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ADIABATIC VARIATION AND BEHAVIOR OF INVARIANT CURVE

(General Criterion for Adiabatic Manipulation of RF System)

Ken Takayama

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Abstract

Considerations about the motion of the Courant and Snyder invariant curve/1/ are presented, which lead to an essential and practical procedure for estimating the adiabaticity of a linear harmonic oscillator. As a cosequence, an adiabatic ratio which is a measure of the adiabaticity is obtained for an example, by using a normalized time which is related to the oscillation period in the initial state.

1. Introduction

We shall consider a time-dependent harmonic oscillator described by the Hamiltonian

$$H(x, p; t) = \frac{1}{2}\left[p^2 + \lambda(t) x^2 \right], \tag{1-1}$$

where $\lambda(t)$ is the time-varying parameter. It is well known that for sufficiently slow change in the parameter the action variable of system (1-1) is an approximate constant, which we desinate as an "adiabatic constant". The proof for adiabatic invariance of the "action integral" has been given in many places in the literature/2/. Unfortunately, the adiabatic theorem does not tell quantitatively how slow the change in the parameter must be for the adiabatic theorem to hold.

Practically, we would expect to calculate the increase in the action integral when the change in the Hamiltonian is nearly adiabatic and even abrupt. Approximate methods to calculate the increase have been presented by several authors/2-6/. However, they have limits on their application and the calculations are intricate.

We point out that the system (1-1) has an exact dynamical invariant independent of the change in the parameter, which is designated as the "Courant and Snyder invariant"

$$I(x, p; t) = \frac{1}{2\beta(t)}\left\{ x^2 + \left[\frac{1}{2}\dot{\beta}(t)x - \beta(t)p\right]^2 \right\}, \tag{1-2}$$

where $\beta(t)$ satisfies the auxiliary differential equation

$$\frac{1}{2}\beta\ddot{\beta} - \frac{1}{4}\dot{\beta}^2 + \lambda(t)\beta^2 = 1. \tag{1-3}$$

When $\lambda(t)$ is constant, the invariant I is exactly identical with the action variable of the system (1-1) if we choose the initial condition

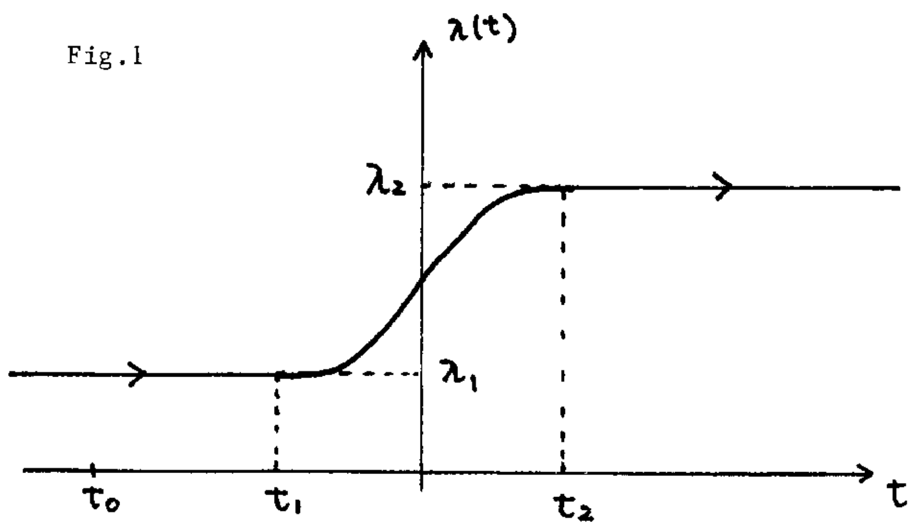
$$\beta(-\infty) = 1/\sqrt{\lambda}\ , \quad \dot{\beta}(-\infty) = 0. \tag{1-4}$$

For a time-varying function $\lambda(t)$, from (1-2), we know that a sequence of infinite phase points which have a certain constant value of I at an arbitrary time behaves as a deformable moving ellipse in the phase space (x,p,;t) after that time. The form of such a ellipse, called an "invariant curve" in the following, is uniquely determined by the auxiliary differential equation (1-3) alone.

