



MOSER-LIKE TRANSFORMATIONS USING THE LIE TRANSFORM

Leo Michelotti

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Abstract

We present the Deprit-Hori-Kamel recursive algorithm for carrying out canonical transformations that eliminate non-secular terms of a Hamiltonian. The method is illustrated in the context of accelerator theory by application to three sample problems.

How Alexander wept when he had no more worlds to conquer, everybody knows --- or has some reason to know by this time, the matter having been rather frequently mentioned.

--- Charles Dickens,
Bleak House (1853)

1. INTRODUCTION.

Like many other good ideas, the one which accelerator physicists call "Moser's transformation"¹ was in fact developed and studied in some detail by Poincare. According to Jupp (p.413),

"In his *Methodes Nouvelles de la Mechanique Celeste*, Poincare (1893) describes techniques to 'eliminate' successively non-resonant periodic terms from the Hamiltonian of a dynamical system. Each elimination is achieved by means of a canonical transformation of variables, which is constructed using a generating function depending upon the old angle variables and the new momentum variables. After all the periodic terms have been removed in this way, the final Hamiltonian is purely secular."

The method was modified by von Zeipel in 1916 and is called, by the rest of the world, the "Poincare-von Zeipel procedure". Even so, it had antecedents. Giacaglia (pp.55,47) writes, somewhat ambiguously,

"It is a recognized fact, although several times not mentioned, that the averaging methods were introduced by Lindstedt (1882), though it is not clear whether his ideas stemmed from the efforts of Euler (1750) in the solution of the problem of motion of the moon. ... In his celebrated '*Methodes Nouvelles*', vol.2, [Poincare] developed a canonical analog of

1. /But not the superconvergent procedure.

Linstedt's method which, even after a superficial look, proves to be a very elaborate generalization. However, it is obvious that the main idea of Poincare's development comes from Delaunay and some remarks of Tisserand on Delaunay's Lunar Theory [Theorie du Mouvement de la Lune, 1867]."

This work of Delaunay's culminated an effort that appears superhuman by modern standards. Deprit, Henrard, and Rom (p.1569) state that

"Delaunay worked at his theory without any assistance, by hand, for some 20 years continuously (sic); his literal calculations cover two volumes in quarto of 400 pages each; he alone proofread them."

Although he is not frequently mentioned outside celestial mechanics, Delaunay's influence on the physics of his age was considerable. Among his other accomplishments, he seems to have been the one who invented action-angle variables. According to Lanczos (pp.254,245),

"Delaunay invented a beautiful method for treating separable systems which satisfy the additional condition that the stream lines of the separated phase planes (q_k, p_k) are closed lines. He considers a canonical transformation whose position coordinates are the "action variables" J_k defined by the areas enclosed by the stream lines. The J_k are constants for the actual motion while the negatives of the conjugate momenta, the "angle variables" ω_k change linearly with the time t . The partial derivatives of E with respect to the J_j give n new constants which are the frequencies ν_j of the motion. ... At first sight Delaunay's theory seems rather technical and involved. Yet it was this procedure ... which opened the eyes of physicists to the power of the Hamiltonian methods."

Somewhere in history this association was reversed. We now think of the action variable as the "momentum", although a vestige of the original ordering may be contained in the terminology "action-angle", rather than "angle-action".

It has been remarked by many authors that the Poincare-von Zeipel procedure suffers from a serious disadvantage. Because the generating

function is written in a mixed system of variables, the transformation from the new, "averaged" variables to the old, "exact" variables is only defined implicitly. Practically, then, carrying out the transformation to better than lowest order is accomplished more in principle than in practice. Beginning with a theoretical paper by Hori (1966), Lie transforms provided a new, alternative "averaging" procedure in which transformation equations were explicit and could be developed recursively to any order. Further, the algorithm was written completely in terms of nested Poisson brackets, explicitly providing invariance under canonical transformations and thereby assuring that it could be implemented without alteration using any convenient system of conjugate variables. Finally, the new theory possessed the almost unique distinction of being "not known to Poincare, a thing hard to discover in perturbation theories"², indeed in dynamics as a whole. That alone would serve to make it exciting.

Although it is true that the current renaissance in Lie transforms can be traced to Hori, it was not until Deprit's work, three or four years later, that the world took notice that something new had come on the scene. Working independently, Deprit (1969) wrote his own algorithm, and, in 1970, he and his collaborators linked it to a modern computer algebra program (MACSYMA) and reproduced Delaunay's monumental calculations. (The dramatic result of this double checking was that in

2. /Giacaglia, p.144.

twenty years of effort Delaunay had made only one mistake --- amounting to writing $147-90+9 = 46$ --- at the 9th order, all other errors resulting from its propagation through other terms.) Deprit's method was studied further by Kamel and Dewar, and recently it has been used in plasma physics by Dewar, Kaufmann, Littlejohn, and Cary.

Also using Lie operators in plasma physics, Abarbanel recently has exhibited perturbation series free of small denominators (1980) and has worked on calculating diffusion in phase space for chaotic systems (1982). His ideas are exciting and may prove useful for accelerator problems, but we shall not consider them here, as they have not yet been crystallized into an algorithm.

In accelerator theory, Dragt and his coworkers, using a different approach, have been systematically exploiting Lie transforms for constructing transfer maps through accelerators with nonlinear elements. A version of PROGRAM MARYLIE, the first fruit of their labor, is now available for general use and is rapidly gaining acceptance by accelerator physicists. Although the underlying theoretical framework of this program is that of the celestial mechanics and plasma physics work, the application is very different: MARYLIE constructs a symplectic mapping which is to be iterated while the others generate an averaged hamiltonian which operates in continuous time.

When the dynamics are integrable, Deprit's algorithm is a simple, systematic procedure for generating invariant KAM surfaces to any desired order. These surfaces provide a useful tool for solving the

dynamic aperture problem when the fundamental limitation comes from distortion of elliptically cross-sectioned emittance tori.³ Of course, nonlinear systems are almost never integrable, and in lieu of major, unforeseen breakthroughs, fast numerical tracking algorithms (coupled with good interactive graphics) will be essential for detailed dynamical studies. Even in such cases, however, it may prove useful to strike a balance between "analytic" and "numerical" methods so as to increase the useful information obtained per computational cycle. As an example of this, consider numerical quadrature, where it is frequently good practice to smooth the integrand by a judicious choice of variables before entering a numerical procedure. By tailoring the problem to the algorithm in this way one can sometimes achieve increased accuracy with fewer computational steps.

In this memo we shall consider briefly Deprit's algorithm in the context of accelerator theory. The method is presented in Sec. 2, and in Sec. 3 it is applied to three simple, familiar examples by way of illustration. Sec. 4 contains a few irresponsible concluding remarks.

