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TRANSVERSE IMPEDANCE OF A COASTING BEAM IN A CORRUGATED VACUUM CHAMBER AT LOW FREQUENCIES

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I. Introduction

In this note, we try to derive an expression for the transverse impedance between an unbunched particle beam and its surrounding when the cross section of the vacuum chamber varies along the circumference of the machine. Only the situation of frequencies below cutoff and below resonances is considered.

The transverse coupling impedance can be calculated from the electromagnetic fields generated by a perturbation on a beam of charged particles. Since all pertinent equations are linear, it is sufficient to investigate a sinusoidal transverse modulation of charge density with mode n; i.e., with n wavelengths along the machine circumference. The perturbating wave travels with phase velocity $\beta_w c$ whereas the particle velocity is $\beta_p c$. Thus the circular frequency of the perturbating wave is $\omega = k_n \beta_w c = n\beta_w c/R$, where R is the radius of the storage ring. We shall neglect the curvature of the vacuum chamber which we replace by a straight periodic cylindrical pipe radius b with period $2\pi R$ with a cylindrical beam of radius a at the center. Periodically, with distance $2\pi R$, the radius of the pipe is enlarged to the value d over a length g (Figure 1).

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II. Perturbing source and field equations

The perturbing charge density of the beam is

$$\rho = \hat{\rho}\Delta \sin\phi\, \delta(a-r) \exp\left[ik(z-\tfrac{1}{2}g) - i\omega t\right], \tag{2.1}$$

with $k \equiv k_n = n/R$, $\hat{\rho}$ the unperturbed density, ϕ the azimuthal angle of the beam pipe and Δ the maximum transverse displacement at angle $\phi = \frac{1}{2}\pi$. The corresponding perturbing current density is

$$J_z = \rho\beta_p c, \tag{2.2}$$

$$J_r = J_y \sin\phi, \tag{2.3}$$

$$J_\phi = J_y \cos\phi, \tag{2.4}$$

and

$$J_y = i\hat{\rho}\Delta kc(\beta_p - \beta_w)\theta(a-r)\exp\left[ik(z-\tfrac{1}{2}g) - i\omega t\right], \tag{2.5}$$

where y designates the transverse direction at $\phi = \frac{1}{2}\pi$.

Maxwell equations governing the longitudinal components of the electric field $\vec{E}$ and magnetic field $\vec{H}$ are

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)E_z = \frac{1}{\epsilon_0}\frac{\partial\rho}{\partial z} - i\frac{\omega}{c}Z_0 J_z, \tag{2.6}$$

and

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)H_z = -\left(\vec{\nabla}\times\vec{J}\right)_z \tag{2.7}$$

where $Z_0 = 120\pi$ ohms and ε_0 are respectively the impedances and electric permittivity for free space and a time dependence of exp $(-i\omega t)$ has been assumed for all fields. The transverse components $\vec{E}_t$ and $\vec{H}_t$ can be expressed in terms of E_z and H_z by

$$\left(\frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2}\right)\vec{E}_t = \vec{\nabla}_t \frac{\partial E_z}{\partial z} + i\frac{\omega}{c}\vec{\nabla}_t \times Z_0\vec{H}_z - i\frac{\omega}{c}Z_0\vec{J}_t , \tag{2.8}$$

and

$$\left(\frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2}\right)Z_0\vec{H}_t = \vec{\nabla}_t \frac{\partial Z_0 H_z}{\partial z} - i\frac{\omega}{c}\vec{\nabla}_t \times \vec{E}_z - Z_0\vec{\nabla}_z \times \vec{J}_t . \tag{2.9}$$

Due to the presence of the transverse current, Eq. (2.5), H_z is nonvanishing. As a result, we have all components for $\vec{E}$ and $\vec{H}$ and the situation is more complicated than that for a longitudinal perturbing wave.

III. Solution

In the beam region, $0 \leq r \leq a$, the solutions of Eqs. (2.6) and (2.7) are respectively

$$E_z^I = \sum_{m=-\infty}^{\infty} iA_m I_1(\chi_m r)\sin\phi \exp\left[ik_m(z - \tfrac{1}{2}g)\right], \tag{3.1}$$

and

$$Z_0 H_z^I = \sum_{m=-\infty}^{\infty} iA'_m I_1(\chi_m r)\cos\phi \exp\left[ik_m(z - \tfrac{1}{2}g)\right], \tag{3.2}$$

where $\chi_m^2 = k_m^2 - \omega^2/c^2$, $k_m = m/R$ and I_1 is the modified Bessel function of order 1. The time dependent factor $\exp(-i\omega t)$ has been left out for the sake of convenience. Using Eqs. (2.8) and (2.9) we get for the transverse fields,

$$E_\phi^I = \sum_m \left[\frac{k_m A_m}{r\chi_m^2} I_1(\chi_m r) - \frac{\omega}{c\chi_m} A'_m I'_1(\chi_m r) - \delta_{mn}\eta\beta_w\beta_1\right]\cos\phi \exp\left[ik_m(z - \tfrac{1}{2}g)\right], \tag{3.3}$$

and

$$Z_0 H_\phi^I = \sum_m \left[-\frac{k_m}{r\chi_m^2} A'_m I_1(\chi_m r) + \frac{\omega}{c\chi_m} A_m I'_1(\chi_m r) - \delta_{mn}\eta\beta_1\right]\sin\phi \exp\left[ik_m(z - \tfrac{1}{2}g)\right], \tag{3.4}$$

where we have used the notation

$$\eta = Z_0 \hat{\rho} \Delta k^2 c / \chi , \tag{3.5}$$

$$\beta_1 = \beta_P - \beta_W . \tag{3.6}$$

We also define

$$\beta_2 = 1 - \beta_P \beta_W \tag{3.7}$$

for later use. In Eq. (3.5), $\chi \equiv \chi_n$.

