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MEASUREMENT OF BETATRON TUNE AND TUNESPREAD BY SWEEPING A VOLTAGE OF LINEARLY-VARYING FREQUENCY

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I. Introduction

Super-fast transverse dampers[1] are to be installed in the Energy-Doubler to damp out any large-amplitude betatron oscillations due to injection mismatch. In this note, we attempt to investigate the possibility of using the same dampers to measure the betatron tunes as well as the corresponding tunespreads by applying across the kickers of the dampers an alternating voltage whose frequency increases linearly.

II. The Exact Derivation

We are going to work in the Floquet space and consider only the horizontal motion of the beam. The equation of motion of a beam particle is therefore

$$\frac{d^2\eta}{d\phi^2} + Q_0^2\eta = Q_0^2\beta^{3/2}F(\phi), \tag{1}$$

where η is the Floquet displacement of the particle and ϕ its Floquet phase along the Doubler ring. They are related to the actual horizontal displacement x and longitudinal position s along the Doubler ring by

$$\begin{aligned} \eta &= \beta^{-\frac{1}{2}}x, \\ \phi &= \int^s \frac{ds'}{Q_0\beta}, \end{aligned} \tag{2}$$

with Q_0 the horizontal betatron tune and β the beta-function at the point concerned. The external force $F(\phi)$ acting on the beam particle of charge e and the total energy E (or velocity v) is supplied by the kicker. We have

$$F(\phi) = \frac{ec^2 V_0 \Delta}{Ev^2 d} \sum_{n=0}^{\infty} \delta(\phi - \phi_k - 2\pi n)\, e^{i(\omega t + \delta_k)}, \tag{3}$$

where ω is the angular frequency of the voltage applied and

$$\Delta = \ell/\beta_k Q_0$$

is the Floquet phase advance across the kicker which is of length ℓ and plate separation d and is situated at the phase ϕ_k where the beta-function is β_k. In Eq. (4) we have made the assumption that Δ is small so that β_k is roughly constant over the length of the kicker. Because of this, the action of the kicker is represented by δ-functions in Eq.(3). As a result, the voltage applied to the kicker can be written as

$$V = V_0\, e^{i(2\pi n Q_k + \delta_k)}, \tag{4}$$

where δ_k is the phase of the voltage when the particle passes through the kicker the first time (n = 0) and Q_k, related to the frequency ω, plays the role of a "kicker tune" in the case that the frequency of the applied voltage is constant. (The kicker tune is different from Q_k when the frequency changes as explained below.)

Using Eqs. (3) and (4), the equation of motion becomes

$$\frac{d^2\eta}{d\phi^2} + Q_0^2 \eta = Q_0^2 g \Delta \sum_{n=0}^{\infty} \delta(\phi - \phi_k - 2\pi n)\, e^{i(2\pi n Q_k + \delta_k)}, \tag{5}$$

where $g = ec^2 V_0 \beta_k^{3/2}/(Ev^2 d)$. This equation can be solved exactly; for completeness the solution is given in Appendix I. If the particle starts off with $\eta = 0$ and $\eta' \equiv d\eta/d\phi = 0$, after passing through the kicker N times, at a point of observation with Floquet phase ϕ_{obs}, the displacement and its gradient (the particular solution) are

$$\bar{\eta}_N = \sum_{n=0}^{N-1} \sin\left[\delta_k + 2\pi n(Q_k - Q_o) + Q_o(2\pi N + \phi_{obs} - \phi_k)\right]$$
$$- \sum_{n=0}^{N-1} \sin\left[\delta_k + 2\pi n(Q_k + Q_o) - Q_o(2\pi N + \phi_{obs} - \phi_k)\right], \tag{6}$$

$$\bar{\eta}'_N = \sum_{n=0}^{N-1} \cos\left[\delta_k + 2\pi n(Q_k - Q_o) + Q_o(2\pi N + \phi_{obs} - \phi_k)\right]$$
$$+ \sum_{n=0}^{N-1} \cos\left[\delta_k + 2\pi n(Q_k + Q_o) - Q_o(2\pi N + \phi_{obs} - \phi_k)\right]. \tag{7}$$

In above the displacement and its gradient have been normalized to

$$\bar{\eta} = 2\eta/(g\Delta Q_o),$$
$$\bar{\eta}' = 2\eta'/(g\Delta Q_o^2), \tag{8}$$

so that they are dimensionless, depend only on the phase and tune of the kicker and are independent of the kicker's dimensions, the magnitude of the applied voltage and the energy of the particle. Equations (6) and (7) can be further simplified by choosing the point of observation at the entrance of the kicker; i.e., $\phi_{obs} = \phi_o$.

If the frequency of the voltage applied on the kicker is constant in time, (i.e., Q_k is n-independent), the series in Eqs. (6) and (7) can be summed exactly to give

$$\bar{\eta}_N = \sin\left[\delta_k + 2\pi N Q_o + (N-1)\pi(Q_k - Q_o)\right] \sin\pi N(Q_k - Q_o)/\sin\pi(Q_k - Q_o)$$
$$- \sin\left[\delta_k - 2\pi N Q_o + (N-1)\pi(Q_k + Q_o)\right] \sin\pi N(Q_k + Q_o)/\sin\pi(Q_k + Q_o), \tag{9}$$

$$\bar{\eta}'_N = \cos\left[\delta_k + 2\pi N Q_o + (N-1)\pi(Q_k - Q_o)\right] \sin\pi N(Q_k - Q_o)/\sin\pi(Q_k - Q_o)$$
$$+ \cos\left[\delta_k - 2\pi N Q_o + (N-1)\pi(Q_k + Q_o)\right] \sin\pi N(Q_k + Q_o)/\sin\pi(Q_k + Q_o). \tag{10}$$

It is clear that there will be resonances at

$$Q_k = m \pm Q_o , \qquad m = 0, \pm 1, \pm 2, \ldots, \tag{11}$$

or whenever the kicker's tune matches the betatron tune (modulus 1).

