Fermi National Accelerator Laboratory

FN-455

Distortion Functions*

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August 1987

*Lecture given at KEK Oho '87 High Energy Accelerator Seminars, August 17-21, 1987, in Tsukuba, Japan.

Operated by Universities Research Association Inc. under contract with the United States Department of Energy

DISTORTION FUNCTIONS*

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I. INTRODUCTION

This lecture talks about the analytic computation of the distortions of beam shapes in the horizontal and vertical phase spaces due to the introduction of nonlinear multipoles.

When we say that the lattice of a storage ring is perfectly linear, we mean that there are only perfect dipoles and normal quadrupoles. The equations of motion in the horizontal and vertical planes are then linear and decoupled. The lattice is described completely by the beta-functions $\beta_x(s)$ and $\beta_y(s)$ which are periodic around the ring and s is measured along the ideal closed orbit. One can also introduce the other parameters $\alpha_x(s)$ and $\alpha_y(s)$ and the Floquet phases $\psi_x(s)$ and $\psi_y(s)$ which are related to the β's by

$$\alpha_u(s) = -\frac{1}{2}\beta_u'(s) \ , \tag{1.1}$$

and

$$\psi_u(s) = \int^s \frac{ds'}{\beta_u(s')} \ , \tag{1.2}$$

where u can be x or y. After one complete revolution around the ring, the change in ψ_u is $2\pi\nu_u$, where ν_x and ν_y are called the horizontal and vertical betatron tunes. In one of the transverse planes, the displacement U and the displacement angle U' are described by

$$U(s) = \mathcal{A}_u\left(\frac{\beta_u}{\beta_0}\right)^{1/2}\cos(\psi_u + \phi_0) \ ,$$

$$U'(s) = -\mathcal{A}_u(\beta_0\beta_u)^{-1/2}[\alpha_u\cos(\psi_u + \phi_0) + \sin(\psi_u + \phi_0)] \ , \tag{1.3}$$

where ϕ_0 is some initial phase. We have deliberately put in some reference β_0 so that $\mathcal{A}_u$ retains the dimension of some initial amplitude. If we record the positions of the particle at one particular point along the ring for many turns, we find that they lie on an ellipse in the transverse phase space as in Fig. 1. In fact, the equation of the ellipse can be obtained from Eq. (1.3) by eliminating ψ_u,

$$\frac{\mathcal{A}_u^2}{\beta_0} = \frac{U^2 + (\alpha_u U + \beta_u U')^2}{\beta_u} \ . \tag{1.4}$$

The area of the ellipse is

$$\epsilon_u = \frac{\pi\mathcal{A}_u^2}{\beta_0} \ , \tag{1.5}$$

which is called the emittance of the beam in the u-directional phase space and is obviously an invariant of motion. Thus, knowing $\beta_u(s)$ along the ring, one knows exactly the shapes of the beam ellipses in both transverse phase spaces anywhere along the ring and no tracking is required.

From Eq. (1.3), the β's have their physical meanings of oscillation amplitudes. For example, in the bunch-bunch collision region, we want small bunch sizes in both transverse directions and we think of a design with small β's at that location. In short, we understand the patterns of focussing which generate the β's. Therefore,

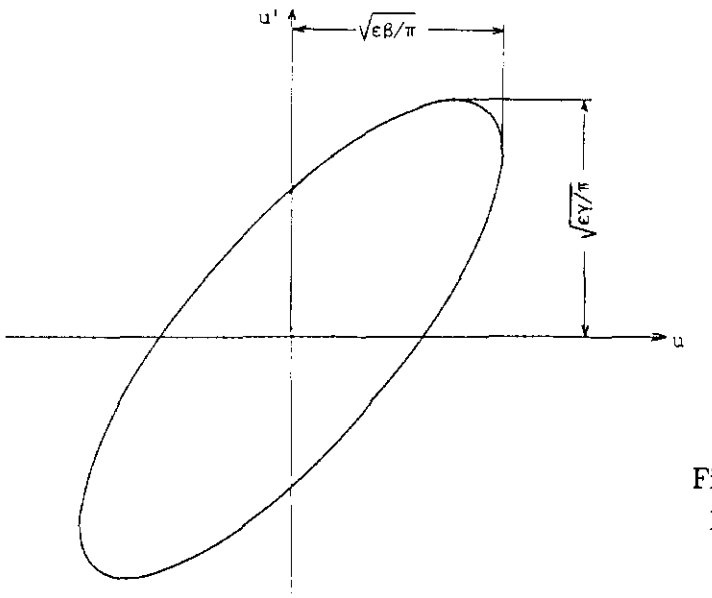

Figure 1: Beam shape in a transverse phase plane. The lattice is perfectly linear, the emittance is ϵ, and $\gamma = (1+\alpha^2)/\beta$.

the β's are in fact qualified as a design tool rather than a computational device for avoiding tracking.

Unfortunately, no machine is perfectly linear. There are systematic sextupole components in dipole fields from steel saturation, remanant fields, persistent currents, eddy currents, and random sextupole components due to field errors. Of course, there are also sextupoles placed around the ring on purpose to counteract the above and to modify chromaticity. Higher multipoles are also possible; for example, the octupole components from beam-beam collision. The theory therefore becomes nonlinear. Does this mean that we shall lose all our prediction of the beam shape by the beta-functions? The answer is no. For a large-size storage ring, the need for sophisticated diagnosis of minor faults demands a rational beam behavior. Such rational behavior is also required for a beam pipe of small bore so that the magnet size and consequently the cost can be reduced. All these imply a machine that is as linear as possible. As a result, perturbation theory can be used away from resonances. Collins[1] has proposed a set of distortion functions for each order of the perturbation. These distortion functions are closed, *i.e.*, periodic. They are independent of the beam amplitude and are very similar to the beta-functions and alpha-functions of the linear theory. Of course, the beam profile is not so simple now, because horizontal and vertical motions are coupled together. So it no longer manifests itself as an ellipse in each transverse phase space. Instead, it becomes a four dimensional hyper-egg and we can only talk about its projections onto the transverse phase planes. However, these distortion functions can give us the exact projections. They can also give us two important numbers: the transverse betatron tuneshifts $\Delta\nu_x$ and $\Delta\nu_y$.

In the Section II, we shall first preview these distortion functions, the formulae for their computation, and how they can be used in computing the phase-space projections of the beam and the tuneshifts. In Section III, we shall derive all the formulae in Section II from a Hamiltonian theory. Lastly, in Section IV, some applications are discussed.

II. DISTORTION FUNCTIONS

If we use the Floquet phase ψ_u as the independent variable instead of s, the motion of a particle in the transverse phase space, Eq. (1.3), becomes

$$u(\psi_u) = \mathcal{A}_u \cos(\phi_u) \ ,$$

$$u'(\psi_u) = -\mathcal{A}_u \sin(\phi_u) \ , \tag{2.1}$$

where the prime is now derivative with respect to ψ_u, which is contained in the instantaneous betatron phase $\phi_u = \psi_u + \phi_0$, and

$$u = \left(\frac{\beta_0}{\beta_u}\right)^{1/2} U \ , \qquad u' - \alpha_u u = (\beta_0 \beta_u)^{1/2} U' \ . \tag{2.2}$$

In this way, the ellipses become circles.