3. /For example, see F. Willeke, "Determination of the Dynamic Aperture of Circular Accelerators by the Perturbation Theory Method", Proc. SSC Workshop, U. of Michigan, Ann Arbor, Dec. 12-17, 1983.

2. PERTURBATION THEORY AND THE LIE TRANSFORM.

In this section the symbols z , z^* , and u will denote a generic set of $2N$ conjugate phase space variables.

$$z, z^*, u = \begin{pmatrix} q \\ p \end{pmatrix} \text{ or } \begin{pmatrix} \gamma \\ \underline{j} \end{pmatrix} \text{ or whatever ...}$$

We are confronted with the problem of solving the presumably complicated dynamics generated by a Hamiltonian $H(z; \phi; \epsilon)$.

$$\frac{dz^*}{d\phi} = \Pi \cdot \frac{\partial H(z^*; \phi; \epsilon)}{\partial z^*}, \quad \Pi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Because we look toward applications to circular accelerators the independent variable ϕ is taken to be cyclic; H is periodic in ϕ with period 2π . This attribute is not essential, however, and in most applications the independent variable is interpreted as (non-cyclic) time. The variable ϵ is a control parameter, the "small parameter" on which the perturbation series will be built. Everything must behave smoothly in a neighborhood of $\epsilon=0$; no catastrophes are allowed in the phase space region of interest. In particular, the limiting hamiltonian

$$H_0 = \lim_{\epsilon \rightarrow 0} H$$

is well defined, and its orbits are presumed known.

Although an orbit of H may be very complicated we assume that its behavior can be averaged, in a sense to be made more precise. The expectation is that this "averaged" orbit will be easier to solve, or at least to study. Its development also is governed by a hamiltonian, say K .

$$\frac{dz}{d\phi} = \mathbb{I} \cdot \frac{\partial K(z; \phi; \epsilon)}{\partial z}$$

Further, by slicing state space at constant values of ϕ we get a one-to-one correspondence between points z on the K -orbits and points z^* on the H -orbits. This mapping $z \rightsquigarrow z^*$, which Dewar (1978) calls the "clothing transformation" in analogy with renormalized quantum field theory, is needed to solve the exact dynamics from the averaged dynamics.

A Lie transform gets at the clothing transformation by defining a new dynamics using \mathcal{E} as the independent variable, with ϕ held fixed. For a hamiltonian system, its equation of motion is written

$$\frac{du}{d\mathcal{E}} = \mathbb{I} \cdot \frac{\partial S(u; \phi; \epsilon)}{\partial u}$$

The fundamental problem is to find a generating function S such that if we apply the boundary condition $u(\mathcal{E}=0) = z$, then $u(\mathcal{E}) = z^*$.

In perturbation theory S is constructed by expanding everything in powers of ϵ about $\epsilon=0$. The algorithm presented below has been derived or explained in various ways by Deprit, Kamel, Nayfeh, Cary,

and others. The development in Nayfeh is recommended, although perhaps less formally elegant than the others, in that he treats Hamiltonian dynamics as a special case of the more general problem of solving first order ordinary differential equations. We will lay out the procedure without derivation and then describe briefly two methods for solving the linear partial differential equation which it spawns.

2.1 Recursive algorithm for hamiltonian systems.

Begin by expanding H , K , and S as power series in ϵ .

$$H = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} H_n, \quad K = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} K_n, \quad S = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} S_{n+1}$$

The functions H_n are known; it is required to find all K_n and S_n order by order. The initial step is trivial.

0th order: $K_0 = H_0$.

Now, define an operator D , acting on real valued functions over state space.

$$Df = \partial f / \partial \phi + \{ f, H_0 \}$$

where $\{ , \}$ is the Poisson bracket,

$$\{a, b\} = \frac{\partial a}{\partial z} \cdot \mathbb{I} \cdot \frac{\partial b}{\partial z}$$

The first order equation is then written as follows.

$$\text{1st order: } DS_1 + K_1 = H_1 .$$

There are two unknown functions. K_1 is determined by requiring a bounded solution for S_1 , unbounded solutions doing violence to the convergence of a perturbative series. (This will become clearer in Section 3 when we consider some concrete examples.)

For $n \geq 2$, the n th order equations are constructed as follows.

$$DS_n + K_n = H_n + \sum_n^H + \sum_n^K \quad (1a)$$

$$\sum_n^H = \sum_{m=1}^{n-1} \binom{n-1}{m-1} \{H_{n-m}, S_m\} \quad (1b)$$

$$\sum_n^K = \sum_{m=1}^{n-1} \binom{n-1}{m-1} K_{n-m, m} \quad (1c)$$

$$K_{1j} = \{K_j, S_1\} \quad (1d)$$

$$K_{ij} = \{K_j, S_i\} - \sum_{m=1}^{i-1} \binom{i-1}{m-1} \{K_{i-m, j}, S_m\}, \quad i \geq 2 \quad (1e)$$

For example, at the fourth order, we first find K_{13} , K_{22} , and K_{31} by evaluating the Poisson brackets of functions constructed at lower

orders, according to the prescription of Eq's. (1d, 1e). These are then put together to form \sum_4^K . \sum_4^H is built similarly, and the two are combined with H_n to write the partial differential equation. For future reference and in order to see the pattern that emerges, the sequence of steps through sixth order are displayed below.

0th order. $K_0 = H_0$

1st order. $DS_1 + K_1 = H_1$ (1f)

2nd order. $K_{11} = \{K_1, S_1\}$

$$DS_2 + K_2 = H_2 + \{H_1, S_1\} + K_{11} \quad (1g)$$

3rd order. $K_{12} = \{K_2, S_1\}$

$$K_{21} = \{K_1, -K_{11}, S_2\}$$

$$DS_3 + K_3 = H_3 + \{H_2, S_1\} + 2\{H_1, S_2\} \\ + K_{21} + 2K_{12} \quad (1h)$$