The solution in the 'pipe region' with $a \lesssim r \lesssim b$ differs from Eqs. (3.1) to (3.4) only by the absence of the source terms and by the presence of the terms containing the modified Bessel function of the second kind $K_1(\chi_m r)$:

$$E_z^{II} = \sum_m i\left[B_m I_1(\chi_m r) + C_m K_1(\chi_m r)\right] \sin\phi \exp\left[ik_m(z - \tfrac{1}{2}g)\right], \tag{3.8}$$

$$Z_0 H_z^{II} = \sum_m i\left[B_m' I_1(\chi_m r) + C_m' K_1(\chi_m r)\right] \cos\phi \exp\left[ik_m(z - \tfrac{1}{2}g)\right], \tag{3.9}$$

$$\begin{aligned} E_\phi^{II} = \sum_m \Big\{ & \frac{k_m}{r\chi_m^2}\left[B_m I_1(\chi_m r) + C_m K_1(\chi_m r)\right] \\ & - \frac{\omega}{c\chi_m}\left[B_m' I_1'(\chi_m r) + C_m' K_1'(\chi_m r)\right]\Big\} \cos\phi \exp\left[ik_m(z - \tfrac{1}{2}g)\right], \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} Z_0 H_\phi^{II} = \sum_m \Big\{ & -\frac{k_m}{r\chi_m^2}\left[B_m' I_1(\chi_m r) + C_m' K_1(\chi_m r)\right] \\ & + \frac{\omega}{c\chi_m}\left[B_m I_1'(\chi_m r) + C_m K_1'(\chi_m r)\right]\Big\} \sin\phi \exp\left[ik_m(z - \tfrac{1}{2}g)\right]. \end{aligned} \tag{3.11}$$

We can compute B_m, B_m', C_m, and C_m' in terms of A_m and A_m' by matching E_z, E_ϕ, H_z, and H_ϕ at $r = a$ leading to

$$E_z^{II} = \sum_m i\left[A_m I_1(\chi_m r) - \delta_{mn}\frac{\chi^2 a}{k}\eta\beta_2 T(\chi r)\right]\sin\phi\,\exp\left[ik_m(z-\tfrac{1}{2}g)\right], \tag{3.12}$$

$$Z_0 H_z^{II} = \sum_m i\left[A'_m I_1(\chi_m r) - \delta_{mn}\frac{\chi^2 a}{k}\eta\beta_1 T(\chi r)\right]\cos\phi\,\exp\left[ik_m(z-\tfrac{1}{2}g)\right], \tag{3.13}$$

$$\begin{aligned} E_\phi^{II} = \sum_m \Big\{ \frac{k_m}{r\chi_m^2} A_m I_1(\chi_m r) - \frac{\omega}{c\chi_m} A'_m I'_1(\chi_m r) + \delta_{mn}\Big[\chi a\eta\beta_W\beta_1 T'(\chi r) \\ - \frac{a}{r}\eta\beta_2 T(\chi r)\Big]\Big\}\cos\phi\,\exp\left[ik_m(z-\tfrac{1}{2}g)\right], \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} Z_0 H_\phi^{II} = \sum_m \Big\{ -\frac{k_m}{r\chi_m^2} A'_m I_1(\chi_m r) + \frac{\omega}{c\chi_m} A_m I'_1(\chi_m r) + \delta_{mn}\Big[\frac{a}{r}\eta\beta_1 T(\chi r) \\ - \chi a\eta\beta_W\beta_2 T'(\chi r)\Big]\Big\}\sin\phi\,\exp\left[ik_m(z-\tfrac{1}{2}g)\right], \end{aligned} \tag{3.15}$$

where

$$T(\chi r) = I_1(\chi a)K_1(\chi r) - I_1(\chi r)K_1(\chi a). \tag{3.16}$$

We assume, for the moment, that the walls of the vacuum chamber are perfectly conducting. Thus the longitudinal fields in the cavity region, $b<r<d$, can be written as

$$E_z^{III} = \sum_{s=0}^{\infty} iD_s R(\Gamma_s r)\cos\alpha_s z\,\sin\phi, \tag{3.17}$$

and

$$Z_0 H_z^{III} = \sum_{s=0}^{\infty} iF_s\, S(\Gamma_s r)\sin\alpha_s z\,\cos\phi, \tag{3.18}$$