We will now consider the case when the parameter λ changes from a constant value λ_1 to another constant value λ_2 in a finite period. It is noted that such cases often appear in real situations. That is

$$\lambda(t) = \begin{cases} \lambda_1 & (\dot{\lambda}(t_1)=0) \quad \text{for } t \leqq t_1, \\ \lambda_2 & (\dot{\lambda}(t_2)=0) \quad \text{for } t \geqq t_2. \end{cases} \tag{1-5}$$

Fig.1



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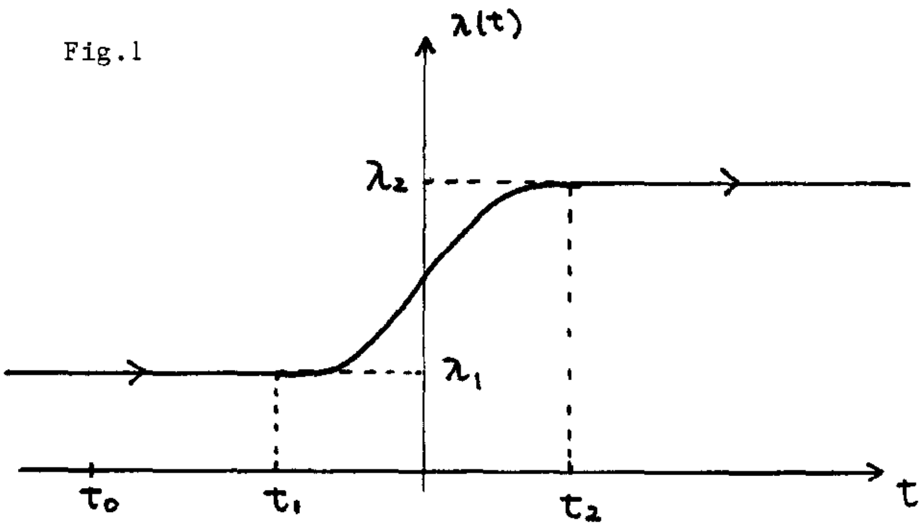


We consider the invariant curve described in the term

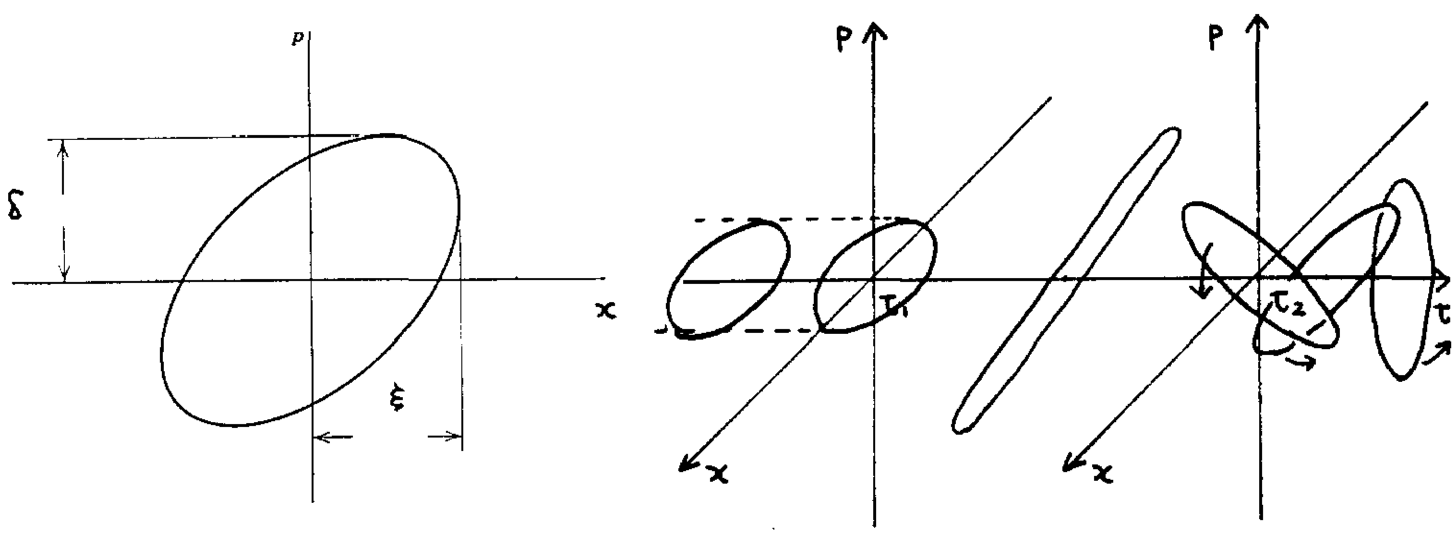
$$I(x, p; t_0) = I_0 \qquad t_0 < t_1 , \tag{1-6}$$

with a constant I_0. The quantity I_0 is equal to the value of the action variable of infinite phase points which comprise the invariant curve, as mentioned above. So if we continue to examine the motion of the invariant curve at $t > t_2$, we find the exact time-evolution of such a sequence of infinite points which once possessed the same action variable J. This fact will give us useful informations about a change in the action variable/7/. Furthermore, it may enable us to estimate the adiabaticity of the system (1-1).

2. Motion of Invariant Curve and Adiabatic Ratio

We may characterize an ellipse or an invariant curve by two parameters of ξ , δ which are a function of $\beta(t)$ and $\dot{\beta}(t)$ (See Fig.2).