In our case the frequency of the voltage across the kicker plates changes linearly in time. Such a variation can be represented by writing

$$Q_k = Q_{ko} + n\bar{Q}_k \tag{12}$$

where $\bar{Q}_k$ is the increment of the voltage frequency per revolution and Q_{ko} is the frequency at the beginning. The instantaneous tune of the kicker or applied voltage is defined as the change in the kicker's phase per revolution:

$$\frac{d(nQ_k)}{dn} = Q_{ko} + 2n\bar{Q}_k \tag{13}$$

Again resonances occur whenever the kicker instantaneous tune matches the betatron tune (modulus 1). Since the frequency is changing, we expect the resonances to occur at some n_Rth revolution where n_R is given by

$$Q_{ko} + 2n_R\bar{Q}_k = m \pm Q_o , \tag{14}$$

or

$$n_R = (m \pm Q_o - Q_{ko})/2\bar{Q}_k , \tag{15}$$

with m equal to any integer. In Eqs. (14) and (15) and below, Q_o and Q_{ko} denote their noninteger parts only. Figure 1 shows the shape of the envelope of the lateral displacement near one resonance. It will be nice if we know the positions and the heights of the extrema because this will give us information of the position of the resonance and also the betatron tune. In the next section, we shall attempt to compute these positions and heights.

III. Approximation and a universal formula

Let us now concentrate on one particular resonance only; we choose the one at the n_Rth turn with

$$n_R = (m + Q_0 - Q_{k0})/2\bar{Q}_k . \tag{16}$$

As a result, only one series from each of Eqs. (6) and (7) is needed. They become

$$\bar{\eta}_N = \sum_{n=0}^{N-1} \sin\left[\delta_k + 2\pi N Q_0 - 2\pi n_R^2 \bar{Q}_k + 2\pi (n-n_R)^2 \bar{Q}_k\right], \tag{17}$$

$$\bar{\eta}_N' = \sum_{n=0}^{N-1} \cos\left[\delta_k + 2\pi N Q_0 - 2\pi n_R^2 \bar{Q}_k + 2\pi (n-n_R)^2 \bar{Q}_k\right]. \tag{18}$$

The envelope of the normalized displacement

$$R_N = \left(\bar{\eta}_N^2 + \bar{\eta}_N'^2\right)^{\frac{1}{2}} \tag{19}$$

after N passages through the kicker is independent of δ_k according to Eqs. (17) and (18). Thus if only the envelope is required, we can choose for simplicity

$$\delta_k + 2\pi N Q_0 - 2\pi n_R^2 \bar{Q}_k = 0 \tag{20}$$

and reduce Eqs. (17) and (18) to

$$\tilde{\eta}_N = \sum_{n=0}^{N-1} \sin 2\pi (n-n_R)^2 \bar{Q}_k , \tag{21}$$

$$\tilde{\eta}_N' = \sum_{n=0}^{N-1} \cos 2\pi (n-n_R)^2 \bar{Q}_k , \tag{22}$$

where the "tildes" on $\tilde{\eta}_N$ and $\tilde{\eta}_N'$ imply that they are no longer the physical displacement and displacement gradient since the initial kicker phase δ_K, which should be a constant, has been chosen differently for different turn number. However, the envelope R_N is still a physical quantity since it is δ_K-independent under the approximation that we can neglect the other series.

We next approximate the series by Fresnel integrals

$$S(x) = \int_0^x \sin \tfrac{1}{2}\pi t^2 dt \,, \qquad C(x) = \int_0^x \cos \tfrac{1}{2}\pi t^2 dt \,, \tag{23}$$

and Eqs. (21) and (22) become

$$\tilde{\eta}_N = \left\{ S[2\bar{Q}_k^{\frac{1}{2}}(n_R + \tfrac{1}{2})] + S[2\bar{Q}_k^{\frac{1}{2}}(N - n_R - \tfrac{1}{2})] \right\} / 2\bar{Q}_k^{\frac{1}{2}} \,, \tag{24}$$

$$\tilde{\eta}'_N = \left\{ C[2\bar{Q}_k^{\frac{1}{2}}(n_R + \tfrac{1}{2})] + C[2\bar{Q}_k^{\frac{1}{2}}(N - n_R - \tfrac{1}{2})] \right\} / 2\bar{Q}_k^{\frac{1}{2}} . \tag{25}$$

The accuracy of the approximation is measured by the smallness of $2\bar{Q}_k^{\frac{1}{2}}$ which is the size of a step and is made continuous in the integrals. In our case, $\bar{Q}_k = 0.0001$ will be used so that the approximation is very good.