4th order $K_{13} = \{K_3, S_1\}$

$$K_{22} = \{K_2, S_2\} - \{K_{12}, S_1\}$$

$$K_{31} = \{K_1, S_3\} - 2\{K_{11}, S_2\} - \{K_{21}, S_1\}$$

$$DS_4 + K_4 = H_4 + \{H_3, S_1\} + 3\{H_2, S_2\} \\ + 3\{H_1, S_3\} + K_{31} \quad (1i) \\ + 3K_{22} + 3K_{13}$$

$$\begin{aligned}
\text{5th order: } K_{14} &= \{K_4, S_1\} \\
K_{23} &= \{K_3, S_2\} - \{K_{13}, S_1\} \\
K_{32} &= \{K_2, S_3\} - 2\{K_{12}, S_2\} - \{K_{22}, S_1\} \\
K_{41} &= \{K_1, S_4\} - 3\{K_{11}, S_3\} - 3\{K_{21}, S_2\} \\
&\quad - \{K_{31}, S_1\} \\
DS_5 + K_5 &= H_5 + \{H_4, S_1\} + 4\{H_3, S_2\} + 6\{H_2, S_3\} \\
&\quad + 4\{H_1, S_4\} + K_{41} + 4K_{32} + 6K_{23} \\
&\quad + 4K_{14} \tag{1j}
\end{aligned}$$

$$\begin{aligned}
\text{6th order: } K_{15} &= \{K_5, S_1\} \\
K_{24} &= \{K_4, S_2\} - \{K_{14}, S_1\} \\
K_{33} &= \{K_3, S_3\} - 2\{K_{13}, S_2\} - \{K_{23}, S_1\} \\
K_{42} &= \{K_2, S_4\} - 3\{K_{12}, S_3\} - 3\{K_{22}, S_2\} \\
&\quad - \{K_{32}, S_1\} \\
K_{51} &= \{K_1, S_5\} - 4\{K_{11}, S_4\} - 6\{K_{21}, S_3\} \\
&\quad - 4\{K_{31}, S_2\} - \{K_{41}, S_1\} \\
DS_6 + K_6 &= H_6 + \{H_5, S_1\} + 5\{H_4, S_2\} + 10\{H_3, S_3\} \\
&\quad + 10\{H_2, S_4\} + 5\{H_1, S_5\} + K_{51} + 5K_{42} \\
&\quad + 10K_{33} + 10K_{24} + 5K_{15} \tag{1k}
\end{aligned}$$

Storing the intermediate array K_{ij} requires a memory that grows quadratically with the order of the calculation. It is possible to redesign the algorithm for linear growth by making the calculations in each order call upon previous solutions only, not upon intermediate results. To this end it is useful to introduce a frequently used operator notation. To any real valued function f over phase space we associate the operator $L(f)$, itself acting on real valued functions over phase space, defined by

$$L(f)g = \{g, f\} .$$

These operators are derivations,

$$L(f)gh = (L(f)g)h + g(L(f)h)$$

and in fact are called Lie derivatives.⁴ Note in passing that $D = \partial/\partial\psi + L(H_0)$.

Now introduce the operators D_m defined recursively as follows.

$$D_1 = L(S_1)$$

$$D_m = L(S_m) - \sum_{k=1}^{m-1} \binom{m-1}{k-1} L(S_k) D_{m-k}$$

*. For whimsical reasons of his own, V. I. Arnold calls them fisherman's derivatives. Dragt's unique notation for ' $L(f)$ ' is ' $\{ : f :$ '.

It is easy to prove by induction that

$$K_{m,n-m} = O_m K_{n-m}$$

Making this substitution in Eq. (1c) leads to the new, desired form. For example, at fifth order, and with the condensed notation $L_m := L(S_m)$,

$$\begin{aligned} \sum_5^K &= 4 L_1 K_4 \\ &+ 6 (L_2 - L_1^2) K_3 \\ &+ 4 (L_3 - L_1 (L_2 - L_1^2) - 2L_2 L_1) K_2 \\ &+ 1 (L_4 - L_1 (L_3 - L_1 (L_2 - L_1^2) - 2L_2 L_1) \\ &\quad - 3L_2 (L_2 - L_1^2) - 3L_3 L_1) K_1 \end{aligned}$$

Although conserving memory, this form of the algorithm wastes time by duplicating previous computations. For example, the function $(L_2 - L_1^2) K_2$ which appears in the expansion for \sum_5^K was previously computed as the second term appearing in \sum_4^K .

Solving these equations yields the averaged hamiltonian and the generating function, but the dressing transformation itself must still be written out. That also is done systematically, order by order.

$$z^* = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} z_n(z; \phi) \quad (2a)$$

$$\text{0th order: } z_0(z; \phi) = z, \quad (2b)$$

$$\text{1st order: } z_1(z; \phi) = I \cdot \partial S_1(z; \phi) / \partial z \quad (2c)$$

For $n \geq 2$:

$$z_n = I \cdot \frac{\partial S_n}{\partial z} + \sum_{m=1}^{n-1} \binom{n-1}{m} z_{m, n-m}, \quad \text{where} \quad (2d)$$

$$z_{1,j} = \{z_j, S_1\} \quad (2e)$$

$$z_{i,j} = \{z_j, S_i\} - \sum_{m=1}^{i-1} \binom{i-1}{m-1} \{z_{i-m, j}, S_m\} \quad (2f)$$

This looks very much like the algorithm for constructing the S_n 's with the important difference that only one S_n appears in Eq. (1e) while all previously constructed S_n 's appear in Eq. (2f).

2.2 Constructing solutions.

To find the functions S_n we must solve partial differential equations of the form

$$DS_n(z, \phi) = \text{rhs}_n(z, \phi). \quad (3)$$

Two methods will be considered: integration along orbits and expansion in eigenfunctions.

2.2.1 Integration along orbits.

The differential operator D is, by Hamilton's equations, the total derivative along the direction of the local H_0 -orbit. Solutions to the partial differential equations can be obtained therefore by integrating the inhomogeneous term along these orbits. This approach is actually an instance of the method of characteristics.

Let $\mathcal{Z}(z; \phi, \phi_i)$ be the orbit that passes through z at $\phi = \phi_i$.

$$\frac{\partial \mathcal{Z}(z; \phi, \phi_i)}{\partial \phi} = \mathbb{I} \cdot \frac{\partial H_0(u; \phi)}{\partial u} \Big|_{u = \mathcal{Z}(z; \phi, \phi_i)}$$

$$\mathcal{Z}(z; \phi_i, \phi_i) = z$$

We then have the following

ASSERTION: The general solution to the partial differential equation

(3) is

$$S_n(z; \phi) = \int^{\phi} d\phi' \text{ rhs}_n(\mathcal{Z}(z; \phi', \phi), \phi')$$

+ any constant
of H_0 motion

The proof of this is easy and will be omitted. There is about this integral a whiff, albeit subdued and disguised, of resonances and small denominators. For, suppose that \mathcal{Z} possesses a period in ϕ

commensurate with 2π . Then the integrand would be a periodic function of ϕ . Unless K_n be chosen to annihilate the integral over one period, the solution would grow indefinitely with increasing ϕ .

