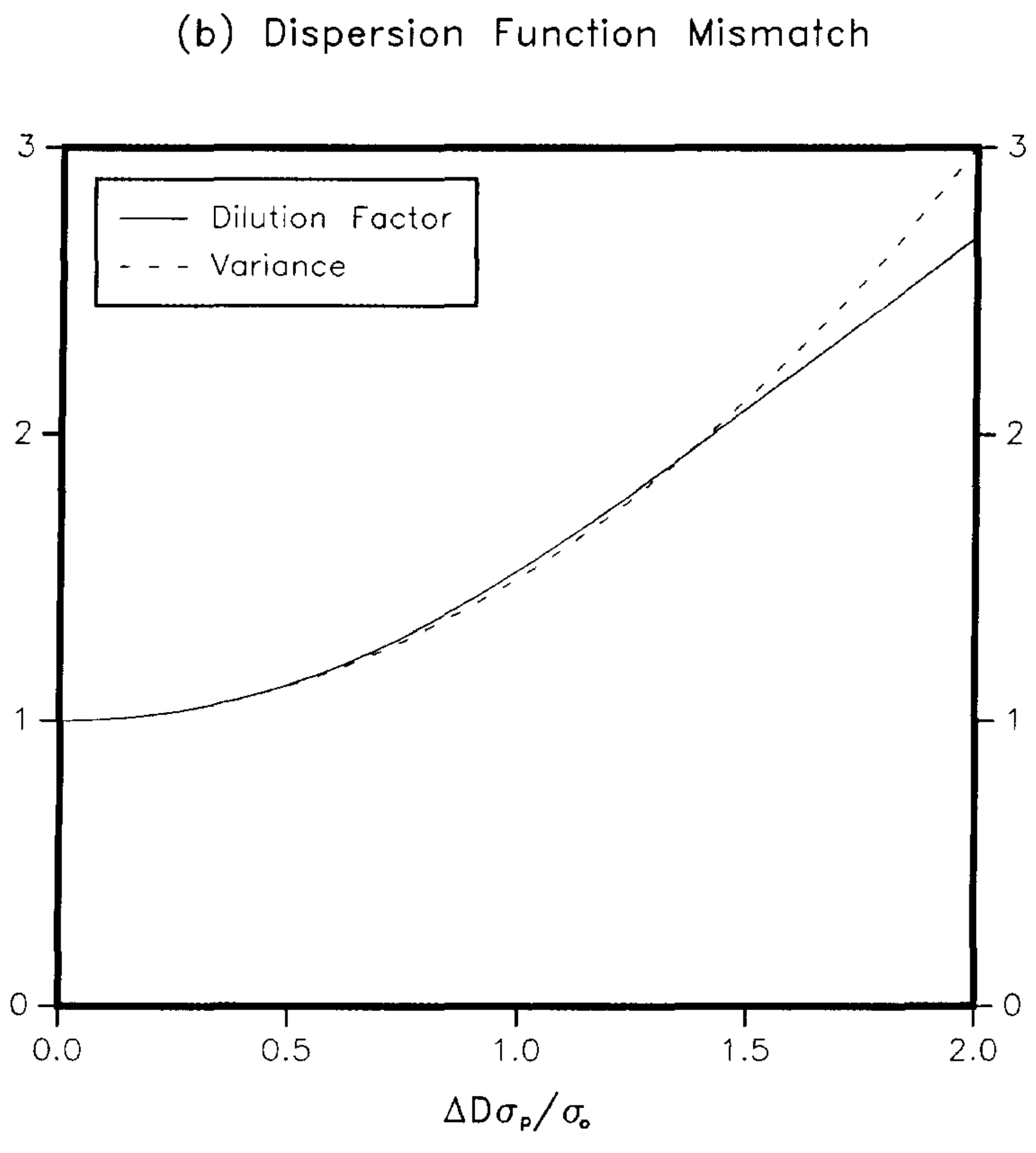
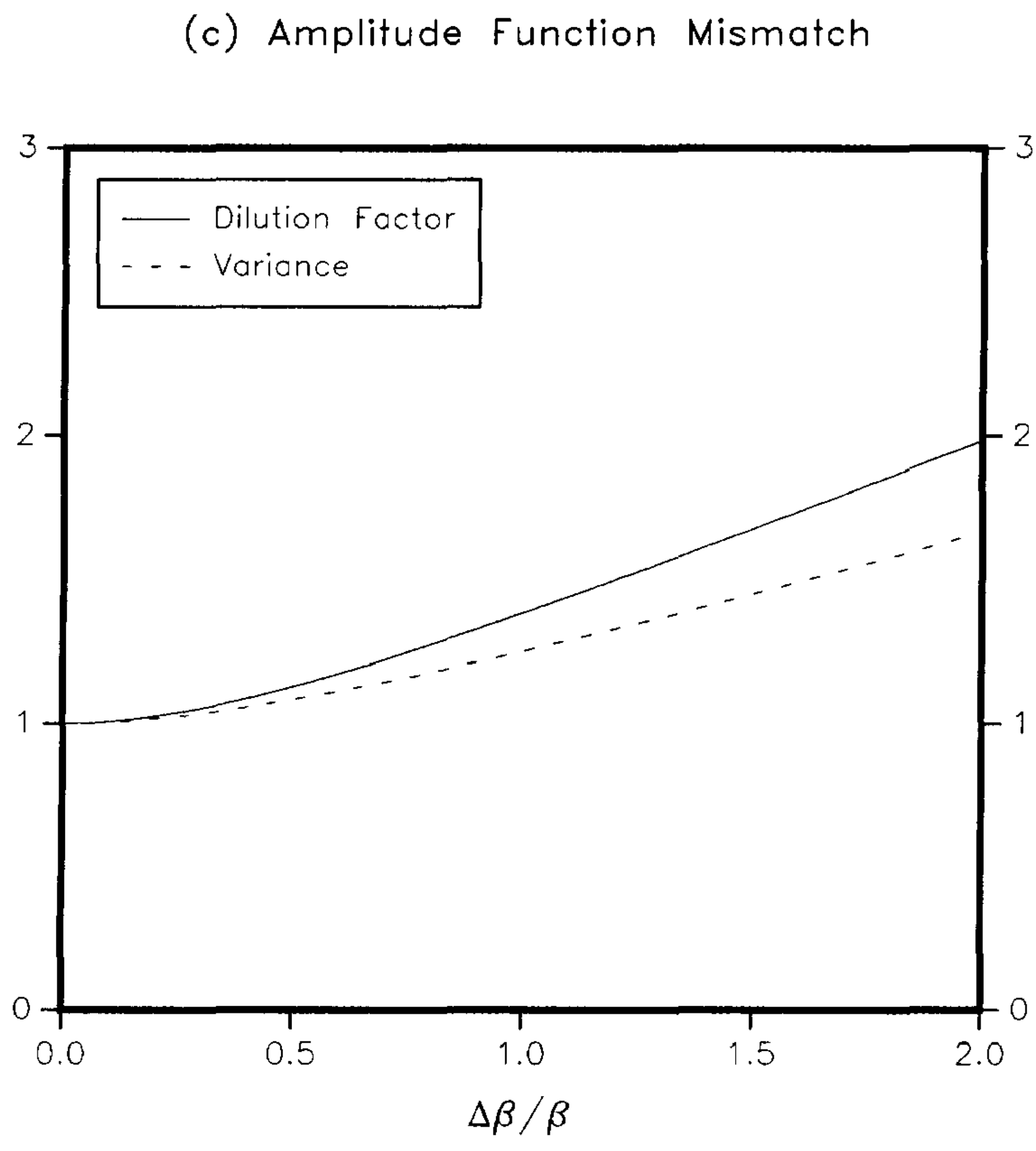