Equations (24) and (25) are plotted in Figure 2 with $\bar{Q}_k = 0.0001$ and $n_R = 100$. The envelope of the displacement R_N for turn number N is represented by the distance from the origin O to the point N on the Fresnel spiral according to Eq. (19). We see that the envelope oscillates at the beginning and increases very sharply near the resonance $n_R = 100$. It increases to a maximum near n = 160 and then oscillates with decaying amplitudes and finally settle down at a magnitude of $2\bar{Q}_k^{\frac{1}{2}} \times OA$. This variation of R_N is exactly that depicted in Figure 1.

When $2\bar{Q}_k^{\frac{1}{2}} n_R \gg 1$, Eqs. (24) and (25) can be further simplified to

$$2\bar{Q}_k^{\frac{1}{2}} \tilde{\eta}_N = \tfrac{1}{2} + S[2\bar{Q}_k^{\frac{1}{2}}(N - n_R - \tfrac{1}{2})] \,, \tag{26}$$

$$2\bar{Q}_k^{\frac{1}{2}} \tilde{\eta}'_N = \tfrac{1}{2} + C[2\bar{Q}_k^{\frac{1}{2}}(N - n_R - \tfrac{1}{2})] . \tag{27}$$

The maximum fractional error transmitted to the resulting R_N is $(2^{3/2}\pi\bar{Q}_k^{1/2}n_R)^{-1}$. In our example, $2\bar{Q}_k^{1/2}n_R = 2$; the maximum error is therefore ~11% which will be much smaller for the second resonance discussed in the following section. The errors of the positions of the extrema relative to n_R are much less. With this approximation, the envelope scales with $2\bar{Q}_k^{\frac{1}{2}}$. In other words, if we define a function

$$y = \left\{ \left[\tfrac{1}{2} + S(x) \right]^2 + \left[\tfrac{1}{2} + C(x) \right]^2 \right\}^{1/2} \tag{28}$$

and solve for the ith extremum y_i, at position x_i (for $x > 0$), the corresponding ith extremum of the actual displacement envelope is given by

$$R_i = y_i / 2\bar{Q}_k^{\frac{1}{2}} \qquad \text{at} \qquad N_i - n_R - \tfrac{1}{2} = x_i / 2\bar{Q}_k^{\frac{1}{2}}. \tag{29}$$

Knowing the exact positions of the maximum and minima, n_R can be determined very accurately, providing that $\bar{Q}_k$, the kicker-tune increment, is known accurately. Further if Q_{ko}, the initial kicker's tune, is known the betatron tune Q_o can be deduced using Eq. (16). Because the first resonance may also be given by

$$n_R = (m - Q_o - Q_{ko}) / 2\bar{Q}_k \ , \tag{30}$$

using this method, we may also get a result of $1-Q_o$, although the true betatron tune is Q_o. The first few extrema of Eq. (28) are tabulated in Table I.

It should be pointed out that Eqs. (21) and (22) contain many resonances corresponding to various values of m in Eq. (16). The approximation by Fresnel integrals leave us with a single resonance only.

IV Many particles with a tunespread Q_s

Again, we consider here only the resonance at n_R. We can also choose δ_k given by Eq. (20), where Q_0 is now the main betatron tune of the particles. The envelope of the average displacement is

$$\langle R_N \rangle = \left(\langle \tilde{\eta}_N \rangle^2 + \langle \tilde{\eta}_N' \rangle^2 \right)^{\frac{1}{2}}. \tag{31}$$

Here,

$$\langle \tilde{\eta}_N \rangle = \sum_{n=0}^{N-1} \int d(\Delta Q) f(\Delta Q) \sin\left(2\pi \tilde{n}^2 \bar{Q}_K + 2\pi \bar{N} \Delta Q\right), \tag{32}$$

$$\langle \tilde{\eta}_N' \rangle = \sum_{n=0}^{N-1} \int d(\Delta Q) f(\Delta Q) \cos\left(2\pi \tilde{n}^2 \bar{Q}_K + 2\pi \bar{N} \Delta Q\right), \tag{33}$$

where $\bar{n} = n - n_R$, $\bar{N} = N - n_R$ and $f(\Delta Q)$ is the tune distribution of the particles. If $f(\Delta Q)$ is symmetric, Eqs. (32) and (33) becomes

$$\begin{pmatrix} \langle \tilde{\eta}_N \rangle \\ \langle \tilde{\eta}_N' \rangle \end{pmatrix} = \langle \cos 2\pi \bar{N} \Delta Q \rangle \begin{pmatrix} \tilde{\eta}_N \\ \tilde{\eta}_N' \end{pmatrix}, \tag{34}$$

where $\tilde{\eta}_N$ and $\tilde{\eta}_N'$ are just the single-particle displacement and displacement gradient given by Eqs. (21) and (22) or Eqs. (24) and (25) or Eqs. (26) and (27), while

$$\langle \cos 2\pi \bar{N} \Delta Q \rangle = \int d(\Delta Q) f(\Delta Q) \cos 2\pi \bar{N} \Delta Q. \tag{35}$$

Thus the effect of a tunespread is completely contained in a factor that is the cosine transform of the betatron-tune distribution function $f(\Delta Q)$. In Table II, we list $\langle \cos 2\pi N \Delta Q \rangle$ for some tune distributions. We see that this factor decreases with $\bar{N}$ in all cases as it should be due to the tunespread.