The arbitrary 'constant of H_0 -motion' is fixed by demanding that the solution have the appropriate 2π periodicity in ϕ . In the particular case

$$H_0(\underline{\gamma}, \underline{J}; \phi) = F(\underline{J}) \quad , \quad \underline{\omega}(\underline{J}) \equiv \frac{\partial F}{\partial \underline{J}}$$

the solution is specialized to

$$S_n(\underline{\gamma}, \underline{J}; \phi) = \int^{\phi} d\phi' \text{rhs}_n(\underline{\gamma} + \underline{\omega}(\underline{J}) \cdot (\phi' - \phi), \underline{J}; \phi') \quad (4) \\ + \text{func}(\underline{\gamma} - \underline{\omega}(\underline{J})\phi, \underline{J})$$

where 'func' represents that function which will make S_n periodic.

2.2.2 Expansion in eigenfunctions.

Like all linear operators, D possesses eigenfunctions. By using these as a basis, the differential equations (3) can be expanded and solved algebraically. We will specialize considerations of this section to the unperturbed hamiltonian ⁵

5. /A small warning: this violates the crucial hypothesis of the KAM theorem, $\det \text{hess } H_0 \neq 0$.

$$H_0(\underline{\gamma}, \underline{J}; \phi) = \underline{\nu} \cdot \underline{J}$$

Our operator D then becomes

$$D = \frac{\partial}{\partial \phi} + \underline{\nu} \cdot \frac{\partial}{\partial \underline{\gamma}}$$

acting on the space of functions periodic in ϕ and in all the γ_i 's. The eigenfunctions of D on this space are the complex exponentials

$$D \cdot \exp i(m\phi + \underline{p} \cdot \underline{\gamma}) = i(m + \underline{\nu} \cdot \underline{p}) \cdot \exp i(m\phi + \underline{p} \cdot \underline{\gamma})$$

where m is an integer, and \underline{p} is a list of integers. (Note that D is an anti-Hermitian operator.) Written in component form, Eq. (3) becomes

$$i(m + \underline{\nu} \cdot \underline{p}) S_{n; m \underline{p}}(\underline{J}) = \text{rhs}_{n; m \underline{p}}(\underline{J})$$

where we formally acknowledge that the components depend on the action variables, \underline{J} .

For all m and \underline{p} that satisfy the resonant condition $m + \underline{\nu} \cdot \underline{p} = 0$ we must choose the (m, \underline{p}) component of K_n so that $\text{rhs}_{n; m \underline{p}} = 0$. Even in the absence of non-trivial resonances, we must at least choose the average term $K_{n; \epsilon_0}$ according to this criterion.

To use this representation in the solution algorithm that was laid out in Eq. (1) and Eq. (2), we must obtain the components of Poisson brackets. Since we will always be dealing with polynomials in the action variables, it is sufficient to consider brackets of the following form.

$$\{ J_k^a f(\underline{\gamma}, \phi), J_l^b g(\underline{\gamma}, \phi) \}; m, \underline{p}$$

$$= i J_k^{a-1} J_l^{b-1} \sum_{\langle m, \underline{p} \rangle} (b J_k p_l' - a J_l p_k'') f_{m', \underline{p}'} g_{m'', \underline{p}''}$$

The notation $\sum_{\langle m, \underline{p} \rangle}$ signifies a sum carried out over all $m', m'', \underline{p}', \underline{p}''$ such that $m' + m'' = m$ and $\underline{p}' + \underline{p}'' = \underline{p}$. For one dimensional problems this expression simplifies to

$$i J^{a+b-1} \sum_{\langle m, p \rangle} (b p' - a p'') f_{m', p'} g_{m'', p''} \quad (5)$$

3. EXAMPLES.

To illustrate the Lie transform perturbation series, we shall look at three sample problems in one dimension: (1) zeroth harmonic sextupole and octupole terms (to 4th order), (2) quadrupole field errors (to 2nd order), and (3) arbitrary sextupole and octupole terms (to 2nd order). Our purpose here is certainly not to study these systems --- and we will by no means make a thorough job of them --- but only to use them as settings for realizing Deprit's algorithm.

3.1 Zeroth harmonic sextupole and octupole terms.

The hamiltonian for an accelerator with zeroth harmonic sextupoles and octupoles can be expressed in action-angle form as follows.

$$H = \nu J + \epsilon J^{3/2} \sin^3 \gamma + \frac{1}{2} \epsilon^2 \kappa J^2 \sin^4 \gamma$$

That is,

$$H_0 = \nu J$$

$$\begin{aligned} H_1 &= J^{3/2} \sin^3 \gamma \\ &= J^{3/2} \left(\frac{3}{4} \sin \gamma - \frac{1}{4} \sin 3\gamma \right) \end{aligned}$$

$$\begin{aligned} H_2 &= \kappa J^2 \sin^4 \gamma \\ &= \kappa J^2 \left(\frac{3}{8} - \frac{1}{2} \cos 2\gamma + \frac{1}{8} \cos 4\gamma \right) \end{aligned}$$

Introducing the parameter κ is a device for expressing an intrinsically two-parameter problem in a one-parameter formalism. For this to be legitimate, κ should not be too large.

The first order equation to be solved is

$$\left(\frac{\partial}{\partial \phi} + \nu \frac{\partial}{\partial \gamma}\right) S_1 + K_1 = J^{3/2} \left(\frac{3}{4} \sin \gamma - \frac{1}{4} \sin 3\gamma\right)$$

Its solution can be written by inspection, but we will use trajectory integration just to illustrate the method in this simple case. First of all, there is no growth term --- that is, neither $\sin \gamma$ nor $\sin 3\gamma$ has a non-zero average --- so we must choose $K_1 = 0$. Then using Eq. (4) we write the solution for S_1 as

$$\begin{aligned} S_1 &= \int_{\phi_0}^{\phi} d\phi' \left[\frac{3}{4} \sin(\gamma + \nu(\phi' - \phi)) - \frac{1}{4} \sin 3(\gamma + \nu(\phi' - \phi)) \right] \times J^{3/2} \\ &\quad + \text{func}(\gamma - \nu\phi, J) \\ &= (J^{3/2}/\nu) \left[-\frac{3}{4} (\cos \gamma - \cos(\gamma - \nu\phi + \nu\phi_0)) \right. \\ &\quad \left. + \frac{1}{12} (\cos 3\gamma - \cos(\gamma - \nu\phi + \nu\phi_0)) \right] \\ &\quad + \text{func}(\gamma - \nu\phi, J) \end{aligned}$$