with

$$\alpha_s = \pi s/g, \qquad s = 0, 1, 2, \ldots, \tag{3.19}$$

$$\Gamma_s^2 = \alpha_s^2 - \omega^2/c^2, \tag{3.20}$$

$$R(\Gamma r) = K_1(\Gamma d)I_1(\Gamma r) - I_1(\Gamma d)K_1(\Gamma r), \tag{3.21}$$

$$S(\Gamma r) = K_1'(\Gamma d) I_1(\Gamma r) - I_1'(\Gamma d) K_1(\Gamma r), \tag{3.22}$$

so that $R(\Gamma d) = S'(\Gamma d) = 0$. The corresponding transverse fields are

$$E_\phi^{III} = \sum_{s=0}^{\infty} \left[\frac{i\alpha_s}{r\Gamma_s^2} D_s R(\Gamma_s r) - \frac{\omega}{c\Gamma_s} F_s S'(\Gamma_s r) \right] \sin\alpha_s z \cos\phi \tag{3.23}$$

and

$$Z_0 H_\phi^{III} = \sum_{s=0}^{\infty} \left[\frac{i\alpha_s}{r\Gamma_s^2} F_s S(\Gamma_s r) + \frac{\omega}{c\Gamma_s} D_s R'(\Gamma_s r) \right] \cos\alpha_s z \sin\phi . \tag{3.24}$$

The final step in the field calculation is the matching of the tangential field components at $r = b$, i.e.,

$$E_z^{II} = \begin{cases} E_z^{III} & 0 \le z \le g \\ 0 & g \le z \le 2\pi R, \end{cases} \tag{3.25}$$

$$E_\phi^{II} = \begin{cases} E_\phi^{III} & 0 \le z \le g \\ 0 & g \le z \le 2\pi R, \end{cases} \tag{3.26}$$

$$H_z^{II} = H_z^{III} \qquad 0 \le z \le g, \tag{3.27}$$

$$H_\phi^{II} = H_\phi^{III} \qquad 0 \le z \le g, \tag{3.28}$$

all evaluated at $r = b$.

For simplification, we introduce the following abbreviations:

$$\bar{A}_m = A_m I_1(\chi_m b),$$

$$\bar{A}_m' = A_m' I_1(\chi_m b),$$

$$\bar{B}_m = \delta_{mn} \chi^2 a \eta \beta_z T(\chi b)/k,$$

$$\bar{B}_m' = \delta_{mn}\chi^2 a\eta\beta_1 T(\chi b)/k,$$

$$I_{mm} = I_1'(\chi_m b)/[\chi_m b I_1(\chi_m b)],$$

$$\bar{C}_m = \delta_{mn}\chi a\eta\beta_2 T'(\chi b)/(kb),$$

$$\bar{C}_m' = \delta_{mn}\chi a\eta\beta_1 T'(\chi b)/(kb),$$

$$\bar{D}_s = D_s R(\Gamma_s b),$$

$$\bar{F}_s = F_s S(\Gamma_s b),$$

$$K_{ss} = R'(\Gamma_s b)/[\Gamma_s b R(\Gamma_s b)],$$

$$P_{ss} = S'(\Gamma_s b)/[\Gamma_s b S(\Gamma_s b)],$$

$$U_{mm} = k_m c/(\chi_m^2 \omega b^2),$$

$$V_{ss} = \alpha_s c/(\Gamma_s^2 b^2 \omega). \tag{3.29}$$

Then Eqs. (3.25) to (3.28) are translated into

$$\sum_m (\bar{A}_m - \bar{B}_m) \exp[ik_m(z - \tfrac{1}{2}g)] = \begin{cases} \sum_s \bar{D}_s \cos\alpha_s z & 0 \leq z \leq g \\ 0 & g \leq z \leq 2\pi R, \end{cases} \tag{3.30}$$

$$\sum_m \left[U_{mm}(\bar{A}_m - \bar{B}_m) - (\bar{A}_m' I_{mm} - \bar{C}_m')\right] \exp[ik_m(z - \tfrac{1}{2}g)]$$

$$= \begin{cases} \sum_s (iV_{ss}\bar{D}_s - \bar{F}_s P_{ss}) \sin\alpha_s z & 0 \leq z \leq g \\ 0 & g \leq z \leq 2\pi R, \end{cases} \tag{3.31}$$

$$\sum_m (\bar{A}'_m - \bar{B}'_m) \exp\left[ik_m(z - \tfrac{1}{2}g)\right] = \sum_s \bar{F}_s \sin\alpha_s z, \qquad 0 \le z \le g \tag{3.32}$$

and

$$\sum_m \left[(\bar{A}_m I_{mm} - \bar{C}_m) - U_{mm}(\bar{A}'_m - \bar{B}'_m)\right] \exp\left[ik_m(z - \tfrac{1}{2}g)\right]$$
$$= \sum_s (iV_{ss}\bar{F}_s + \bar{D}_s K_{ss}) \cos\alpha_s z, \qquad 0 \le z \le g \tag{3.33}$$