With the introduction of some weak sextupoles, the circles are distorted into

$$u = \delta u + (\mathcal{A}_u + \delta\mathcal{A}_u)\cos(\phi_u + \delta\phi_u) \ ,$$

$$u' = \delta u' - (\mathcal{A}_u + \delta\mathcal{A}_u)\sin(\phi_u + \delta\phi_u) \ . \tag{2.3}$$

The change in closed orbit is

$$\delta x = 2(\mathcal{A}_y^2 \bar{B} - \mathcal{A}_x^2 B_1) \ , \qquad \delta x' = 2(\mathcal{A}_y^2 \bar{A} - \mathcal{A}_x^2 A_1) \ , \tag{2.4}$$

$$\delta y = 0 \ , \qquad \delta y' = 0 \ . \tag{2.5}$$

The changes in phase-space circles are

$$\delta\mathcal{A}_x = \mathcal{A}_x^2(G_3 - G_1) - \mathcal{A}_y^2(G_+ - G_-) \ ,$$

$$\delta\phi_x = \mathcal{A}_x(F_3 + F_1) - (\mathcal{A}_y^2/\mathcal{A}_x)(F_+ + F_-) \ ,$$

$$\delta\mathcal{A}_y = -2\mathcal{A}_x\mathcal{A}_y(G_+ + G_-) \ ,$$

$$\delta\phi_y = -2\mathcal{A}_x(2\bar{F} + F_+ + F_-) \ . \tag{2.6}$$

In above, the functions F_1, G_1, F_3, G_3, $\bar{F}$, F_+, G_+, F_-, and G_- are given in terms of the distortion functions B_1, A_1, B_3, A_3, $\bar{B}$, $\bar{A}$, B_+, A_+, B_-, and A_- in Table 1. We note that the F and G are just like a rotation of the vector (B_α, A_α) by a generic angle α. By generic, we mean, for example, the substitution of $\alpha = 2\phi_y + \phi_x$ in F_+ and G_+. The strengths of the sextupoles m_α are defined as

$$s = \left(\frac{\beta_x^3}{\beta_0}\right)^{1/2} S \qquad \bar{s} = \left(\frac{\beta_x \beta_y^2}{\beta_0}\right)^{1/2} S \ , \tag{2.7}$$

where

$$S = \lim_{\ell \to 0} \left[\frac{B_y'' \ell}{2(B\rho)}\right] \ . \tag{2.8}$$

name B_α A_α	angle α	strength m_α	tune ν_α	$F(\alpha)$ $G(\alpha)$
B_1 A_1	ϕ_x	$s/4$	ν_x	$F_1 = A_1 \cos\alpha + B_1 \sin\alpha$ $G_1 = A_1 \sin\alpha - B_1 \cos\alpha$
B_3 A_3	$3\phi_x$	$s/4$	$3\nu_x$	$F_3 = A_3 \cos\alpha + B_3 \sin\alpha$ $G_3 = A_3 \sin\alpha - B_3 \cos\alpha$
$\bar{B}$ $\bar{A}$	ϕ_x	$\bar{s}/4$	ν_x	$\bar{F} = \bar{A} \cos\alpha + \bar{B} \sin\alpha$
B_+ A_+	$2\phi_y + \phi_x$	$\bar{s}/4$	$2\nu_y + \nu_x$	$F_+ = A_+ \cos\alpha + B_+ \sin\alpha$ $G_+ = A_+ \sin\alpha - B_+ \cos\alpha$
B_- A_-	$2\phi_y - \phi_x$	$\bar{s}/4$	$2\nu_y - \nu_x$	$F_- = A_- \cos\alpha + B_- \sin\alpha$ $G_- = A_- \sin\alpha - B_- \cos\alpha$

Table I: Distortion functions for first order sextupoles

Here, $B_y'' x$ is the local gradient of sextupole field, ℓ its length, and $(B\rho)$ the magnetic rigidity of the particle.

Each pair of distortion functions can be computed using the following criteria. In the region between two sextupoles, (B_α, A_α) rotates like a vector by the angle α. On passing a sextupole, B is continuous while A jumps by an amount m_α which may be $s/4$ or $\bar{s}/4$. Finally, (B_α, A_α) have to close after one revolution of the ring. The explicit formulae for the set (B_α, A_α) at location ψ_α are

$$B_\alpha(\psi_\alpha) = \frac{1}{2\sin\pi\nu_\alpha} \sum_k m_\alpha \cos(|\psi_{\alpha k} - \psi_\alpha| - \pi\nu_\alpha) \qquad 0 \le |\psi_{\alpha k} - \psi_\alpha| \le 2\pi\nu_\alpha \ ,$$

$$A_\alpha(\psi_\alpha) = B_1'(\psi_\alpha) \qquad\qquad 0 < |\psi_{\alpha k} - \psi_\alpha| < 2\pi\nu_\alpha \ , \tag{2.9}$$

where the summation is over the location of each sextupole k.

The first-order perturbation produces no tuneshifts. The lowest contribution comes from the second order:

$$2\pi\Delta\nu_x = -\frac{\mathcal{A}_x^2}{2} \sum_k (B_3 s + 3B_1 s)_k - \mathcal{A}_y^2 \sum_k (B_+\bar{s} + B_-\bar{s} - 2B_1\bar{s})_k \ ,$$

$$2\pi\Delta\nu_y = -\mathcal{A}_x^2 \sum_k (B_+\bar{s} + B_-\bar{s} - 2B_1\bar{s})_k - \frac{\mathcal{A}_y^2}{2} \sum_k (B_+\bar{s} - B_-\bar{s} + 4\bar{B}\bar{s})_k \ . \tag{2.10}$$

Similar expressions can be written for skew quadrupoles, skew sextupoles, normal octupoles, skew octupoles, etc.

III. HAMILTONIAN DERIVATION

Collins[1] has given a derivation of all the formulae in Section II. However, his derivation includes an *a priori* assumption of the closure (periodicity) for the distortion functions. An alternative derivation is the Hamiltonian method.[2,3]

III.1 Floquet Transformation and Action-angle variables

We start from the Hamiltonian describing the motion of a single beam particle,

$$H_1 = \frac{1}{2}[P_x^2 + K_x(s)X^2] + \frac{1}{2}[P_y^2 + K_y(s)Y^2] + \frac{B_y''}{6(B\rho)}(X^3 - 3XY^2) \ , \tag{3.1}$$

where P_x and P_y are the canonical momenta conjugate to the horizontal and vertical displacements X and Y, $K_x(s)$ and $K_y(s)$ are proportional to the restoring forces due to the ring's curvature and the field gradients of the quadrupoles. The last term gives only the normal-sextupole potential. Other terms, such as skew quadrupole, skew sextupole, and higher multipoles have also been considered[4] but will not be included here.