The arbitrary function is now chosen to cancel the unwanted terms and to make S_1 periodic in ϕ as well as γ . We are left with

$$S_1 = (J^{3/2}/\nu) \left[-\frac{3}{4} \cos Y + \frac{1}{12} \cos 3Y \right]$$

At the second order, because $K_1 = 0$, Eq. (1g) is simplified to

$$\left(\frac{\partial}{\partial \phi} + \nu \frac{\partial}{\partial Y} \right) S_2 + K_2 = H_2 + \{H_1, S_1\} \quad (6)$$

The Poisson bracket is evaluated easily.

$$\begin{aligned} \{H_1, S_1\} &= \frac{3}{16\nu} J^2 \left(-5 + 5 \cos 3Y \cos Y + 3 \sin 3Y \sin Y \right) \\ &= (3J^2/4\nu) \left(-\frac{5}{4} + \frac{1}{4} \cos 4Y + \cos 2Y \right) \end{aligned}$$

K_2 must be chosen to cancel the average term on the rhs of Eq. (6), because the presence of such a term would make S_2 grow without bound.

$$K_2 = - (J^2/\nu) \left(\frac{15}{16} - \frac{3}{8} \nu K \right)$$

The solution for S_2 is written by inspection.

$$S_2 = J^2 \left[\frac{1}{4\nu} \left(\frac{3}{2\nu} - K \right) \sin 2Y + \frac{1}{32\nu} \left(\frac{3}{2\nu} + K \right) \sin 4Y \right]$$

Pushing on to third order, we first note that since $K_1 = K_{11} = 0$, then $K_{21} = 0$. The evaluation of K_{12} is straightforward.

$$K_{12} = \frac{3}{16} (J^{5/2}/v^2) \left(\frac{5}{2} - v\kappa \right) (3\sin\gamma - \sin 3\gamma)$$

The other two brackets that we need are evaluated below.

$$\{H_2, S_1\} = \frac{\kappa}{8v} J^{5/2} (\sin 5\gamma + 3\sin 3\gamma - 14\sin \gamma)$$

$$\begin{aligned} \{H_1, S_2\} = (J^{5/2}/v^2) & \left[\frac{1}{128} (6v\kappa - 27) \sin 5\gamma \right. \\ & + (3v\kappa/16) \sin 3\gamma \\ & \left. - \frac{1}{128} (102v\kappa - 135) \sin \gamma \right] \end{aligned}$$

Putting these into Eq. (1h) and solving yields the result.

$$\begin{aligned} S_3 = (J^{5/2}/v^3) & \left[\frac{1}{320} (27 - 14v\kappa) \cos 5\gamma \right. \\ & + \frac{1}{16} (5 - 6v\kappa) \cos 3\gamma \\ & \left. - \frac{1}{64} (315 - 286v\kappa) \cos \gamma \right] \end{aligned}$$

Note that there are no average terms to be cancelled, and thus $K_3 = 0$.

To see the next contribution to K we must go to fourth order.

The terms K_{13} and K_{31} vanish. The evaluation of K_{22} is tedious but straightforward. We will only give the result here.

$$\begin{aligned}
K_{22} = (J^3/\nu^3) & \left[\left(-\frac{15}{256} + \frac{3}{128} \nu\kappa \right) \cos 6\gamma \right. \\
& + \left(\frac{45}{128} + \frac{3}{32} \nu\kappa - \frac{3}{32} (\nu\kappa)^2 \right) \cos 4\gamma \\
& + \left(-\frac{225}{256} - \frac{75}{128} \nu\kappa + \frac{3}{8} (\nu\kappa)^2 \right) \cos 2\gamma \\
& \left. + \left(\frac{75}{32} - \frac{15}{16} \nu\kappa \right) \right]
\end{aligned}$$

The two nonvanishing Poisson brackets which contribute to fourth order are as follows.

$$\begin{aligned}
\{H_2, S_2\} = J^3 \frac{\kappa}{\nu^2} & \left[-\frac{1}{64} (2\nu\kappa - 9) \cos 6\gamma \right. \\
& - \frac{3}{64} (2\nu\kappa + 3) \cos 4\gamma \\
& + \frac{3}{64} (14\nu\kappa - 15) \cos 2\gamma \\
& \left. - \frac{1}{64} (34\nu\kappa - 45) \right]
\end{aligned}$$

$$\begin{aligned}
\{H_1, S_3\} = (J^3/\nu^3) & \left[\frac{3}{512} (38\nu\kappa - 47) \cos 6\gamma \right. \\
& - \frac{69}{128} (6\nu\kappa - 7) \cos 4\gamma \\
& - \frac{3}{512} (726\nu\kappa - 743) \cos 2\gamma \\
& \left. + \frac{15}{128} (62\nu\kappa - 67) \right]
\end{aligned}$$

Now choose K_4 to cancel the average part of Eq. (ii).

$$\begin{aligned}
 K_4 &= 3 \langle \{H_2, S_2\} \rangle + 3 \langle \{H_1, S_3\} \rangle + 3 \langle K_{22} \rangle \\
 &= - (J^3/\nu^3) \left[\frac{2115}{128} - \frac{675}{32} \nu\kappa + \frac{51}{32} (\nu\kappa)^2 \right]
 \end{aligned}$$

It is left to the reader to finish the calculation for S_4 from the expressions already given.

Putting the pieces together, then, the averaged Hamiltonian to fourth order can be written:

$$\begin{aligned}
 K(J) &= \nu J \left[1 - \frac{1}{2} \left(\frac{15}{16} - \frac{3}{8} \nu\kappa \right) \left(\frac{\epsilon J^{1/2}}{\nu} \right)^2 \right. \\
 &\quad \left. - \frac{1}{24} \left(\frac{2115}{128} - \frac{675}{32} \nu\kappa + \frac{51}{32} (\nu\kappa)^2 \right) \left(\frac{\epsilon J^{1/2}}{\nu} \right)^4 \right. \\
 &\quad \left. + O(\epsilon^6) \right]
 \end{aligned}$$

3.2 Quadrupole field errors.

The application of Deprit's algorithm to this linear problem is going to be cumbersome and obscure in comparison to the elegant treatment of Courant and Snyder, which takes no more than a few lines. This is not surprising: linear problems are solved best by linear methods. Again, our purpose is only to observe the working of an unfamiliar algorithm within a familiar setting.