The solution is straightforward. Equations (3.30) and (3.31) give

$$\bar{A}_m - \bar{B}_m = \alpha \sum_{s=0}^{\infty} N_{ms} \bar{D}_s, \tag{3.34}$$

$$U_{mm}(\bar{A}_m - \bar{B}_m) - (\bar{A}'_m I_{mm} - C'_m) = -i\alpha \sum_{s=0}^{\infty} M_{ms}(iV_{ss}\bar{D}_s - \bar{F}_s P_{ss}), \tag{3.35}$$

whereas Eqs. (3.32) and (3.33) give

$$\tfrac{1}{2}(1 + \delta_{os})\bar{F}_s = i \sum_{m=-\infty}^{\infty} M^*_{sm}(\bar{A}'_m - \bar{B}'_m), \tag{3.36}$$

$$\tfrac{1}{2}(1 + \delta_{os})(iV_{ss}\bar{F}_s + \bar{D}_s K_{ss}) = \sum_{m=-\infty}^{\infty} N^*_{sm}\left[(\bar{A}_m I_{mm} - \bar{C}_m) - U_{mm}(\bar{A}'_m - \bar{B}'_m)\right]. \tag{3.37}$$

In above,

$$M_{ms} = \frac{\pi s/2}{(\pi\alpha m)^2 - (\pi s/2)^2} \begin{cases} \sin\pi\alpha m, & s \text{ even} \\ -i\cos\pi\alpha m, & s \text{ odd} \end{cases} \tag{3.38}$$

$$N_{ms} = \frac{\pi\alpha m}{(\pi\alpha m)^2 - (\pi s/2)^2} \begin{cases} \sin\pi\alpha m, & s \text{ even} \\ -i\cos\pi\alpha m, & s \text{ odd} \end{cases} \tag{3.39}$$

and $\alpha = g/2\pi R$ is the circumference factor or the fraction of the ring with enlarged cross section. It will be more convenient to write the above expressions in matrix form:

$$\bar{A} - \bar{B} = \alpha N \bar{D}, \tag{3.40}$$

$$U(\bar{A}-\bar{B}) - (I\bar{A}' - \bar{C}') = \alpha M(V\bar{D} + iP\bar{F}), \tag{3.41}$$

$$Q^{-1}\bar{F} = iM^{\dagger}(\bar{A}' - \bar{B}'), \tag{3.42}$$

$$Q^{-1}(iV\bar{F} + K\bar{D}) = N^{\dagger}[(I\bar{A} - \bar{C}) - U(\bar{A}' - \bar{B}')], \tag{3.43}$$

where

$$Q^{-1}_{ss'} = \tfrac{1}{2}(\delta_{ss'} + \delta_{os}\delta_{os'}). \tag{3.44}$$

Here only M and N are full matrices, U,I,P,V and K are diagonal, $\bar{A}$,$\bar{A}'$,$\bar{D}$,$\bar{F}$ are column matrices whereas $\bar{B}$,$\bar{B}'$,$\bar{C}'$ and $\bar{C}$ are column matrices with only one entry m = n.

Finally $\bar{F}$ and $\bar{D}$ can be solved from Eqs. (3.42) and (3.43) and substituted in Eqs. (3.40) and (3.41) to get

$$\bar{A} - \bar{B} = \alpha(NQK^{-1}VM^{\dagger} - NQK^{-1}N^{\dagger}U)(\bar{A}' - \bar{B}') + \alpha NQK^{-1}N^{\dagger}(I\bar{A} - \bar{C}), \tag{3.45}$$

and

$$\begin{aligned} U(\bar{A}-\bar{B}) - (I\bar{A}' - \bar{C}') = {} & \alpha[M(V^2QK^{-1} - PQ)M^{\dagger} - MVQK^{-1}N^{\dagger}U](\bar{A}' - \bar{B}') \\ & + \alpha MVQK^{-1}N^{\dagger}(I\bar{A} - \bar{C}). \end{aligned} \tag{3.46}$$

IV Transverse coupling impedance

The transverse force per unit charge experienced by the beam particle over one period is

$$F = \int_0^{2\pi R} dz \left(\vec{E}^I + \vec{\beta}_p \times Z_0 \vec{H}^I\right)\Big|_{r=0,\ \omega t = k(z-g/2)}$$

$$= \int_0^{2\pi R} dz \left[E_\phi^I(\phi=0) - \beta_p Z_0 H_\phi^I(\phi=\tfrac{\pi}{2}) \right]_{r=0,\ \omega t = k(z-g/2)}$$

$$= \pi R \gamma_w (\beta_2 A_n + \beta_1 A_n' + 2\eta\beta_1^2/\gamma_w), \tag{4.1}$$

where $\gamma_w = (1-\beta_w{}^2)^{-\frac{1}{2}}$. We notice that only the fields with the same mode number as the perturbation contribute to the coupling impedance; the other modes just average to zero. The transverse impedance is defined as

$$Z_t = -iF/(\beta_p I_0 \Delta), \tag{4.2}$$

where I_0 is the unperturbed beam current, or

$$Z_t = -i\,\frac{\pi R \gamma_w}{\beta_p I_0 \Delta} (\beta_2 A_n + \beta_1 A_n' + 2\eta\beta_1^2/\gamma_w). \tag{4.3}$$