Without the sextupole term, the Hamiltonian (3.1) describes two ellipses in the transverse phase spaces as given by Eq. (1.3). We next perform a canonical transformation into the Floquet space so that the ellipses become circles as given by Eq. (2.1). The generating function is

$$G_1(x, P_x, y, P_y; s) = \sum_{u=x,y} \left[-\left(\frac{\beta_u}{\beta_0}\right)^{1/2} P_u u + \frac{\beta_u'}{4\beta_0} u^2 \right] \ . \tag{3.2}$$

The new Hamiltonian becomes

$$H_2 = \frac{R}{2\beta_x}\left(\beta_0 p_x^2 + \frac{x^2}{\beta_0}\right) + \frac{R}{2\beta_y}\left(\beta_0 p_y^2 + \frac{y^2}{\beta_0}\right) + \frac{RB_y''}{6(B\rho)}\left[\left(\frac{\beta_x}{\beta_0}\right)^{3/2} x^3 - 3\left(\frac{\beta_x\beta_y^2}{\beta_0^3}\right)^{1/2} xy^2\right] \ . \tag{3.3}$$

In above, the independent variable s has been changed to the more convenient $\theta = s/R$, where R is the average radius of the storage ring. Use has also been made of the relation

$$\frac{1}{2}\beta_u\beta_u'' - \frac{1}{4}\beta_u'^2 + K_u\beta_u^2 = 1 \ , \tag{3.4}$$

which defines the beta-functions in terms of the restoring forces K_x and K_y. This Hamiltonian is now solved exactly to zero order in sextupole strength by canonical transformation to the action-angle variables I_x, a_x and I_y, a_y. The generating function

$$G_2(a_x, p_x, a_y, p_y; \theta) = \sum_{u=x,y} \frac{1}{2}\beta_0 p_u^2 \cot[Q_u(\theta) + a_u] \tag{3.5}$$

is used to obtain the transformation

$$u = (2I_u\beta_0)^{1/2}\cos[Q_u(\theta) + a_u] \ ,$$

$$\beta_0 p_u = -(2I_u\beta_0)^{1/2}\sin[Q_u(\theta) + a_u] \ , \tag{3.6}$$

where $Q_u(\theta) = \psi_u(\theta) - \nu_u\theta$, $\beta_0 p_u = du/d\psi_u$ and is denoted by u' in below. The definition of the Floquet phase ψ_u is given by Eq. (1.2). After the transformation, the new Hamiltonian becomes

$$H_3 = \nu_x I_x + \nu_y I_y + \text{ sextupole terms} \ . \tag{3.7}$$

III.2 Expansion into Harmonics

We note that $Q_u(\theta)$ is periodic. Thus, treating I_u and a_u as θ-independent, the sextupole terms

$$\begin{aligned}\Delta H_3|_{\text{sex}} = (2I_x\beta_x)^{3/2}\frac{RB_y''}{24(B\rho)}[\cos 3(Q_x + a_x) + 3\cos(Q_x + a_x)] \\ -(2I_x\beta_x)^{1/2}(2I_y\beta_y)\frac{RB_y''}{8(B\rho)}[2\cos(Q_x + a_x) \\ + \cos(2Q_y + Q_x + 2a_y + a_x) + \cos(2Q_y - Q_x + 2a_y - a_x)]\end{aligned} \tag{3.8}$$

can be expanded into harmonics. Take for example the $\cos(Q_x + a_x)$ term, which can be written as

$$\begin{aligned}(2I_x\beta_x)^{3/2}\frac{RB_y''}{48(B\rho)}\left[e^{i(Q_x + a_x)} + e^{-i(Q_x + a_x)}\right] = \\ (2I_x)^{3/2}\beta_0^{1/2}\sum_{m=-\infty}^{\infty}\left(\xi_m e^{-im\theta} + \xi_m^* e^{-im\theta}\right) \ ,\end{aligned} \tag{3.9}$$

where the harmonic amplitude is

$$\xi_m = \frac{1}{48\pi}\int_0^{2\pi} d\theta \frac{RB_y''}{2(B\rho)}\left(\frac{\beta_x^3}{\beta_0}\right)^{1/2} e^{i(Q_x + a_x) + im\theta} \ . \tag{3.10}$$

For a thin sextupole of length ℓ at location k, the strength is defined as

$$s_k = \lim_{\ell\to 0}\left[\frac{B_y''\ell}{2(B\rho)}\left(\frac{\beta_x^3}{\beta_0}\right)^{1/2}\right]_k \quad \text{or} \quad \bar{s}_k = \lim_{\ell\to 0}\left[\frac{B_y''\ell}{2(B\rho)}\left(\frac{\beta_x\beta_y^2}{\beta_0}\right)^{1/2}\right]_k \ . \tag{3.11}$$

Then we get

$$\xi_m = \frac{1}{48\pi}\sum_k s_k e^{i(Q_x + m\theta)_k} \ . \tag{3.12}$$

Doing this for every term, we get

$$\begin{aligned}\Delta H_3|_{\text{sex}} = (2I_x)^{3/2}\beta_0^{1/2}\sum_m(A_{3m}\sin q_{3m} + 3A_{1m}\sin q_{1m}) \\ -(2I_x)^{1/2}(2I_y)\beta_0^{1/2}\sum_m(2B_{1m}\sin p_{1m} + B_{+m}\sin p_{+m} + B_{-m}\sin p_{-m}) \ ,\end{aligned} \tag{3.13}$$

where

$$
\begin{aligned}
q_{1m} &= a_x + \alpha_{1m} - m\theta \ , \\
q_{3m} &= 3a_x + \alpha_{3m} - m\theta \ , \\
p_{1m} &= a_x + \beta_{1m} - m\theta \ , \\
p_{\pm m} &= (2a_y \pm a_x) + \beta_{\pm m} - m\theta \ ,
\end{aligned}
\tag{3.14}
$$

and

$$
\begin{aligned}
A_{1m} e^{i\alpha_{1m}} &= \frac{i}{24\pi} \sum_k s_k e^{i(Q_x + m\theta)_k} \ , \\
A_{3m} e^{i\alpha_{3m}} &= \frac{i}{24\pi} \sum_k s_k e^{i(3Q_x + m\theta)_k} \ , \\
B_{1m} e^{i\beta_{1m}} &= \frac{i}{8\pi} \sum_k \bar{s}_k e^{i(Q_x + m\theta)_k} \ , \\
B_{\pm m} e^{i\beta_{\pm m}} &= \frac{i}{8\pi} \sum_k \bar{s}_k e^{i(2Q_y \pm Q_x + m\theta)_k} \ .
\end{aligned}
\tag{3.15}
$$

In above, the harmonic amplitudes A_{1m}, A_{3m}, B_{1m}, $B_{\pm m}$, and the phases α_{1m}, α_{3m}, β_{1m}, $\beta_{\pm m}$ are real numbers.