Because we'll want to compare the results obtained in this section to known, correct answers, let us start a little further back than in the previous section in order to establish the notation. Assume a hamiltonian of the form

$$H(x, p; \varepsilon; s) = \frac{1}{2} (p^2 + G(s)x^2) + \varepsilon \cdot \frac{1}{2} \tilde{g}(s)x^2$$

The functions $G(s)$ and $\tilde{g}(s)$ are equi-periodic. We now (1) make a Floquet transformation using the betatron functions β and $\psi = \nu\phi = \int \frac{ds}{\beta}$ associated with H_0 , and (2) change the independent variable from s to ϕ . The transformation equations are given by

$$x = [2\nu\beta(\phi)]^{1/2} \sin\gamma$$

$$p = [2\nu/\beta(\phi)]^{1/2} \left(\frac{1}{2\nu} \frac{d \ln \beta(\phi)}{d\phi} \sin\gamma + \cos\gamma \right)$$

and the new form of the hamiltonian is

$$H(\gamma, J; \varepsilon; \phi) = \nu J [1 + \varepsilon g(\phi) \beta^2(\phi) \sin^2 \gamma]$$

where $g(\phi(s)) = \tilde{g}(s)$. Because we are dealing with a purely linear system we expect that

$$\begin{aligned} K(J) &= (\nu + \Delta\nu) J \\ &= (\nu + \Delta\nu^{(1)} + \Delta\nu^{(2)} + \dots) J \end{aligned}$$

where $\Delta\nu^{(k)} = O(\varepsilon^k)$. We shall evaluate $\Delta\nu$ through second order.

To ease the notation, define the function

$$F(\phi; \phi_0) \equiv \int_{\phi_0}^{\phi} d\phi' \frac{\nu}{2} g(\phi') \beta^2(\phi')$$

The parametric dependence of F on ϕ_0 will frequently be suppressed. Reexpress the hamiltonian in terms of this function

$$\begin{aligned} H_0 &= \nu J \\ H_1 &= J \frac{dF}{d\phi} (1 - \cos 2\gamma) \\ &\equiv J h_1(\gamma, \phi) \end{aligned}$$

In writing the first order equations, it is convenient to define

$$S_1 \equiv \int s_1(r, \phi) \quad \text{and} \quad K_1 \equiv \int k_1(r, \phi)$$

so that Eq. (1f) becomes

$$\left(\frac{\partial}{\partial \phi} + \nu \frac{\partial}{\partial Y} \right) s_1 = \frac{dF}{d\phi} (1 - \cos 2Y) - k_1$$

We will break this up into two pieces, say $s_1 = s_1^a + s_1^b$, where s_1^a is independent of Y .

The solution for s_1^a is written easily by inspection.

$$\begin{aligned} s_1^a &= F(\phi; \phi_0) - k_1(\phi - \phi_0) \\ &\equiv F_r(\phi; \phi_0) \end{aligned}$$

The constant k_1 must be chosen so that F_r is periodic, and thus bounded. That is, we must have $F_r(\phi_0 + 2\pi; \phi_0) = 0$, and therefore

$$\begin{aligned} 2\pi k_1 &= F(\phi_0 + 2\pi; \phi_0) \\ &= \frac{\nu}{2} \oint d\phi g(\phi) \beta^2(\phi) \end{aligned} \tag{7a}$$

This gives us the first order tune shift arising from perturbations in the quadrupole fields.

$$\Delta \nu^{(1)} = \epsilon k_1 = \frac{\nu}{4\pi} \oint d\phi \epsilon g(\phi) \beta^2(\phi) \tag{7b}$$

The differential equation for s_1^b ,

$$\left(\frac{\partial}{\partial \phi} + \nu \frac{\partial}{\partial \gamma} \right) s_1^b = - \frac{dF}{d\phi} \cos 2\gamma$$

could be solved by integrating along a trajectory and adding an appropriate null space function to make the solution periodic, as was done in the last section. We shall short circuit this process and just write the final, easily verified answer.

$$s_1^b = - \frac{1}{2 \sin 2\pi\nu} \int_{\phi}^{\phi+4\pi} dF(\phi') \sin 2[\gamma + \nu(\phi' - \phi) - \pi\nu]$$

The function is represented as a Lebesgue-Stieltjes integral with pseudo-length $dF(\phi)$. Periodicity is confirmed by using the boundary condition $dF(\phi + 2\pi) = dF(\phi)$.

Because $H_2 = 0$, the second order Eq. (1g) becomes

$$DS_2 \equiv \{H_1 + K_1, S_1\} - K_2$$

As before, we factor out the J dependence explicitly and define

$$S_2 \equiv J s_2(\gamma, \phi) \quad , \quad K_2 \equiv J k_2(\gamma, \phi)$$

Because k_1 is a constant, the second order differential equation gets rewritten as

$$\left(\frac{\partial}{\partial \phi} + \nu \frac{\partial}{\partial \gamma}\right) s_2 = \frac{\partial h_1}{\partial \gamma} s_1 - (h_1 + k_1) \frac{\partial s_1}{\partial \gamma} - k_2$$

$$\equiv \text{rhs}(\gamma, \phi)$$

Consider solving this by the method of integration along orbits. The number k_2 must be chosen to eliminate unbounded growth of s_2 . The troublesome term comes from the combination of circular functions in both h_1 and s_1 .

$$\begin{aligned} & \text{term in } \frac{\partial h_1}{\partial \gamma} s_1 - h_1 \frac{\partial s_1}{\partial \gamma} \\ &= \frac{-1}{2 \sin 2\pi \nu} \frac{dF}{d\phi} \int_{\phi}^{\phi+2\pi} dF(\phi') \left[2 \sin 2\gamma \sin 2(\gamma + \nu(\phi' - \phi) - \pi \nu) \right. \\ & \quad \left. + 2 \cos 2\gamma \cos 2(\gamma + \nu(\phi' - \phi) - \pi \nu) \right] \\ &= - \frac{dF/d\phi}{\sin 2\pi \nu} \int_{\phi}^{\phi+2\pi} dF(\phi') \cos [2\nu(\phi' - \phi - \pi)] \end{aligned}$$

Applying Eq. (4) produces the growth term, s_2^a .

$$s_2^a(\gamma, \phi) = - \frac{1}{\sin 2\pi \nu} \int_{\phi_0}^{\phi} dF(\phi'') \int_{\phi''}^{\phi''+2\pi} dF(\phi') \cos [2\nu(\phi' - \phi'' - \pi)] - k_2(\phi - \phi_0)$$

That this could grow indefinitely is demonstrated easily by noting that the integrand is invariant under the diagonal translation $\phi' \rightarrow \phi' + 2\pi$, $\phi'' \rightarrow \phi'' + 2\pi$. Because of this, $s_2^a(\gamma, \phi_0 + 2\pi n) = n \cdot s_2^a(\gamma, \phi_0 + 2\pi)$. To eliminate the growth we must make s_2^a periodic by choosing

$$2\pi k_2 = -\frac{1}{\sin 2\pi\nu} \int_{\phi''}^{\phi''+2\pi} dF(\phi'') \int_{\phi''}^{\phi'+2\pi} dF(\phi') \cos 2\nu(\phi' - \phi'' - \pi) \quad (8a)$$