We can compare the peaks and valleys of this displacement envelope with the corresponding standard one ($\Delta Q = 0$) in Table I to obtain the suppression factor $\langle\cos 2\pi \bar{N}\Delta Q\rangle$ and then derive the tune distribution $f(\Delta Q)$ by a cosine transform. As is well-known, in order to know $f(\Delta Q)$ exactly, we need $\langle\cos 2\pi\bar{N}\Delta Q\rangle$ for all $\bar{N}$. However, if only the root-mean-squared tunespread defined by

$$(\Delta Q_{RMS})^2 = \frac{1}{2\pi^2}\,\frac{\partial}{\partial \bar{N}^2}\langle \cos 2\pi \bar{N}\Delta Q\rangle \Big|_{\bar{N}=0} \tag{36}$$

is required, we only need to compare the first few peaks and valleys after the resonance.

V Adjacent resonances

When adjacent resonances are included, we need to consider both series in Eq. (6) or Eq. (7). Suppose the first two resonances occur at the n_{R+}th and n_{R-}th turns after turning on the kicker, where the plus and minus signs correspond to the plus and minus signs in Eq. (15). We want to point out that sometimes the n_{R-}-resonance will occur before the n_{R+}-resonance, (see Appendix II). After including a tunespread, Eqs. (6) and (7) become

$$\begin{pmatrix} \langle\bar{\eta}_N\rangle \\ \langle\bar{\eta}'_N\rangle \end{pmatrix} = \langle \cos 2\pi \bar{N}_+\Delta Q\rangle \begin{pmatrix} \cos\Phi_+ & \sin\Phi_+ \\ -\sin\Phi_+ & \cos\Phi_+ \end{pmatrix} \begin{pmatrix} \tilde{\eta}_{N_+} \\ \tilde{\eta}'_{N_+} \end{pmatrix} \tag{37}$$

$$- \langle \cos 2\pi \bar{N}_-\Delta Q\rangle \begin{pmatrix} \cos\Phi_- & \sin\Phi_- \\ \sin\Phi_- & -\cos\Phi_- \end{pmatrix} \begin{pmatrix} \tilde{\eta}_{N_-} \\ \tilde{\eta}'_{N_-} \end{pmatrix}, \tag{38}$$

where

$$\Phi_{\pm} = \delta_k \pm 2\pi N Q_0 - 2\pi n_{R\pm}^2 ,$$

and $\bar{N}_{\pm} = N - n_{R\pm}$. Similar to Eqs. (21) and (22), $\tilde{\eta}_{N\pm}$ and $\tilde{\eta}'_{N\pm}$ are given by

$$\tilde{\eta}_{N\pm} = \sum_{n=0}^{N-1} \sin 2\pi (n-n_{R\pm})^2 \bar{Q}_k \quad \text{and} \quad \tilde{\eta}'_{N\pm} = \sum_{n=0}^{N-1} \cos 2\pi (n-n_{R\pm})^2 \bar{Q}_k . \tag{39}$$

Up to now Eqs. (37) and (38) are exact with the only assumption that the tune distribution $f(\Delta Q)$ is symmetric. If we are interested in the first two resonances only (at the $n_{R\pm}$th turn), $\tilde{\eta}_{N\pm}$ and $\tilde{\eta}'_{N\pm}$ can again be approximated by Fresnel integrals

$$2\bar{Q}_k^{\frac{1}{2}} \tilde{\eta}_{N\pm} = \frac{1}{2} + S\left[2\bar{Q}_k^{\frac{1}{2}}(N - n_{R\pm} - \tfrac{1}{2})\right] , \tag{40}$$

$$2\bar{Q}_k^{\frac{1}{2}} \tilde{\eta}'_{N\pm} = \frac{1}{2} + C\left[2\bar{Q}_k^{\frac{1}{2}}(N - n_{R\pm} - \tfrac{1}{2})\right] , \tag{41}$$

where the assumption that $2\bar{Q}_k^{\frac{1}{2}} n_{R\pm}$ are big has been made. As written in Eqs. (6) and (7), $\bar{\eta}_N$ is the <u>difference</u> of two series but $\bar{\eta}'_N$ is the <u>sum</u> of two series. As a result, the resulting envelope is no longer independent of the initial kicker's phase δ_k. However, when the tunespread is big enough, $\sigma_{RMS} > 1/|n_{R+}-n_{R-}|$, the displacement decays back to nearly zero before reaching the second resonance. Thus, the two resonance are decoupled. In this case, the envelope will again be independent of δ_k and the peaks and valleys for each resonance will be identical. Otherwise the two resonances will mix and the values of the peaks and valleys will be modified a bit. However, their positions are modified by very little. As will be shown in the next section, we can damp the lateral displacement of the bunch to zero between two resonances. As a result these resonances can always be made to decouple.

The corresponding peaks and valleys of the resonances are always separated by $\Delta n = |n_{R+} - n_{R-}|$. By measuring many of these separations, Δn can be known very accurately and so is the betatron tune which is equal to one of the four following values: (see Appendix II)

$$Q_0 = \begin{cases} \bar{Q}_k \Delta n \\ 1 - \bar{Q}_k \Delta n \\ \frac{1}{2} \pm \bar{Q}_k \Delta n \end{cases} . \qquad (42)$$

If we know the initial kicker-voltage tune Q_{ko}, as is shown at the end of Section III, these four possibilities of Q_0 can be reduced to two.