III.3 Moser Transformation

For the first-order beam shape, we can solve the equations of motion obtained from the Hamiltonian H_3 to the first order. However, because we are interested in the second-order tuneshifts also, it will be advantageous for us to make another canonical transformation from (a_u, I_u) to (b_u, J_u) so that the J_u's become constants of motion up to first order in s_k or $\bar{s}_k$. This is called a Moser transformation with generating function (derived in the Appendix)

$$
\begin{aligned}
G_3(a_x, J_x, a_y, J_y; \theta) &= a_x J_x + a_y J_y \\
&\quad - (2J_x)^{3/2} \beta_0^{1/2} \sum_m \left(\frac{A_{3m}}{m - 3\nu_x} \cos q_{3m} + \frac{3A_{1m}}{m - \nu_x} \cos q_{1m} \right) \\
&\quad + (2J_x)^{1/2} (2J_y) \beta_0^{1/2} \sum_m \left(\frac{2B_{1m}}{m - \nu_x} \cos p_{1m} + \frac{B_{+m}}{m - \nu_+} \cos p_{+m} + \frac{B_{-m}}{m - \nu_-} \cos p_{-m} \right) \ ,
\end{aligned}
\tag{3.16}
$$

where $\nu_\pm = 2\nu_y \pm \nu_x$. By definition, the new Hamiltonian is

$$
H_4 = \nu_x J_x + \nu_y J_y + \Delta H_4|_{\text{sex}} \ , \tag{3.17}
$$

where $\Delta H_4|_{\text{sex}}$ does not contain any zero-order or first-order terms s_k or $\bar{s}_k$. The first-order changes in I_u and a_u are therefore given by

$$
\delta I_u = I_u - J_u = \frac{\partial G_3}{\partial a_u} - J_u \ ,
$$

$$\delta a_u = a_u - b_u = a_u - \frac{\partial G_3}{\partial J_u} \ . \tag{3.18}$$

Explicitly, they are

$$\begin{aligned}
\delta I_x &= (2I_x)^{3/2}\beta_0^{1/2}\sum_m \left(\frac{3A_{3m}}{m-3\nu_x}\sin q_{3m} + \frac{3A_{1m}}{m-\nu_x}\sin q_{1m}\right) \\
&-(2I_x)^{1/2}(2I_y)\beta_0^{1/2}\sum_m \left(\frac{2B_{1m}}{m-\nu_x}\sin p_{1m} + \frac{B_{+m}}{m-\nu_+}\sin p_{+m} - \frac{B_{-m}}{m-\nu_-}\sin p_{-m}\right) \ , \\
\delta I_y &= -2(2I_x)^{1/2}(2I_y)\beta_0^{1/2}\sum_m \left(\frac{B_{+m}}{m-\nu_+}\sin p_{+m} + \frac{B_{-m}}{m-\nu_-}\sin p_{-m}\right) \ , \\
\delta a_x &= 3(2I_x\beta_0)^{1/2}\sum_m \left(\frac{A_{3m}}{m-3\nu_x}\cos q_{3m} + \frac{A_{1m}}{m-\nu_x}\cos q_{1m}\right) \\
&-(2I_x)^{-1/2}(2I_y)\beta_0^{1/2}\sum_m \left(\frac{2B_{1m}}{m-\nu_x}\cos p_{1m} + \frac{B_{+m}}{m-\nu_+}\cos p_{+m} + \frac{B_{-m}}{m-\nu_-}\cos p_{-m}\right) \ , \\
\delta a_y &= -2(2I_x\beta_0)^{1/2}\sum_m \left(\frac{2B_{1m}}{m-\nu_x}\cos p_{1m} + \frac{B_{+m}}{m-\nu_+}\cos p_{+m} + \frac{B_{-m}}{m-\nu_-}\cos p_{-m}\right) \ .
\end{aligned} \tag{3.19}$$

These are related to the changes in amplitudes and phases. Recalling from Eq. (3.6) that

$$\begin{aligned}
u &= \mathcal{A}_u \cos[Q_u(\theta) + a_u] \ , \\
u' &= -\mathcal{A}_u \sin[Q_u(\theta) + a_u] \ ,
\end{aligned} \tag{3.20}$$

where

$$\mathcal{A}_u = (2I_u\beta_0)^{1/2} \ , \tag{3.21}$$

we have changes in amplitudes

$$\delta\bar{\mathcal{A}}_u = \left(\frac{\beta_0}{2I_u}\right)^{1/2}\delta I_u \ . \tag{3.22}$$

A bar has been put on top of $\delta\bar{\mathcal{A}}_u$ because it is defined by

$$\begin{aligned}
u &= (\mathcal{A}_u + \delta\bar{\mathcal{A}}_u)\cos(\phi_u + \delta\bar{\phi}_u) \ , \\
u' &= -(\mathcal{A}_u + \delta\bar{\mathcal{A}}_u)\sin(\phi_u + \delta\bar{\phi}_u) \ ,
\end{aligned} \tag{3.23}$$

which is different from the $\delta\mathcal{A}_u$ defined in Eq. (2.3) where the closed-orbit distortion terms have been included. As for the angle variable a_u, if we solve the Hamiltonian H_3, we get

$$\frac{da_u}{d\theta} = \frac{\partial H_3}{\partial I_u} = \nu_u + \text{ sextupole terms} \ . \tag{3.24}$$

Thus, for the unperturbed part,

$$a_u(\theta) = \nu_u\theta + \text{ constant} \ . \tag{3.25}$$

Here, the constant should be chosen as $\phi_u - \psi_u$, where $\phi_u(\theta)$ is the instantaneous betatron phase and $\psi_u(\theta)$ is the Floquet phase designating the location at the point θ. Although both of them depend on θ, their difference is θ-independent. Such a choice of the constant is necessary, because substitution of

$$a_u = \nu_u\theta - \psi_u + \phi_u = \phi_u - Q_u \tag{3.26}$$

into Eq. (3.20) gives

$$\begin{aligned} u &= \mathcal{A}_u \cos\phi_u \ , \\ u' &= -\mathcal{A}_u \sin\phi_u \ , \end{aligned} \tag{3.27}$$

exactly the same as Eq. (2.1), where ϕ_u really denotes the instantaneous betatron phase. Therefore, the change in the angle variable a_u is just the change in the instantaneous phase aside from closed-orbit distortion; or

$$\delta\bar{\phi}_u = \delta a_u \ . \tag{3.28}$$

III.4 Summation of Harmonics and Distortion Functions

The solution (3.19) involves summations over the harmonics m. It is obvious that when ν_x, $3\nu_x$, or $\nu_\pm$ is equal to an integer, the solution blows up. In fact, these are the first-order resonances of the sextupoles. When we are close to a particular resonance, we just take the term with an m that is closest to that ν_α and forget the rest. This is the way to look at the situation near a resonance. In the actual operation of the storage ring, however, we are always far away from all resonances except possibly during extraction. Then, all the harmonics are necessary. It is nice that, except right at a resonance, the summations over m can actually be performed using the formula

$$\sum_{m=-\infty}^{\infty} \frac{e^{i(m\theta + b)}}{m-\nu} = \begin{cases} -\pi \csc \pi\nu \, e^{i[b + \nu(\theta - \pi)]} & 0 < \theta < 2\pi \ , \\ -\pi \cot \pi\nu \, e^{ib} & \theta = 0 \ . \end{cases} \tag{3.29}$$