We note in passing that

$$\Delta\nu^{(2)} = \frac{1}{2} \varepsilon^2 k_2 \quad (8b)$$

For completeness, we shall write the the full differential equation for s_2 .

$$\begin{aligned} Ds_2 = & -\frac{dF/d\phi}{\sin 2\pi\nu} \int_{\phi}^{\phi+2\pi} dF(\phi') \cos 2\nu(\phi' - \phi - \pi) \\ & + \frac{1}{\sin 2\pi\nu} \left(\frac{dF}{d\phi} + k_1 \right) \int_{\phi}^{\phi+2\pi} dF(\phi') \cos 2[\gamma + \nu(\phi' - \phi - \pi)] \\ & + 2 \frac{dF}{d\phi} F_r(\phi) \sin 2\gamma - k_2 \end{aligned}$$

The solution of this is

$$\begin{aligned} s_2 = & s_2^a + s_2^b + s_2^c + \text{func}(\gamma - \nu\phi) \\ s_2^b = & \frac{1}{\sin 2\pi\nu} \int_{\phi_0}^{\phi} (dF(\phi'') + k_1 d\phi'') \int_{\phi''}^{\phi'+2\pi} dF(\phi') \cos 2[\gamma + \nu(\phi' - \phi - \pi)] \\ s_2^c = & 2 \int_{\phi_0}^{\phi} dF(\phi'') F_r(\phi'') \sin 2(\gamma + \nu(\phi'' - \phi)) \end{aligned}$$

where func is chosen to make s_2 periodic.

We now want to compare our results to those of Courant and Snyder (Eq. 4.37). Written in our notation, their expression for the perturbed tune is given below.

$$\begin{aligned} \cos \mu - \cos \mu_0 &= -\epsilon \sin 2\pi\nu F(2\pi; 0) \\ &+ \epsilon^2 \cdot 2 \int_0^{2\pi} dF(\phi'') \int_{\phi''}^{2\pi} dF(\phi') \sin \nu(\phi' - \phi'') \sin \nu(2\pi - (\phi' - \phi'')) \\ &+ \mathcal{O}(\epsilon^3) \end{aligned}$$

where $2\pi\nu = \mu_0$, and $\mu = 2\pi(\nu) + \Delta\nu^{(1)} + \Delta\nu^{(2)} + \dots$. To make the comparison, we must expand $\cos \mu - \cos \mu_0$ to second order.

$$\begin{aligned} \cos \mu - \cos \mu_0 &= -2\pi \sin 2\pi\nu \cdot \Delta\nu^{(1)} \\ &- 2\pi \sin 2\pi\nu \cdot \Delta\nu^{(2)} - \frac{1}{2} \cos 2\pi\nu \cdot (2\pi \Delta\nu^{(1)})^2 \\ &+ \mathcal{O}(\epsilon^3) \end{aligned}$$

We can already identify the first order term using Eq. (7). The second is given by the following.

$$\begin{aligned} &- 2\pi \sin 2\pi\nu \cdot \Delta\nu^{(2)} - \frac{1}{2} \cos 2\pi\nu \cdot (2\pi \Delta\nu^{(1)})^2 \quad (9) \\ &= \epsilon^2 \int_0^{2\pi} dF(\phi'') \int_{\phi''}^{2\pi} dF(\phi') [\cos 2\nu(\phi' - \phi'' - \pi) - \cos 2\pi\nu] \end{aligned}$$

where use has been made of the sine product formula. It is easily seen that the terms involving $\cos 2\pi v$ on both sides of this equation are identical.

$$\begin{aligned}
 & -\epsilon^2 \cos 2\pi v \int_{\phi''=0}^{2\pi} \int_{\phi'=0}^{2\pi} dF(\phi') dF(\phi'') \\
 & = -\epsilon^2 \cos 2\pi v \cdot \frac{1}{2} \int_{\phi''=0}^{2\pi} \int_{\phi'=0}^{2\pi} dF(\phi') dF(\phi'') \\
 & = -\frac{1}{2} \cos 2\pi v \left[\epsilon F(2\pi; 0) \right]^2 \\
 & = -\frac{1}{2} \cos 2\pi v (2\pi \Delta v^{(1)})^2
 \end{aligned}$$

incorporating this into Eq. (9) yields the result

$$-2\pi \sin 2\pi v \cdot \Delta v^{(2)} = \epsilon^2 \int_{\phi''=0}^{2\pi} \int_{\phi'=0}^{2\pi} dF(\phi'') dF(\phi') \cos 2v(\phi' - \phi'' - \pi) \quad (10)$$

This is equivalent to our previous result provided that the double integral in Eq. (10) is half that in Eq. (8a). That demonstration is easy. First note that

$$\int_{\phi'=0}^{\phi''+2\pi} = \int_{\phi'=0}^{2\pi} + \int_{\phi'=2\pi}^{2\pi+\phi''}$$

Then, all that is needed is the following line of development.

$$\begin{aligned}
& \int_{\phi''=0}^{2\pi} \int_{\phi'=2\pi}^{\phi''+2\pi} dF(\phi'') dF(\phi') \cos 2\nu (\phi' - \phi'' - \pi) \\
&= \int_{\phi''=0}^{2\pi} \int_{\phi'=0}^{\phi''} dF(\phi'') dF(\phi') \cos 2\nu (\phi' - \phi'' + \pi) \\
&= \int_{\phi'=0}^{2\pi} \int_{\phi''=0}^{\phi'} dF(\phi'') dF(\phi') \cos 2\nu (\phi'' - \phi' + \pi) \\
&= \int_{\phi''=0}^{2\pi} \int_{\phi'=0}^{2\pi} dF(\phi'') dF(\phi') \cos 2\nu (\phi' - \phi'' - \pi)
\end{aligned}$$

~~It is seen in some remarks that the formal requirement of a~~
~~bounded transformation is the most essential condition,~~
~~expressing this happens in quantum mechanics when energy~~
~~levels of a system are related by requiring a periodic solution~~
~~of the equation to converge and thus to be finite. This~~
~~could be more precisely the condition that the solution~~
~~converges.~~

3.3 Sextupoles and octupoles with arbitrary harmonics.

We consider now a Hamiltonian with arbitrary sextupole and octupole terms.