The main errors in determining Q_0 come from (1) error in $\bar{Q}_k$ and (2) error in reading off the turn numbers of the peaks and valleys since what we are seeing actually are the displacements but not the envelope. We therefore have

$$|\delta Q_0| = \bar{Q}_k (\delta \Delta n) + (\delta \bar{Q}_k) \Delta n , \qquad (43)$$

where Δn is of the order of $(4\bar{Q}_k)^{-1}$. For $\bar{Q}_k = 0.0001$, the first peak and the first valley are separated by ~33 turns (see Table I). Thus $\delta\Delta N$ should be less than 5 say, which will be much less if much more measurements between the corresponding peaks and valleys of the two resonances are made. Thus we get

$$|\delta Q_0| \lesssim 0.0005 + \delta \bar{Q}_k / 4\bar{Q}_k ,$$

which indicates that the main error lies in the accuracy of $\bar{Q}_k$. If we wish to measure Q_0 up to 0.001, we need $\delta\bar{Q}_k/\bar{Q}_k < 0.004$.

VI Remarks

In order to exhibit a resonance, the kicker has to kick the beam laterally by an amount

$$X = \frac{1}{2} g \Delta \beta_{obs}^{\frac{1}{2}} \eta_{max}$$

where $\eta_{max} = (1.65)/2\bar{Q}_k^{\frac{1}{2}}$ (see Table I) is the maximum Floquet displacement for a resonance. For a kicker of length ℓ = 120 cm, gap d = 5 cm, voltage applied V_0 = 0.5 k, beam particles of energy E = 150 GeV, betatron tune Q_0 = 19.38, and beta-functions at kicker and observation point $\beta_k = \beta_{obs} = 10^4$ cm, we get maximum displacement X = 6.4 mm. This displacement is not small when compared with the beam radius which is ~3 mm. If we allow the center of the beam to return back to the center of the beampipe due to the tune-spread suppression factor of Eq. (35), the radius of the beam will be increased to ~9.4 mm. Thus, by kicking the beam several times in order to see several resonances, the beam size will be increased by very much or the bunch area diluted. This will eventually lead to the loss of the bunch. In order to avoid this, we suggest the following operation procedure:

(1) We would like the kicker to kick one particular bunch only. This is in fact the way the superfast dampers[1] are designed. For a bunch area of 0.3 eV and RF voltage 2.16 MV, a 150 GeV bunch has a r.m.s. length of ~0.6 ns and is separated from the next bunch by ~19 ns. Thus a kicker that is on for ~5 ns and has a risetime of ~5 ns will do the job well.

(2) After monitoring the first peak after a resonance, the kicker is allowed to continue its operation for ~250 turns bringing us information of ~20 peaks and valleys which are more than enough for the determination of the betatron tune and tunespread. Then, the super-fast damper is turned on for ~500 turns. This will be enough to push the bunch back to the center of the beampipe. Since the turn number is small, essentially there

will not be any dilution of the bunch area due to tunespread which may have a typical value of 0.0005. After that, we can wait for the second resonance. In this way, not only bunch-area dilution can be avoided; at the same time we can make adjacent resonances decoupled so that the derivation of Section III can be applied.

Appendix I

We are going to solve for the particular solution of Eq. (3); i.e., the initial condition of $\eta = 0$ and $\eta' = 0$ at $\phi<0$ is assumed.

In general, η may grow with ϕ; for example, when the kicker tune Q_k is very close to the betatron tune Q_o. We assume that there exists a positive number a such that

$$\begin{aligned} &\eta(\phi)e^{-a\phi} \longrightarrow 0 \,, \\ &\eta'(\phi)e^{-a\phi} \longrightarrow 0 \,, \qquad \text{as } \phi \to \infty \end{aligned} \tag{A.1}$$

Then $\eta(\phi)e^{-a\phi}$ can be Fourier expanded as

$$\eta(\phi)e^{-a\phi} = \int_{-\infty}^{\infty} \hat{\eta}(Q)\, e^{-iQ\phi} dQ \tag{A.2}$$

or

$$\hat{\eta}(Q) = \frac{1}{2\pi}\int_0^{\infty} \eta(\phi)\, e^{i(Q+ia)\phi} d\phi \,. \tag{A.3}$$

The equation of motion (3), after Fourier transformed becomes

$$\left[Q_o^2-(Q+ia)^2\right]\hat{\eta}(Q) = \frac{Q_o^2 g\Delta}{2\pi}\sum_{n=0}^{\infty} e^{i(2\pi nQ_k+\delta_k)}\, e^{i(Q+ia)(\phi_k+2\pi n)} \tag{A.4}$$

Making use of Eqs. (A.2) and (A.4), we obtain

$$\eta(\phi) = \frac{Q_o^2 g\Delta}{2\pi}\sum_{n=0}^{\infty} e^{i(2\pi nQ_k+\delta_k)} \int_{-\infty+ia}^{\infty+ia} dQ\, e^{iQ(\phi_k+2\pi n-\phi)}/(Q_o^2-Q^2) \,, \tag{A.5}$$

where the path of integration runs horizontally above all the poles in the complex Q-plane. Between the Nth passage and N+1th passage of the kicker, i.e., $\phi_k+2\pi N>\phi>\phi_k+2\pi(N-1)$ we close the path of integration by going around the upper hemisphere when n>N (which gives zero because of casuality) and around the lower hemisphere when n<N. The result is

$$\eta(\phi) = \frac{1}{2} i Q_0 g \Delta \sum_{n=0}^{N-1} e^{i\delta_k} \Big[e^{i2\pi n(Q_k+Q_0) - iQ_0(\phi-\phi_0)} - e^{i2\pi n(Q_k-Q_0) + iQ_0(\phi-\phi_0)} \Big]$$

for $\phi_k+2\pi N>\phi>\phi_k+2\pi(N-1)$. The summation above sums up the contribution of each passage through the kicker; $n=\bar{n}$ corresponds to the $\bar{n}$+1th passage.