The enlarged portion of the vacuum chamber is usually only a very small portion of the whole ring; i.e., $\alpha \ll 1$. When the wavelength of the perturbation is much longer than the length g of the enlarged portion, i.e., $n \ll \frac{1}{\alpha} = 2\pi R/g$, our solution (3.45) and (3.46) can be expanded in powers in α. Neglecting $O(\alpha^2)$, we get

$$\bar{A}-\bar{B} = \alpha\left(NQK^{-1}UM^{t} - NQK^{-1}N^{t}U\right)\left(I^{-1}\bar{C}'-\bar{B}'\right) + \alpha NQK^{-1}N^{t}\left(I\bar{B}-\bar{C}\right), \tag{4.4}$$

$$I\bar{A}'-\bar{C}' = \alpha\left[UNQK^{-1}VM^{t} - UNQK^{-1}N^{t}U + M(PQ-QK^{-1}V^{2})M^{t} + MVQK^{-1}N^{t}U\right]\cdot$$
$$\cdot\left(I^{-1}\bar{C}'-\bar{B}'\right) + \alpha\left(UNQK^{-1}N^{t} - MVQK^{-1}N^{t}\right)\left(I\bar{B}-\bar{C}\right). \tag{4.5}$$

We first consider the trivial case of $\alpha = 0$, corresponding to a smooth vacuum chamber. We find

$$\bar{A} = \bar{B}, \tag{4.6}$$

$$\bar{A}' = I^{-1}\bar{C}', \tag{4.7}$$

or

$$A_n = ka\eta\beta_2 T(\chi b)/[\gamma_w^2 I_1(\chi b)], \tag{4.8}$$

$$A_n' = ka\eta\beta_1 T'(\chi b)/[\gamma_w^2 I_1'(\chi b)]. \tag{4.9}$$

Putting in the small argument approximation for the Bessel functions, we arrive at

$$Z_t = \frac{iRZ_0}{\beta_p^2\gamma_p^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right) \tag{4.10}$$

in agreement with the well-known formula for the transverse impedance of a perfectly conducting cylindrical vacuum chamber. The convention used here is: positive imaginary part implies capacitive.

The first order α correction can be obtained by using Eqs. (3.45) and (3.46), from which the first order correction in A_n and A_n' can be written as

$$\delta A_n = 2\alpha \frac{a^2}{b^2} \frac{\eta}{\gamma_w} (\beta_p \beta_w W_0 + W_1), \tag{4.11}$$

$$\delta A_n' = 2\alpha \frac{a^2}{b^2} \frac{\eta}{\gamma_w} (\beta_p W_0 + \beta_w W_1 - \beta_1 W_2/\gamma_w^2), \tag{4.12}$$

where

$$W_0 = -\frac{\sin^2 \pi\alpha n}{(\pi\alpha n)^2} \frac{R(\Gamma_0 b)}{\Gamma_0 b R'(\Gamma_0 b)}, \tag{4.13}$$

$$W_1 = \sum_{s=1}^{\infty} \left[1 - (-1)^s \cos 2\pi\alpha n\right] \frac{(\pi s/2)^2}{\left[(\pi\alpha n)^2 - (\pi s/2)^2\right]^2} \frac{R(\Gamma_s b)}{\Gamma_s b R'(\Gamma_s b)}, \tag{4.14}$$

and

$$W_2 = \sum_{s=1}^{\infty} \left[1 - (-1)^s \cos 2\pi\alpha n\right] \frac{(\pi\alpha n)^2}{\left[(\pi\alpha n)^2 - (\pi s/2)^2\right]^2} \frac{\Gamma_s b S'(\Gamma_s b)}{S(\Gamma_s b)}. \tag{4.15}$$

Under the conditions $\pi\alpha n \ll 1$ and $nd/R \ll 1$, we can make the approximation $\Gamma_s \approx \pi s/g$ and sum W_1 and W_2 over odd values of s only. Thus the transverse coupling impedance differs from the smooth beam pipe value of Eq. (4.10) by

$$\begin{aligned} \delta Z_t &= -i \frac{\pi R \gamma_w}{\beta_p I \Delta} (\beta_2 \delta A_n + \beta_1 \delta A_n') \\ &= -i\alpha \frac{2 R Z_0}{\beta_p^2 b^2} (\beta_p^2 W_0 + W_1 - \beta_1^2 W_2). \end{aligned} \tag{4.16}$$

In the case of bellows, with $g \ll \pi^2 b/32$, we find

$$W_0 = \frac{S^2-1}{S^2+1}, \tag{4.17}$$

with S = d/b while W_1 and W_2 can be neglected. Substituting Eq. (4.17) in Eq. (4.16), we get the transverse impedance for a bellow

$$\delta Z_t = -i\,\frac{Z_0 g}{\pi b^2}\cdot\frac{S^2-1}{S^2+1}. \tag{4.18}$$

For pairs of cross section variations for which $g \gg \pi b$, summing up W_1 and W_2 using small argument approximations for R, R', S and S', we get

$$\begin{aligned} W_1 &= -\frac{S^2-1}{S^2+1}, \\ W_2 &= -\frac{1}{3}(\pi\alpha n)^2\,\frac{S^2-1}{S^2+1}, \end{aligned} \tag{4.19}$$