Let us try one term in Eq. (3.19):

$$\sum_m \frac{A_{1m} e^{iq_{1m}}}{m - \nu_x} = \frac{i}{24\pi} \sum_k s_k \sum_m \frac{e^{i(Q_{xk} + m\theta_k - m\theta + a_x)}}{m - \nu_x} \ .$$

We get

$$\begin{aligned} &\frac{-i}{24 \sin \pi\nu_x} \sum_k s_k e^{i(a_x - \nu_x\theta + \psi_{xk} - \pi\nu_x)} && 0 < \theta_k - \theta < 2\pi \ , \\ &\frac{-i}{24 \sin \pi\nu_x} \sum_k s_k e^{i(a_x - \nu_x\theta + \psi_{xk} + \pi\nu_x)} && 0 < \theta - \theta_k < 2\pi \ , \\ &\frac{-i}{24 \sin \pi\nu_x} \sum_k s_k \cos \pi\nu_x \, e^{i(a_x - \nu_x\theta + \psi_{xk})} && \theta = \theta_k \ . \end{aligned}$$

All these can be grouped together as

$$\frac{1}{3}[-iB_1(\psi_x) + A_1(\psi_x)]e^{i\phi_x} \ , \tag{3.30}$$

where the Eq. (3.26) has been used for a_u and the set of distortion functions is

$$B_1(\psi_x) = \frac{1}{2\sin\pi\nu_x}\sum_k \frac{s_k}{4}\cos(|\psi_{xk}-\psi_x| - \pi\nu_x) \qquad 0 \le |\psi_{xk}-\psi_x| \le 2\pi\nu_x \ ,$$

$$A_1(\psi_x) = B_1'(\psi_x) \qquad\qquad 0 < |\psi_{xk}-\psi_x| < 2\pi\nu_x \ . \tag{3.31}$$

Note that the restriction $0 \le |\psi_{xk}-\psi_x| \le 2\pi\nu_x$ demands the summation over k for one complete revolution of the ring only. Therefore, at the same location but after one revolution, ψ_x becomes $\psi_x + 2\pi\nu_x$ and all the ψ_{xk}'s have to increase by $2\pi\nu_x$ also in order to satisfy the restriction. Thus, the distortion functions have exactly the same values after one revolution. In other words, they are closed or periodic.

Similarly, the other sums lead to

$$\sum_m \frac{A_{3m}e^{iq_{3m}}}{m-3\nu_x} = \frac{1}{3}(-iB_3 + A_3)e^{i3\phi_x} \ ,$$

$$\sum_m \frac{B_{1m}e^{ip_{1m}}}{m-\nu_x} = (-i\bar{B} + \bar{A})e^{i\phi_x} \ ,$$

$$\sum_m \frac{B_{\pm m}e^{ip_{\pm m}}}{m-\nu_\pm} = (-iB_\pm + A_\pm)e^{i\phi_\pm} \ , \tag{3.32}$$

where $\phi_\pm = 2\phi_y \pm \phi_x$. The distortion functions introduced in Eq. (3.32) are given by

$$B_3(3\psi_x) = \frac{1}{2\sin 3\pi\nu_x}\sum_k \frac{s_k}{4}\cos 3(|\psi_{xk}-\psi_x| - \pi\nu_x) \qquad 0 \le |\psi_{xk}-\psi_x| \le 2\pi\nu_x \ ,$$

$$A_3(3\psi_x) = B_3'(3\psi_x) \qquad\qquad 0 < |\psi_{xk}-\psi_x| < 2\pi\nu_x \ ,$$

$$\bar{B}(\psi_x) = \frac{1}{2\sin\pi\nu_x}\sum_k \frac{\bar{s}_k}{4}\cos(|\psi_{xk}-\psi_x| - \pi\nu_x) \qquad 0 \le |\psi_{xk}-\psi_x| \le 2\pi\nu_x \ ,$$

$$\bar{A}(\psi_x) = \bar{B}'(\psi_x) \qquad\qquad 0 < |\psi_{xk}-\psi_x| < 2\pi\nu_x \ ,$$

$$B_\pm(\psi_\pm) = \frac{1}{2\sin\pi\nu_\pm}\sum_k \frac{\bar{s}_k}{4}\cos(|\psi_{\pm k}-\psi_\pm| - \pi\nu_\pm) \qquad 0 \le |\psi_{\pm k}-\psi_\pm| \le 2\pi\nu_\pm \ ,$$

$$A_\pm(\psi_\pm) = B_\pm'(\psi_\pm) \qquad\qquad 0 < |\psi_{\pm k}-\psi_\pm| < 2\pi\nu_\pm \ , \tag{3.33}$$

where $\psi_\pm = 2\psi_y \pm \psi_x$ and the prime denotes differentiation with respect to the argument. We see that Eqs. (3.31) and (3.33) conform with the definition of the generic distortion functions of Eq. (2.9). It is obvious that all the A's jump by an amount $m_\alpha = s_k/4$ or $\bar{s}_k/4$ across a thin sextupole at location k due to the differentiation of the variable inside the absolute-value delimiters while the B's are continuous but exhibit a cusp. Knowing B_α and A_α at location ψ_α, their values at another location $\psi_\alpha' = \psi_\alpha + \alpha$ are given by

$$B_\alpha' = \frac{1}{2\sin\pi\nu_\alpha}\cos(\psi_\alpha + \alpha - \psi_{\alpha k} - \pi\nu_\alpha) \ ,$$

$$A'_\alpha = -\frac{1}{2\sin\,\pi\nu_\alpha}\sin(\psi_\alpha + \alpha - \psi_{\alpha k} - \pi\nu_\alpha)\ , \tag{3.34}$$

where we have assumed for simplicity that there is only one sextupole at location $\psi_{\alpha k}$ which is less than both ψ_α and ψ'_α, (the situation of many sextupoles can be dealt with similarily). Also, the prime in Eqs. (3.34) and and (3.35) below does not imply differentiation. Expanding the sine and cosine, one easily gets

$$\begin{pmatrix} B'_\alpha \\ A'_\alpha \end{pmatrix} = \begin{pmatrix} \cos\,\alpha & \sin\,\alpha \\ -\sin\,\alpha & \cos\,\alpha \end{pmatrix} \begin{pmatrix} B_\alpha \\ A_\alpha \end{pmatrix}\ , \tag{3.35}$$

implying just a rotation of the vector (B_α, A_α) by the angle α. Thus, a set of distortion functions obeys the three criteria set out in Section 2, which provide an easier and more physical way for its derivation.