$$H = H_0 + \varepsilon H_1 + \frac{1}{2} \varepsilon^2 H_2$$

$$H_0 = \nu J$$

$$\begin{aligned} H_1 &= A(\phi) J^{3/2} \sin^3 \gamma = A(\phi) J^{3/2} \left(\frac{3}{4} \sin \gamma - \frac{1}{4} \sin 3\gamma \right) \\ &\equiv J^{3/2} h_1(\gamma, \phi) \end{aligned}$$

$$\begin{aligned} H_2 &= B(\phi) J^2 \sin^4 \gamma \\ &= B(\phi) J^2 \left(\frac{3}{8} - \frac{1}{2} \cos 2\gamma + \frac{1}{8} \cos 4\gamma \right) \\ &\equiv J^2 h_2(\gamma, \phi) \end{aligned}$$

The transform equations will be solved by expansion in eigenfunctions -- that is, Fourier decomposition.

$$A(\phi) = \sum_m a_m e^{im\phi}, \quad a_{-m} = a_m^*$$

$$B(\phi) = \sum_m b_m e^{im\phi}, \quad b_{-m} = b_m^*$$

$$h_k(\gamma, \phi) = \sum_{m,p} h_{k,mp} e^{i(m\phi + p\gamma)}, \quad k = 1, 2$$

$$h_{k,-m-p} = h_{k,mp}^*$$

The only nonvanishing components of h_1 and h_2 are

$$\begin{aligned} h_{1;m,\pm 1} &= \pm 3a_m/8i & , & & h_{1;m,\pm 3} &= \mp a_m/8i \\ h_{2;m,0} &= 3b_m/8 & , & & h_{2;m,\pm 2} &= -b_m/4 \\ & & & & h_{2;m,\pm 4} &= b_m/16 \end{aligned} \quad (11)$$

The first order equation is

$$z^2(m + \nu p) S_{1;mp} + K_{1;mp} = H_{1;mp}$$

We must choose $K_{1;00} := H_{1;00} = 0$. We shall also assume that no resonance terms are important, so that $m + \nu p$ vanishes only for $m = p = 0$. Therefore, $K_{1;mp} = 0$ for all m, p , or more succinctly

$$K_1 = 0 .$$

If we define $S_{1;mp} \equiv J^{3/2} s_{1;mp}$, then the nonvanishing components are

$$\begin{aligned} s_{1;m,\pm 1} &= \mp \frac{3}{8} \frac{a_m}{m \pm \nu} \\ s_{1;m,\pm 3} &= \pm \frac{1}{8} \frac{a_m}{m \pm 3\nu} \end{aligned}$$

Going to second order, we must consider the equation

$$i(m+\nu p) S_{2;mp} + K_{2;mp} = H_{2;mp} + \{H_1, S_1\}_{;mp}$$

The Poisson bracket is evaluated using Eq. (5).

$$\begin{aligned} \{H_1, S_1\}_{;mp} &= \{J^{3/2} h_1, J^{3/2} s_1\}_{;mp} \\ &= i \frac{3}{2} J^2 \sum_{\langle m,p \rangle} (p' - p'') h_{1;m'p'} s_{1;m''p''} \\ &= \frac{3}{2} J^2 \sum_{\langle m,p \rangle} \frac{p' - p''}{m'' + \nu p''} h_{1;m'p'} h_{1;m''p''} \end{aligned}$$

Again, projecting out the average term gives us K_2 .

$$\begin{aligned} K_{2;00} &= J^2 \left[\frac{3}{8} b_0 - 3 \sum_{m,p} \frac{p}{m + \nu p} |h_{1;mp}|^2 \right] \\ &\equiv J^2 k_{2;00} \end{aligned}$$

Note that we have used $h_{1,-m-p} = h_{1;mp}^*$. A little further evaluation, using Eq's. (11) leads to

$$\begin{aligned} k_{2;00} &= \frac{3}{8} b_0 - \frac{15}{16} \frac{|a_0|^2}{\nu} \\ &\quad + \frac{27\nu}{16} \sum_{m=1}^{\infty} \left(\frac{1}{m^2 - \nu^2} + \frac{1}{m^2 - 9\nu^2} \right) |a_m|^2 \end{aligned}$$

When $b_0 = 0$ this expression agrees with Eq. (5.6) of Cole (1969). One special case is worth mentioning. Suppose that for some $|a|$, all $|a_m| = |a|$. Then the sum can be done using

$$\cot \pi x = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{k=1}^{\infty} \frac{1}{x^2 - k^2}$$

The result is

$$k_{2;cc} = \frac{3}{8} b_0 - \frac{27|a|^2}{16} \left(\frac{\pi}{2} \cot \pi v + \frac{\pi}{6} \cot 3\pi v \right)$$

4. REMARKS.

Chief among the questions associated with any perturbation theory is that of convergence. As the Lie series is calculated to higher order in ϵ , larger harmonics in \mathcal{X} are pulled into the generating function and, as a consequence, into the clothing transformation. If the series converges, then the magnitudes of these terms should decrease rapidly with increasing order so that, for example, a Fourier spectrum of the orbit would show only a few dominant frequencies. In general, however, the interval of convergence in ϵ or J will be finite. Near its boundary all the higher order terms become important and the spectrum would be broadband. This is precisely the behavior one has come to expect in approaching chaotic regions of phase space. It is tempting to speculate that divergence of the Lie series and passage through the last KAM boundary coincide. Unfortunately, the situation is more complicated. Dewar (1978) has constructed examples in which the Lie transform is well defined and leads to integrable orbits past the convergence interval of its power series expansion.

A second issue, not unrelated to the first, is the ambiguity inherent to setting up a problem with several important nonlinear terms, within a single parameter perturbation theory. The naive, "natural" ordering blindly associates higher powers of J with higher powers of ϵ , usually in the combination $\epsilon J^{1/2}$. This is not necessarily the best for all problems. There may be instances in which

it would lead to a divergent Lie series whereas restructuring the problem --- changing the association of multipoles with powers of ϵ ; perhaps even combining several multipoles into a single power of ϵ --- would produce a convergent series. One ultimately would like to get around the problem by generalizing to a multi-parameter Lie transformation.

At any rate, setting up problems correctly and interpreting the results will, as always, require judgment; these tasks cannot be absorbed into the formalism. The perturbation series associated with most nonlinear problems in fact almost always diverges, and yet their finite partial sums generally provide useful (in the eyes of the beholder) information. (Perhaps we should think of the Lie series as "asymptotic" in some sense?)

The power of the Deprit-Hori-Kamel algorithm lies in its ability to be automated easily. The examples done for this memo do not sufficiently demonstrate its usefulness. We plan to exercise the method more vigorously and to write programs that apply it to more realistic problems in four dimensional phase space.

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