while W_0 is the same as Eq. (4.17). The result is

$$\delta Z_t = i\alpha\,\frac{2RZ_0}{\beta_p^2\gamma_p^2 b^2}\cdot\frac{S^2-1}{S^2+1}\left[1-\frac{1}{3}\beta_i^2\gamma_p^2(\pi\alpha n)^2\right], \tag{4.20}$$

where $\gamma_p = (1-\beta_p^{\,2})^{-\frac{1}{2}}$, so that the total transverse impedance is

$$\begin{aligned} Z_t = i\,\frac{RZ_0}{\beta_p^2\gamma_p^2}\Big\{&(1-\alpha)\left(\frac{1}{a^2}-\frac{1}{b^2}\right)+\alpha\left(\frac{1}{a^2}-\frac{1}{d^2}\right) \\ &+\frac{\alpha}{d^2}\,\frac{(S^2-1)^2}{S^2+1}-\frac{2}{3}\,\frac{\alpha}{b^2}\,\beta_i^2\gamma_p^2(\pi\alpha n)^2\left(\frac{S^2-1}{S^2+1}\right)\Big\}. \end{aligned} \tag{4.21}$$

The first two terms are just the sum of the contribution of the smooth chamber with radius b and circumference factor $1-\alpha$, and of the smooth chamber with radius d and circumference factor α. To obtain a better estimate of the transverse impedance of a pair of cross section variations, we should not use small argument approximations for the Bessel functions because no matter how small $\pi b/g$ is, $\pi sb/g$ will become big as s increases.

We decompose the transverse impedance into

$$Z_t = (1-\alpha)(Z_t)_b + \alpha(Z_t)_d + (Z_t)_c ,$$

where $(Z_t)_b$ and $(Z_t)_d$ are the transverse coupling impedance of a smooth pipe of radius b and d respectively and $(z_t)_c$ is the contribution of the pair of steps in cross section. From Eq. (4.16),

$$(Z_t)_c = -2i\alpha \frac{RZ_0}{\beta_p^2 b^2}\left[\beta_p^2 W_0 + W_1 - \beta_1^2 W_2 + \frac{1}{2\gamma^2}\left(1-\frac{1}{S^2}\right)\right], \tag{4.22}$$

which can be rewritten as

$$(Z_t)_c = -i\,\frac{gZ_0}{\pi\beta_p^2 b^2}\left[(W_0+W_1) - \beta_1^2 W_2 + \frac{1}{2\gamma^2}\left(1-\frac{1}{S^2}-2W_0\right)\right]. \tag{4.23}$$

We are still under the conditions that $\pi\alpha n \ll 1$ and $nd/R \ll 1$, but g/d can take any value. The second term in Eq. (4.23) contains a factor $(\pi\alpha n)^2$ through W_2 and can therefore be neglected while the third term is small when the energy of the particle beam is high enough. As a result, we are left with

$$(Z_t)_c = -i\,\frac{Z_0 g}{\pi\beta_p^2 b^2}(W_0 + W_1). \tag{4.24}$$

Since $\pi\alpha n \ll 1$, the argument of W_0

$$|\Gamma_0 b|^2 \lesssim (nb/R)^2 = (\pi\alpha n)^2(2b/g)^2 \tag{4.25}$$

is always small, so that small argument approximation can still be used in W_0. What we need to do then is to sum up W_1 numerically using the exact values for the modified Bessel functions. The normalized transverse impedance for the two steps $(Z_t)_c/(-iZ_o/\pi d\beta_p{}^2)$ depends on g/d and S only. It is plotted in Figure 2 as a function of S-1 for various values of g/d. We note that the normalized transverse impedance becomes independent of g/d when g/d $\gtrsim$ 1, and can be fitted to within a few percents by the following expression:

$$(Z_t)_c = -i\frac{Z_o}{\pi d\beta_p^2}\cdot\frac{(S-1)^2}{0.90+0.78S} \cdot \tag{4.26}$$

V. Resistive wall

The resistivity of the walls of the vacuum chamber can be included in our discussion. Equation (3.17) will contain an extra term that does not vanish at the wall, so will Eq. (3.18) and Eqs. (3.23) and (3.24) that follow. The matching of $\vec{E}$ and $\vec{H}$ at r = d will determine the extra terms added. Furthermore, the matching of fields at r = b, Eqs. (3.25) to (3.26), will be more complicated. However, the solution is straightforward and simple. The contribution of the resistive walls leads to an extra transverse impedance

$$(Z_t)_{res} = (1-i)RZ_o\left[(1-\alpha)\frac{\delta_V}{b^3} + \alpha\frac{\delta_E}{d^3}\right], \tag{5.1}$$

where δ_V and δ_E are the skin depths of the walls of the vacuum chamber and those of the enlarged part respectively. Here the finite conductivity of the side walls of the steps has not been included.