Appendix II

The turn numbers n_R at which resonances occur are given by Eq. (15) which is

$$n_R = (m \pm Q_0 - Q_{k0})/2\bar{Q}_k, \qquad m = 0, \pm 1, \pm 2, \cdots,$$

Case I $Q_o > Q_{ko}$ (mod. 1)

For the "+" series, first resonance is at $(Q_o - Q_{ko})/2\bar{Q}_k$. For the "-" series, first resonance is at

$$(1-Q_0-Q_{k0})/2\bar{Q}_k \qquad \text{if} \quad 1-Q_0-Q_{k0}>0,$$

$$(2-Q_0-Q_{k0})/2\bar{Q}_k \qquad \text{if} \quad 1-Q_0-Q_{k0}<0.$$

Thus, when $1-Q_o-Q_{ko}<0$, first resonance is at $(Q_o-Q_{ko})/2\bar{Q}_k$ and when $1-Q_o-Q_{ko}>0$, it is at

$$(Q_0-Q_{k0})/2\bar{Q}_k \qquad \text{if} \quad Q_0<\tfrac{1}{2},$$

$$(1-Q_0-Q_{k0})/2\bar{Q}_k \qquad \text{if} \quad Q_0>\tfrac{1}{2}.$$

Case II $Q_o < Q_{ko}$ (mod. 1)

For the "+" series, first resonance is at $(1+Q_o-Q_{ko})/2\bar{Q}_k$. For the "-" series, the first resonance is at

$$(1-Q_0-Q_{k0})/2\bar{Q}_k \qquad \text{if} \quad 1-Q_0-Q_{k0}>0,$$

$$(2-Q_0-Q_{k0})/2\bar{Q}_k \qquad \text{if} \quad 1-Q_0-Q_{k0}<0.$$

Thus, when $1-Q_o-Q_{ko}>0$, the first resonance is at $(1-Q_o-Q_{ko})/2\bar{Q}_k$, and when $1-Q_o-Q_{ko}<0$, it is at

$$(1+Q_0-Q_{k0})/2\bar{Q}_k \qquad \text{if} \quad Q_0<\tfrac{1}{2},$$

$$(2-Q_0-Q_{k0})/2\bar{Q}_k \qquad \text{if} \quad Q_0>\tfrac{1}{2}.$$

The above is summarized in Table III. We see that there are four different entires in the last column. This leads to the four possible solutions of Q_0 in Eq. (42) after measuring $|n_{R+}-n_{R-}|$.

Reference

[1] C. Moore and R. Rice, Fermilab UPC

EXT. NO	X	Y	TURN NO	ENVELOPE
1	1.21720	1.65556	61.35990	82.77806
2	1.87252	1.24760	94.12595	62.37989
3	2.34449	1.54872	117.72426	77.43612
4	2.73901	1.29858	137.45040	64.92924
5	3.08196	1.51716	154.59791	75.85778
6	3.39134	1.32056	170.06677	66.02809
7	3.67411	1.50071	184.20552	75.03532
8	3.93710	1.33346	197.35491	66.67325
9	4.18323	1.49023	209.66130	74.51154
10	4.41594	1.34219	221.29701	67.10944
11	4.63676	1.48282	232.33800	74.14094
12	4.84772	1.34859	242.88605	67.42943
13	5.04972	1.47722	252.98587	73.86104
14	5.24407	1.35354	262.70371	67.67701
15	5.43136	1.47280	272.06820	73.64004
16	5.61251	1.35752	281.12545	67.87591
17	5.78790	1.46920	289.89491	73.45983
18	5.95821	1.36080	298.41028	68.04024
19	6.12371	1.46618	306.68541	73.30924
20	6.28492	1.36358	314.74586	68.17897
21	6.44204	1.46362	322.60180	73.18096
22	6.59546	1.36596	330.27325	68.29811
23	6.74536	1.46140	337.76789	73.06998
24	6.89203	1.36804	345.10172	68.40190
25	7.03561	1.45945	352.28072	72.97272
26	7.17636	1.36987	359.31793	68.49335
27	7.31436	1.45773	366.21807	72.88658
28	7.44984	1.37149	372.99198	68.57474
29	7.58287	1.45619	379.64343	72.80959
30	7.71363	1.37296	386.18154	68.64778
31	7.84219	1.45480	392.60938	72.74024
32	7.96869	1.37428	398.93472	68.71382
33	8.09320	1.45355	405.16008	72.67734
34	8.21584	1.37548	411.29217	68.77390
35	8.33666	1.45240	417.33306	72.61994
36	8.45577	1.37658	423.28858	68.82887
37	8.57321	1.45135	429.16049	72.56731
38	8.68908	1.37759	434.95388	68.87941
39	8.80340	1.45038	440.67023	72.51880
40	8.91628	1.37852	446.31404	68.92609

Table I. X's and Y's are the standardized positions and heights of the extrema after a resonace. In the third column, $X/2\bar{Q}_k^{\frac{1}{2}}$ are the actual positions of the extrema after the resonance and in the fourth column, $Y/2\bar{Q}_k^{\frac{1}{2}}$ are the actual heights of the normalized evelope at the extrema.