III.5 Closed-orbit Distortion

The sextupole has an average dipole effect on a charged particle, thus distorting the ideal closed orbit. After Floquet transformation, one of the equations of motion derived from the Hamiltonian H_3 is

$$x'' + x = -\frac{B''_y}{2(B\rho)}\left(\frac{\beta_x^3}{\beta_0}\right)^{1/2}\beta_x x^2 + \frac{B''_y}{2(B\rho)}\left(\frac{\beta_x\beta_y^2}{\beta_0}\right)^{1/2}\beta_x y^2\ , \tag{3.36}$$

where x'' is second derivative with respect to the Floquet phase ψ_x. Therefore, across a thin sextupole of strength s or $\bar{s}$, x is continuous but x' jumps by an amount

$$\Delta x' = -sx^2 + \bar{s}y^2\ . \tag{3.37}$$

Substituting Eq. (2.1) for x and y and averaging over the instantaneous phases ϕ_x and ϕ_y, we obtain

$$\Delta x' = -\frac{1}{2}s\mathcal{A}_x^2 + \frac{1}{2}\bar{s}\mathcal{A}_y^2\ . \tag{3.38}$$

Since the vector (x, x') rotates by the angle ϕ_x according to Eq. (3.36) in the region without any sextupole, the distortion, which depends on the amplitudes, obeys exactly the same three criteria as a set of distortion functions. The closed-orbit distortions can be computed in exactly the same way yielding

$$\begin{aligned} \delta x(\psi_x) &= -2\mathcal{A}_x^2 B_1(\psi_x) + 2\mathcal{A}_y^2\bar{B}(\psi_x)\ , \\ \delta x'(\psi_x) &= -2\mathcal{A}_x^2 A_1(\psi_x) + 2\mathcal{A}_y^2\bar{A}(\psi_x)\ . \end{aligned} \tag{3.39}$$

There is no distortion of y or y' for the closed orbit because the right side of the equation for y, similar to Eq. (3.36), is proportional to xy which averages to zero.

III.6 Beam Shape in the Transverse Phase Planes

We are now in a position to compute the distortion of the beam-shape projections. Comparing Eqs. (2.3) and (3.23), we get

$$\delta\mathcal{A}_u = (-\delta u \cos\phi_u + \delta u' \sin\phi_u) + \delta\bar{\mathcal{A}}_u \ ,$$

$$\mathcal{A}_u \delta\phi_u = (\delta u \sin\phi_u + \delta u' \cos\phi_u) + \delta\bar{\phi}_u \ . \tag{3.40}$$

Substituting Eqs. (3.19), (3.22), (3.28), (3.30), and (3.32) into Eq. (3.40), we finally arrive at

$$\begin{aligned}
\delta\mathcal{A}_x &= \mathcal{A}_x^2[(A_3 \sin 3\phi_x - B_3 \cos 3\phi_x) - (A_1 \sin\phi_x - B_1 \cos\phi_x)] \\
&\qquad -\mathcal{A}_y^2[(A_+ \sin\phi_+ - B_+ \cos\phi_+) - (A_- \sin\phi_- - B_- \cos\phi_-)] \ , \\
\delta\phi_x &= \mathcal{A}_x[(A_3 \cos 3\phi_x + B_3 \sin 3\phi_x) + (A_1 \cos\phi_x + B_1 \sin\phi_x)] \\
&\qquad -(\mathcal{A}_y^2/\mathcal{A}_x)[(A_+ \cos\phi_+ + B_+ \sin\phi_+) + (A_- \cos\phi_- + B_- \sin\phi_-)] \ , \\
\delta\mathcal{A}_y &= -2\mathcal{A}_x\mathcal{A}_y[(A_+ \sin\phi_+ - B_+ \cos\phi_+) + (A_- \sin\phi_- - B_- \cos\phi_-)] \ , \\
\delta\phi_y &= -2\mathcal{A}_x[2(\bar{A} \cos\phi_x + \bar{B} \sin\phi_x) + (A_+ \cos\phi_+ + B_+ \sin\phi_+) \\
&\qquad + (A_- \cos\phi_- + B_- \sin\phi_-)] \ .
\end{aligned} \tag{3.41}$$

The above distortions are exactly those given by Eq. (2.6).

III.7 Second-order Tuneshifts

To obtain the second-order tuneshifts, we need to evaluate the second-order sextupole terms in the Hamiltonian H_4. From the generating function G_3 of Eq. (3.16), we get

$$\begin{aligned}
(2I_x)^{3/2} &= (2J_x)^{3/2} + 9(2J_x)^2\beta_0^{1/2} \sum_m \left(\frac{A_{3m}}{m - 3\nu_x} \sin q_{3m} + \frac{A_{1m}}{m - \nu_x} \sin q_{1m} \right) \\
&\quad -3(2J_x)(2J_y)\beta_0^{1/2} \sum_m \left(\frac{2B_{1m}}{m - \nu_x} \sin p_{1m} + \frac{B_{+m}}{m - \nu_+} \sin p_{+m} + \frac{B_{-m}}{m - \nu_-} \sin p_{-m} \right) \ ,
\end{aligned}$$

and similar expression for $(2I_x)^{1/2}(2I_y)$. Then, the second-order terms in the Hamiltonian is

$$\begin{aligned}
\Delta H_4|_{\text{sex}} &= \sum_{m'} (A_{3m'} \sin q_{3m'} + 3A_{1m'} \sin q_{1m'}) \times \\
&\quad \times \left[9(2J_x)^2\beta_0 \sum_m \left(\frac{A_{3m}}{m - 3\nu_x} \sin q_{3m} + \frac{A_{1m}}{m - \nu_x} \sin q_{1m} \right) \right. \\
&\quad \left. -3(2J_x)(2J_y)\beta_0 \sum_m \left(\frac{2B_{1m}}{m - \nu_x} \sin p_{1m} + \frac{B_{+m}}{m - \nu_+} \sin p_{+m} + \frac{B_{-m}}{m - \nu_-} \sin p_{-m} \right) \right] \\
&\quad + \cdots
\end{aligned}$$

Betatron tunes are defined per revolution. We therefore average over θ or take only the θ-independent terms. This leads to

$$\Delta H_4'|_{\rm sex} = \frac{9}{2}(2J_x)^2\beta_0\sum_m\left(\frac{A_{3m}^2}{m-3\nu_x}+\frac{3A_{1m}^2}{m-\nu_x}\right)$$
$$+\frac{1}{2}(2J_y)^2\beta_0\sum_m\left(\frac{4B_{1m}^2}{m-\nu_x}+\frac{B_{+m}^2}{m-\nu_+}+\frac{B_{-m}^2}{m-\nu_-}\right)$$
$$+2(2J_x)(2J_y)\beta_0\sum_m\left(\frac{B_{+m}^2}{m-\nu_+}-\frac{B_{-m}^2}{m-\nu_-}-\frac{6A_{1m}B_{1m}}{m-\nu_x}\cos(\alpha_{1m}-\beta_{1m})\right) . \quad (3.42)$$

Now we need to sum over the harmonics using again Eq. (3.28). A particular term is

$$\sum_m\frac{A_{1m}^2}{m-\nu_x} = -\frac{\beta_0}{576\pi\sin\pi\nu_x}\sum_{kk'}s_ks_{k'}\cos(|\psi_{xk}-\psi_{xk'}|-\pi\nu_x) . \quad (3.43)$$