Figure 10 (Continued)



Summary

The transverse phase space dilution factors due to injection amplitude function, position, and dispersion function errors are given by

$$F_\beta \sim \frac{\sigma^2}{\sigma_o^{\,2}} = 1 + \frac{1}{2}\left[\frac{(\Delta\beta/\beta)_{eq}}{\sqrt{1 + (\Delta\beta/\beta)_{eq}}}\right]^2 = D$$

$$\text{where} \quad \left(\frac{\Delta\beta}{\beta}\right)_{eq} \equiv (D - 1) + \sqrt{D^2 - 1}$$

$$\text{and} \quad D = \frac{1}{2}\left[\beta_1\gamma_2 + \beta_2\gamma_1 - 2\alpha_1\alpha_2\right] \; ;$$

$$F_x \sim \frac{\sigma^2}{\sigma_o^{\,2}} = 1 + \frac{1}{2}\left[\frac{\Delta x_{eq}}{\sigma_o}\right]^2$$

$$\text{where} \quad \Delta x_{eq} \equiv \sqrt{\Delta x^2 + \left(\beta\Delta x' + \alpha\Delta x\right)^2} \; ;$$

$$F_D \sim \frac{\sigma^2}{\sigma_o^{\,2}} = 1 + \frac{1}{2}\left[\frac{\Delta D_{eq}\sigma_p}{\sigma_o}\right]^2$$

$$\text{where} \quad \Delta D_{eq} \equiv \sqrt{\Delta D^2 + \left(\beta\Delta D' + \alpha\Delta D\right)^2} \; .$$

Note: $\sigma_o \equiv \sqrt{\dfrac{\epsilon_n\beta_o}{6\pi(\gamma\beta)}}$

References

1. Proton accelerators are usually equipped with beam damping systems used to damp out betatron oscillations of the beam centroid caused by a position mismatch. These systems do not completely compensate the position mismatch, however, and dilution will still occur. These systems would have no effect on amplitude function or dispersion function mismatches.

2. Abramowitz, M., and I. A. Stegun, Handbook of Mathematical Functions, New York: Dover Publications, Inc., 1970, p. 374.

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