VI. Numerical values for the Energy Doubler

The emittance of the beam in the main ring is ε = 1.5π mm-mrad at ~8.9 GeV and is inversely proportional to the beam energy. The β-oscillation parameter for the Doubler has a maximum of β_{max} ~100 m, so that the beam size radius is

$$a = \begin{cases} 0.30 \text{ cm.} & \text{at } 150 \text{ GeV}, \\ 0.12 \text{ cm.} & \text{at } 1000 \text{ GeV}. \end{cases}$$

Taking the radius of the ring $R = 10^5$ cm and the radius of the vacuum chamber $b = 3.5$ cm, using Eq. (4.10), we arrive at the transverse impedance for a smooth vacuum chamber

$$(Z_t)_{smooth} = \begin{cases} 1.9 \times 10^4 i \ \Omega/\text{cm.} & \text{at } 150 \text{ GeV}, \\ 2.6 \times 10^3 i \ \Omega/\text{cm.} & \text{at } 1000 \text{ GeV}, \end{cases} \tag{6.1}$$

which is capacitive.

The Doubler has roughly 1000 bellows, each of length $g \sim 2.94$ cm and radius $d \sim 4.26$ cm, giving an enlargement ratio $S = d/b \sim 1.22$. Since $32\, g/(\pi^2) \sim 2.75$, Eq. (4.18) cannot be used. This Eq. (4.18) would lead to the incorrect value for 1000 bellows

$$\begin{aligned} \delta Z_t &= -i \frac{Z_0 g}{\pi b^2} \frac{S^2 - 1}{S^2 + 1} \times 1000 \\ &= -i \ 5.7 \times 10^3 \ \Omega/\text{cm.} \end{aligned} \tag{6.2}$$

We can also consider the bellows as 1000 pairs of steps. Here $g/d = .69$. We see from Figure 2 that at $S = 1.22$, the $g/d = .69$ curve is very near to the curves for large g/d; so Eq. (4.26) can be used. It gives for 1000 pairs of steps the correct extra transverse impedance

$$\begin{aligned} (Z_t)_c &= -i \frac{Z_0}{\pi d \beta_p^2} \frac{(S-1)^2}{0.90 + 0.78S} \times 1000 \\ &= -i \ 7.4 \times 10^2 \ \Omega/\text{cm.} \end{aligned} \tag{6.3}$$

which is different from the value in Eq. (6.2). We note that the contribution of the bellows is inductive and is comparable to the transverse impedance of the smooth vacuum chamber.

Finally, using Eq. (5.1), the resistive part of the transverse impedance is found to be

$$(Z_t)_{res} = (1-i)(\mu_R/n)^{1/2} \times 1.7 \times 10^5 \ \Omega/cm. \tag{6.4}$$

which is quite significant when n is near and less than the cutoff mode number and dominates at low frequencies. In Eq. (6.4), μ_R is the relative magnetic permeability of the chamber wall and the conductivity of the chamber wall $\sigma = 13.7 \times 10^3$ mho-cm^{-1} for stainless steel has been used.

The transverse impedance due to the Lambertson magnets has been estimated in a former paper[1]:

$$(Z_t)_{lam} = (1-i)(\mu_R/n)^{1/2} \times 5.6 \times 10^4 \frac{1+\lambda^2}{(1-\lambda^2)^3} \ \ \Omega/cm, \tag{6.5}$$

when is beam is displaced by a fraction λ from the center of the magnet opening. This contribution is of the same order as that of the resistive walls of the beam pipe.

A stability criterion has been derived by Zotter and Sacherer, which is

$$|Z_t| < 4 \frac{E}{e} \frac{\beta_p Q \gamma_p}{I_p R} \left|(n-Q)\eta + Q'\right| \left(\frac{\Delta p}{p}\right), \tag{6.6}$$

where Q is the tune, Q' is the chromaticity and I_p is the peak current which is related to the average current I_{AV} due to M bunches each of r.m.s. length σ_ℓ by

$$I_p = I_{AV} \frac{\sqrt{2\pi} R}{M \sigma_\ell} .$$

For the Doubler, we take $R = 10^5$ cm, dispersion factor $\eta = 0.0028$, $M \sim 1000$, $I_{AV} = 0.15$ amp (2×10^{13} ppp) and obtain

$$\sigma_\ell = 140 S^{1/2} E^{-1/4} V_{RF}^{-1/4} \text{ cm},$$

$$\frac{\Delta p}{p} = 0.026\, S^{1/2} E^{-3/4} V_{RF}^{-1/4},$$

with the RF voltage V_{RF} on MV, E in GeV and the invariant bunch areas in eV-sec. Using V_{RF} = 2.16 MV and S = 0.3 eV-sec and $Q' \ll (n-Q)\eta$, the stability criterion becomes

$$\left|\frac{Z_t}{n-Q}\right| < \begin{cases} 969\ \Omega/\text{cm} & \text{at } 150 \text{ GeV}, \\ 6464\ \Omega/\text{cm} & \text{at } 1000 \text{ GeV}. \end{cases}$$

We note that the above computed Z_t is safe for large n but leads to instability for small $|n-Q|$ and transverse dampers are required to restore stability here.