Written in terms of the distortion functions, we have

$$\sum_m\frac{A_{1m}^2}{m-\nu_x} = -\frac{\beta_0}{72\pi}\sum_k(B_1s)_k ,$$
$$\sum_m\frac{A_{3m}^2}{m-3\nu_x} = -\frac{\beta_0}{72\pi}\sum_k(B_3s)_k ,$$
$$\sum_m\frac{B_{1m}^2}{m-\nu_x} = -\frac{\beta_0}{8\pi}\sum_k(\bar{B}\bar{s})_k ,$$
$$\sum_m\frac{B_{\pm m}}{m-\nu_\pm} = \mp\frac{\beta_0}{8\pi}\sum_k(B_\pm\bar{s})_k ,$$
$$\sum_m\frac{A_{1m}B_{1m}\cos(\alpha_{1m}-\beta_{1m})}{m-\nu_x} = -\frac{\beta_0}{24\pi}\sum_k(\bar{B}s)_k . \quad (3.44)$$

The tuneshifts are given by

$$\Delta\nu_x = \frac{\partial\Delta H_4'}{\partial J_x} \quad \text{and} \quad \Delta\nu_y = \frac{\partial\Delta H_4'}{\partial J_y} . \quad (3.45)$$

Using Eqs. (3.42), (3.43), and (3.44), we obtain exactly the tuneshifts given by Eq. (2.10).

IV. APPLICATIONS

Here, we repeat two examples introduced by Collins[1] illustrating beam-shape distortions and betatron tuneshifts.

IV.1 Beam-shape Distortions

We try to look at the beam shape at a location along the storage ring where all the distortion functions A's are zero. Then we have from Eq. (3.39) closed-orbit distortions

$$\delta x = -2\mathcal{A}_x^2B_1 + 2\mathcal{A}_y^2\bar{B} , \qquad \delta x' = 0 ,$$

$$\delta y = 0 \ , \qquad \delta y' = 0 \ . \tag{4.1}$$

The distortions in the amplitudes obtained from Eq. (3.41) reduce to

$$\delta\mathcal{A}_x = -\mathcal{A}_x^2[B_3 \cos 3\phi_x - B_1 \cos\phi_x] + \mathcal{A}_y^2[B_+ \cos(2\phi_y + \phi_x) - B_- \cos(2\phi_y - \phi_x)] \ ,$$

$$\delta\mathcal{A}_y = 2\mathcal{A}_x\mathcal{A}_y[B_+ \cos(2\phi_y + \phi_x) + B_- \cos(2\phi_y - \phi_x)] \ . \tag{4.2}$$

The projections of the beam shape in the x-x' plane and y-y' plane are plotted in Fig. 2. It is drawn with $\mathcal{A}_x B_3 = 0.1$, $\mathcal{A}_x B_1 = -0.05$, $\mathcal{A}_y B_+ = 0.1$, $\mathcal{A}_y B_- = 0.05$, and $\mathcal{A}_y \bar{B} = 0.1$. We see that the two circles in the linear theory have been distorted into a triangular shape and a rhombic shape. The center of the figure in the x-x'-plot is shifted. Also the thin-line circles become bands. The thickness of the bands is called 'smear', which is a measure of the nonlinearity of the lattice. In this example, the smears can actually be computed. For example, in the x-x' plot, at instantaneous phase $\phi_x = 0$,

$$\delta\mathcal{A}_x = -\mathcal{A}_x^2(B_3 - B_1) + \mathcal{A}_y^2(B_+ - B_-)\cos 2\phi_y \ .$$

Thus, the smear there is

$$\delta\mathcal{A}_x|_{\max} - \delta\mathcal{A}_x|_{\min} = 2\mathcal{A}_y^2|B_+ - B_-| \ .$$

At phase angle $\phi_x = \pi/2$,

$$\delta\mathcal{A}_x = \mathcal{A}_y^2[B_+ \cos(2\phi_y + \pi/2) - B_- \cos(2\phi_y - \pi/2)] \ ,$$

giving a smear of

$$\delta\mathcal{A}_x|_{\max} - \delta\mathcal{A}_x|_{\min} = 2\mathcal{A}_y^2|B_+ + B_-| \ .$$

Similarly, for the y-y' plot,

$$\text{smear} = 4\mathcal{A}_x\mathcal{A}_y|B_+ + B_-| \qquad \text{at } \phi_y = 0 \ ,$$

$$\text{smear} = 4\mathcal{A}_x\mathcal{A}_y|B_+ - B_-| \qquad \text{at } \phi_y = \frac{\pi}{4} \ .$$

IV.2 Tuneshifts

Consider a mock design of the Superconducting Super Collider (SSC). Within each half cell of 40°, there are five superconducting dipoles which have a systematic sextupole at low field. We would like it to be corrected by the chromaticity adjustment of sextupoles at the quads. Is this good enough? In Table II, the beta-functions, the Floquet phases, and the sextupoles at the quads $S_F/2$ and $S_D/2$ are calculated by the usual thin lens formulae. The distortion functions are calculated using Eqs. (3.31) and (3.33) taking the A's at the quads to be zero for closure. Here, β_0 is taken as the maximum β and θ is the bend angle of a half cell.

We have for a full cell,

$$\sum_k (B_3 s)_k = -0.0164\,(\beta_0 b_2 \theta)^2 \ , \qquad \sum_k (B_+ \bar{s})_k = -0.0205\,(\beta_0 b_2 \theta)^2 \ ,$$