$f(x)$	$\langle \cos 2\pi \bar{N} x \rangle$
$\frac{1}{2a}\left[\delta(x+a) - \delta(x-a)\right]$	$\frac{\sin 2\pi \bar{N} a}{2\pi \bar{N} a}$
$\frac{1}{\sqrt{2\pi}\,a} e^{-x^2/2a^2}$	$\frac{1}{\sqrt{2\pi}\,a} e^{-2(\pi \bar{N} a)^2}$
$\frac{a}{\pi} \cdot \frac{1}{x^2+a^2}$	$e^{-2\pi a \bar{N}}$
$\frac{\pi}{a} \cos 2\pi \frac{x}{a} \quad \lvert x \rvert < a$	$\frac{\cos \frac{1}{2}\pi a \bar{N}}{1-(a\bar{N})^2}$

Table II

Conditions	First resonance n_{R+} or n_{R-}	Second resonance n_{R+} or n_{R-}	Turn number between the two resonances $\lvert n_{R+} - n_{R-} \rvert$
$Q_o > Q_{ko}$ $1-Q_o-Q_{ko} > 0$ $Q_o < \frac{1}{2}$	$(Q_o-Q_{ko})/2\bar{Q}_k$	$(1-Q_o-Q_{ko})/2\bar{Q}_k$	$(1-2Q_o)/2\bar{Q}_k$
$Q_o > Q_{ko}$ $1-Q_o-Q_{ko} < 0$	$(Q_o-Q_{ko})/2\bar{Q}_k$	$(2-Q_o-Q_{ko})/2\bar{Q}_k$	$(1+Q_o-Q_{ko})/2\bar{Q}_k$
$Q_o > Q_{ko}$ $1-Q_o-Q_{ko} > 0$ $Q_o > \frac{1}{2}$	$(1-Q_o-Q_{ko})/2\bar{Q}_k$	$(Q_o-Q_{ko})/2\bar{Q}_k$	$(2Q_o-1)/2\bar{Q}_k$
$Q_o < Q_{ko}$ $1-Q_o-Q_{ko} > 0$	$(1-Q_o-Q_{ko})/2\bar{Q}_k$	$(1+Q_o-Q_{ko})/2\bar{Q}_k$	$Q_o/\bar{Q}_k$
$Q_o < Q_{ko}$ $1-Q_o-Q_{ko} < 0$ $Q_o < \frac{1}{2}$	$(1+Q_o-Q_{ko})/2\bar{Q}_k$	$(2-Q_o-Q_{ko})/2\bar{Q}_k$	$(1-2Q_o)/2\bar{Q}_k$
$Q_o < Q_{ko}$ $1-Q_o-Q_{ko} < 0$ $Q_o > \frac{1}{2}$	$(2-Q_o-Q_{ko})/2\bar{Q}_k$	$(1+Q_o-Q_{ko})/2\bar{Q}_k$	$(2Q_o-1)/2\bar{Q}_k$