<u>Reference</u>

[1]K.Y. Ng, Fermilab UPC-149

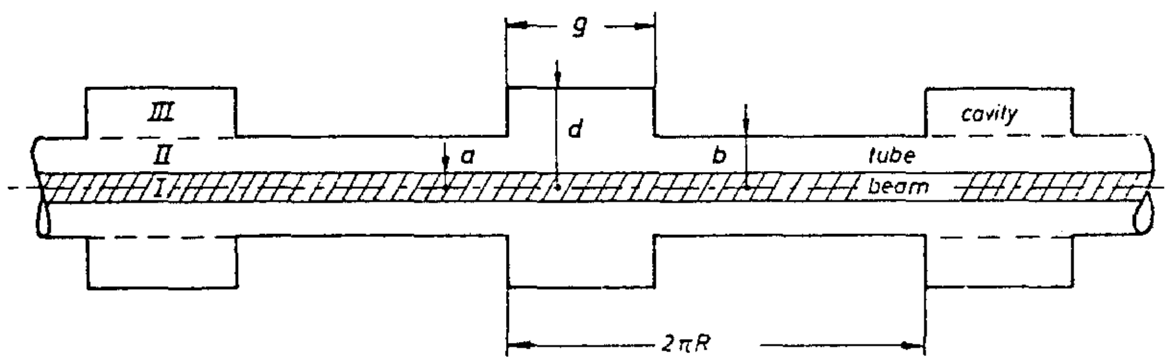

Fig. 1. Schematic cross-section of a few periods of the model geometry.

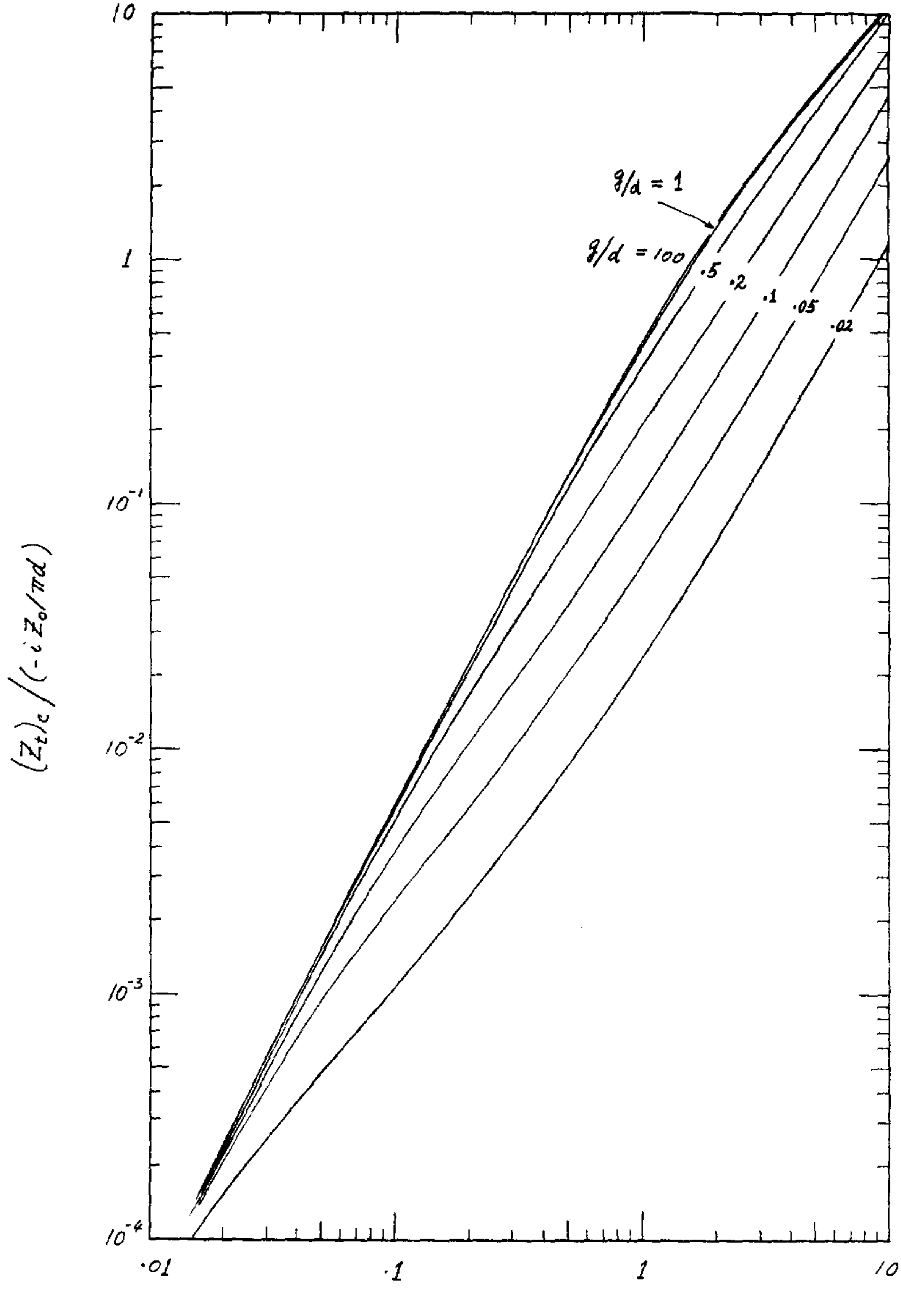

Figure 2