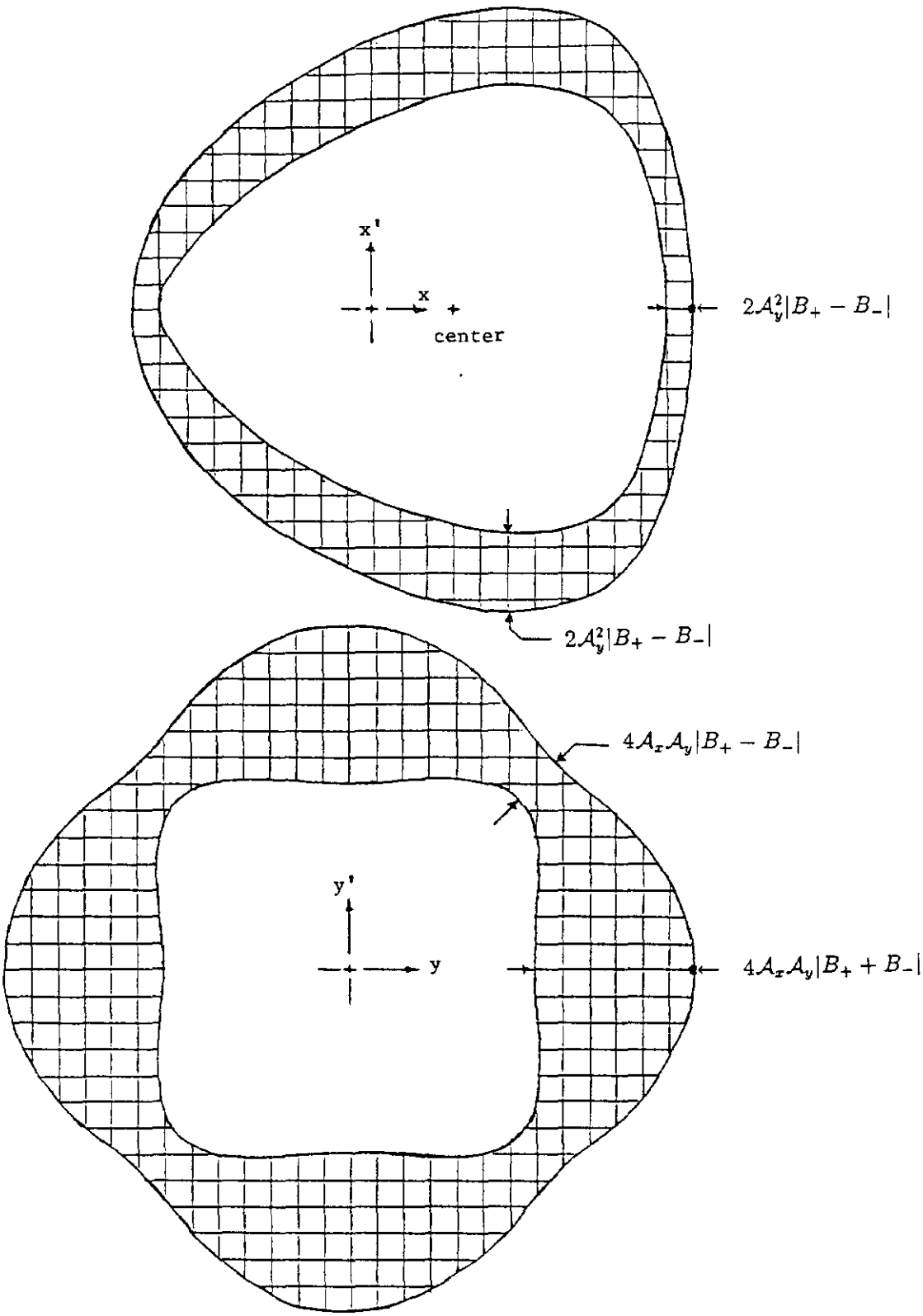

Figure 2: Beam shape projections onto the horizontal and vertical phase planes with the addition of sextupoles.

$$\sum_k (B_1 s)_k = -0.0052\,(\beta_0 b_2 \theta)^2 \ , \qquad \sum_k (B_- \bar{s})_k = \ 0.0018\,(\beta_0 b_2 \theta)^2 \ ,$$
$$\sum_k (B_1 \bar{s})_k = -0.0024\,(\beta_0 b_2 \theta)^2 \ , \qquad \sum_k (\bar{B} \bar{s})_k \ = -0.0101\,(\beta_0 b_2 \theta)^2 \ .$$

The tuneshifts are systematic. For N cells, using Eq. (2.10), they are

$$\Delta\nu_x = (0.0025\mathcal{A}_x^2 + 0.0022\mathcal{A}_y^2) N (\beta_0 b_2 \theta)^2 \ ,$$
$$\Delta\nu_y = (0.0022\mathcal{A}_x^2 + 0.0050\mathcal{A}_y^2) N (\beta_0 b_2 \theta)^2 \ . \tag{4.3}$$

	quad	dipole position					quad	unit
	F	.18	.34	.50	.66	.82	D	
β_x	1.00	.785	.621	.483	.371	.284	.217	β_0
β_y	.217	.284	.371	.483	.621	.785	1.00	β_0
ϕ_x	0	3.48	7.43	12.45	18.96	27.46	40	degrees
ϕ_y	0	12.54	21.04	27.55	32.57	36.51	40	degrees
S	-.325	.200	.200	.200	.200	.200	-.537	$b_2\theta$
s	-.325	.139	.098	.067	.045	.034	-.054	$(\beta_0 b_2 \theta)$
$\bar{s}$	-.071	.050	.058	.067	.076	.084	-.250	$(\beta_0 b_2 \theta)$
B_3	.0293	.0141	.0034	-.0039	-.0074	-.0057	.0032	$(\beta_0 b_2 \theta)$
B_1	.0089	.0035	.0003	-.0017	-.0023	-.0014	.0015	$(\beta_0 b_2 \theta)$
B_+	-.0045	-.0124	-.0119	-.0056	.0054	.0219	.0447	$(\beta_0 b_2 \theta)$
B_-	.0105	.0033	.0014	.0023	.0037	.0033	-.0028	$(\beta_0 b_2 \theta)$
$\bar{B}$	-.0011	-.0022	-.0025	-.0017	.0014	.0081	.0222	$(\beta_0 b_2 \theta)$

Table II: Lattice values including distortion functions for a half-cell of a mock design of the SSC.

In a normal proton ring $\beta_0\theta \sim 3$ meters. For the Tevatron ring, $N = 100$ and $b_2 \sim 1.3/\mathrm{m}^2$. Then, at $\mathcal{A}_x = \mathcal{A}_y = 1$ cm, $\Delta\nu_y = 0.0012$ which is indeed a small number. For the SSC, however, $N = 400$ and $b_2 \sim 33/\mathrm{m}^2$. Again at $\mathcal{A}_x = \mathcal{A}_y = 1$ cm, $\Delta\nu_y > 2$ units, showing that the design is no good. Collins points out that all these can be computed using a hand-held calculator and tracking will add nothing more.

APPENDIX

We take only one part of the Hamiltonian H_3,

$$H_3(a, I) = \nu I + (2I)^{3/2} \sum_m A_{3m} \sin q_{3m} \tag{5.1}$$

and try to find the generating function $G_3(a, J; \theta)$ so that the new Hamiltonian

$$H_4(b, J) = H_3(a, I) + \frac{\partial G_3}{\partial \theta} \tag{5.2}$$

is a constant independent of a and θ when second order in A_{3m} is neglected. In above,

$$q_{3m} = 3a - m\theta + \alpha_{3m} \ , \tag{5.3}$$

and we have left out the subscripts x or y for clarity. Since this is a perturbative canonical transformation, the generating function must be of the form

$$G_3(a, J; \theta) = aJ + \sum_m A'_{3m} \cos q_{3m} \ . \tag{5.4}$$

We therefore get

$$\frac{\partial G_3}{\partial \theta} = \sum_m m A'_{3m} \sin q_{3m} \ , \tag{5.5}$$

and

$$I = \frac{\partial G_3}{\partial a} = J - \sum_m 3A'_{3m} \sin q_{3m} \ . \tag{5.6}$$

Substituting Eqs. (5.5) and (5.6) into Eq. (5.2) and demanding that A_{3m} cancels up to first order, A'_{3m} can be solved and we obtain

$$G_3(a, J; \theta) = aJ - (2J)^{3/2} \sum_m \frac{A_{3m}}{m - 3\nu} \cos q_{3m} \ . \tag{5.7}$$

The other parts of the generating function can be obtained similarily.

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