Table III

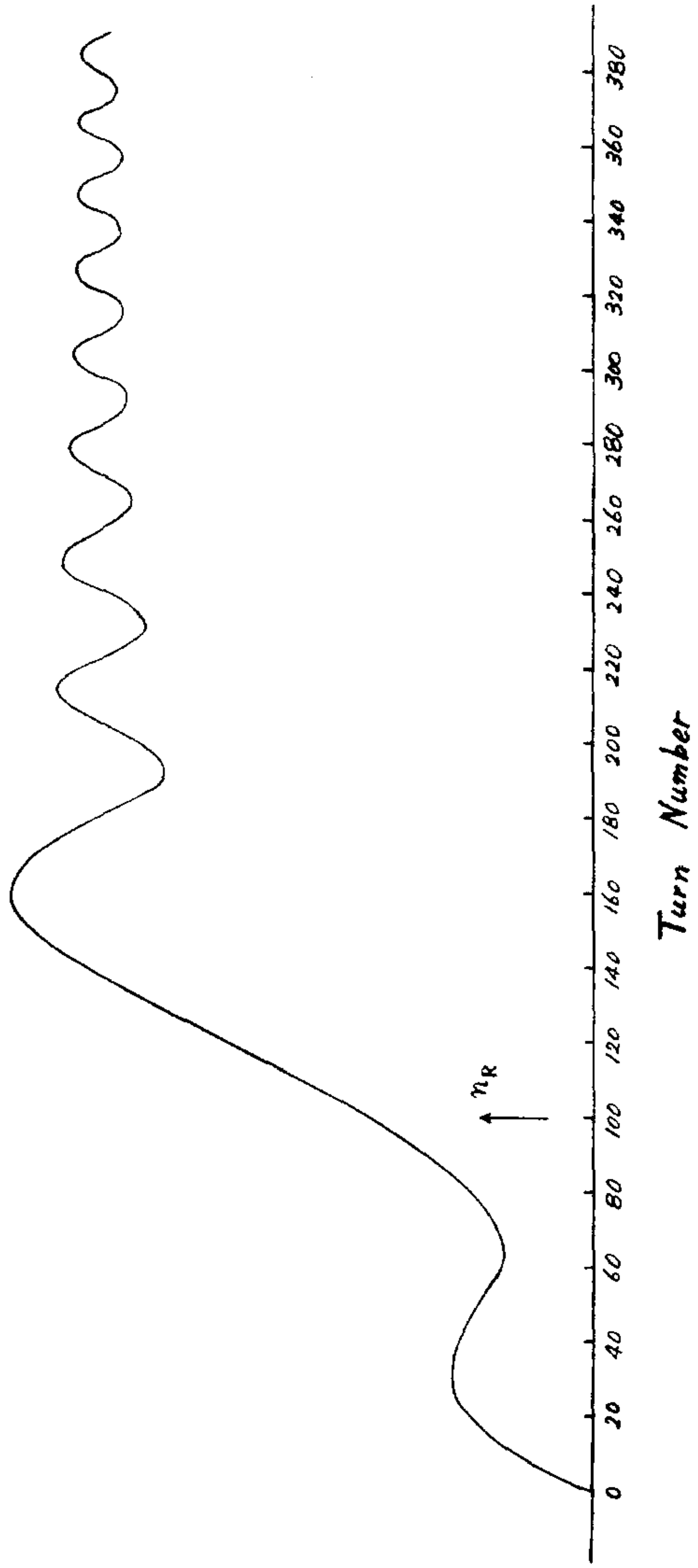
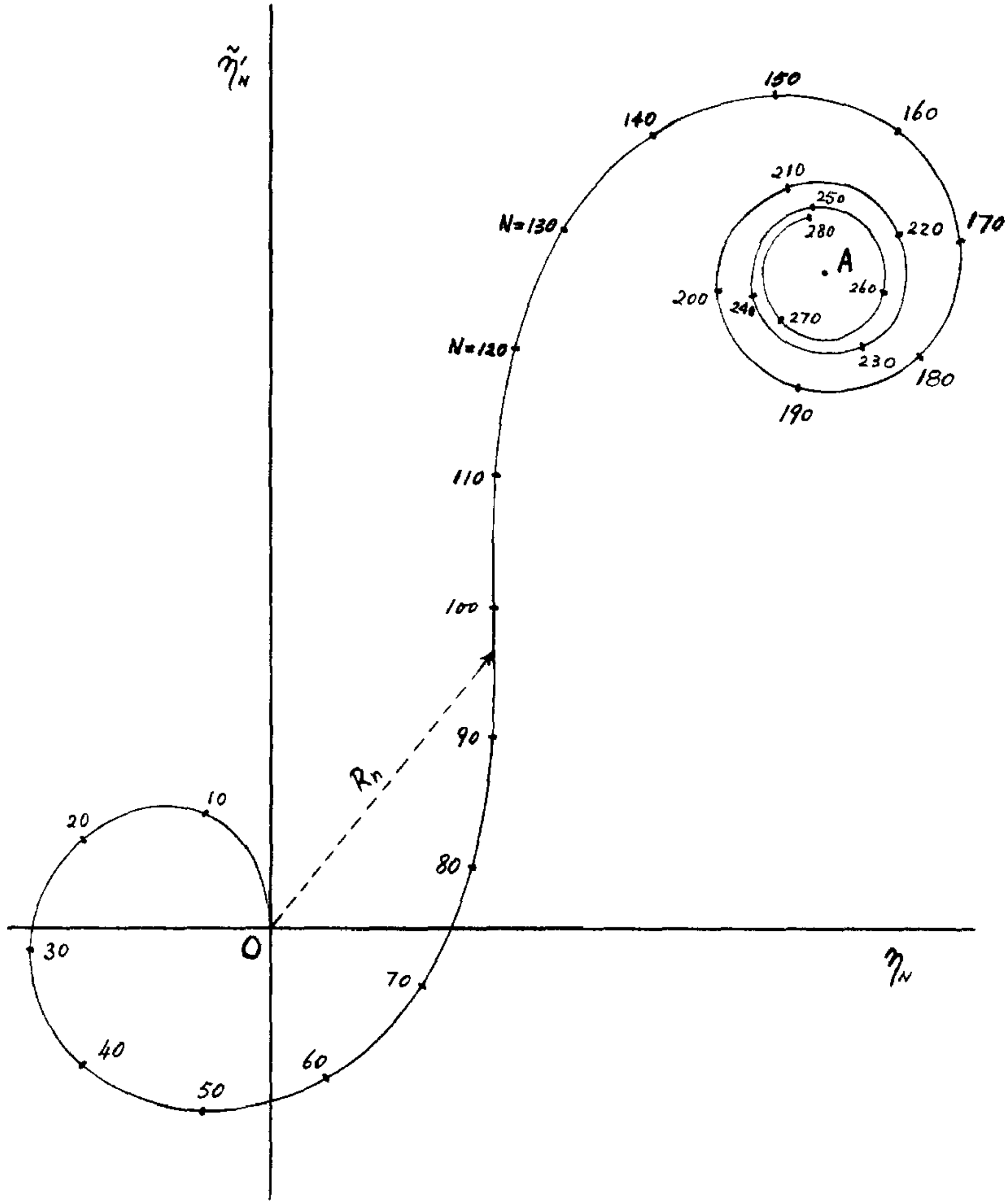

n_R
0
20
40
60
80
100
120
140
160
180
200
220
240
260
280
300
320
340
360
380
Turn Number

Figure 1



$\tilde{\eta}'_N$
η_N
O
A
R_n
10
20
30
40
50
60
70
80
90
100
110
N=120
N=130
140
150
160
170
180
190
200
210
220
230
240
250
260
270
280

Figure 2