Fig.2



If we choose the initial conditions $\beta(t_0)=1/\sqrt{\lambda_1}$, $\dot{\beta}(t_0)=0$, the solution of (1-3) is

$$\beta(t) = 1/\sqrt{\lambda_1}, \quad \dot{\beta}(t) = 0 \qquad t_0 \leqq t \leqq t_1. \tag{2-1}$$

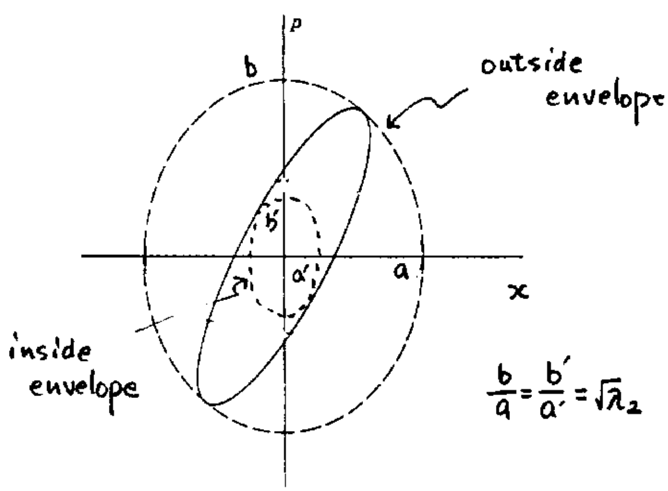
Therefore, before the variation of the Hamiltonian the form of the above ellipse remaines unchanged and its motion is only parallel displacement along the axis of time. At $t=t_1$, the ellipse begins to move, following the time-evolution of $\beta(t)$ which is determined by Eq.(1-3). After the variation of the Hamiltonian, i.e., $t \geqq t_2$, the ellipse continues to move unless ($\beta(t_2)$, $\dot{\beta}(t_2)$) is equal to ($1/\sqrt{\lambda_2}$,0) (See Fig.2).

Now we write the phase space area surrounded by the invariant curve which remains constant in the region $t_0 \leqq t < \infty$, in the term

$$S_0 = 2\pi I_0. \tag{2-2}$$

In addition, we shall define S as the cross section of the outside envelope which the moving ellipse makes after $t=t_2$ (See Fig.3).

Fig.3



If we choose the initial conditions $\beta(t_0)=1/\sqrt{\lambda_1}$, $\dot{\beta}(t_0)=0$, the solution of (1-3) is

$$\beta(t) = 1/\sqrt{\lambda_1} \;, \quad \dot{\beta}(t) = 0 \qquad t_0 \leqq t \leqq t_1 . \tag{2-1}$$

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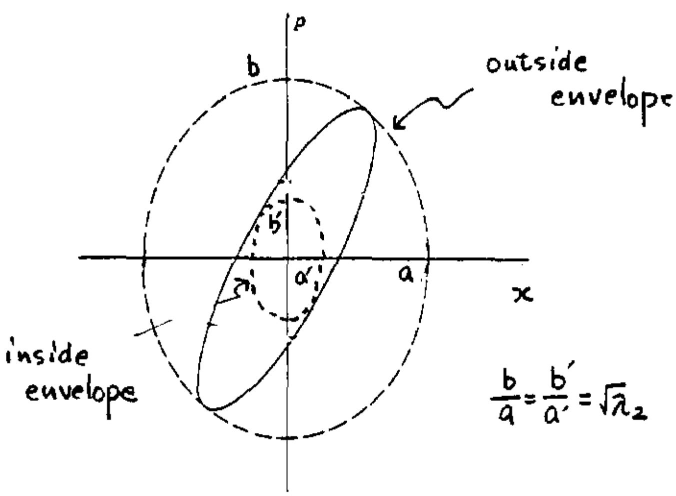
Fig.3



It is noted that the phase area between the outside and inside envelopes is the area effectively occupied by the phase points which have the same action variable J at $t \leqq t_1$. From Fig.3, the cross section S is described in the form

$$S = \pi \,\mathrm{Max}\, \xi(t)\, \mathrm{Max}\, \delta(t) \qquad t \geqq t_2 . \tag{2-3}$$

Then it is trivial to write $\xi(t)$, $\delta(t)$ in the terms of $\beta(t)$, $\dot{\beta}(t)$

$$\xi(t) = \sqrt{2I_0 \beta(t)}, \tag{2-4}$$

$$\delta(t) = \sqrt{2I_0 [1 + \dot{\beta}(t)^2/4]/\beta(t)} . \tag{2-5}$$

Further we shall set

$$\gamma(t) = [1 + \dot{\beta}(t)^2/4]/\beta(t) . \tag{2-6}$$

Here we define the adiabatic ratio r in the term

$$r = \frac{S}{S_0} \quad \text{or} \quad [\,\mathrm{Max}\,\beta(t)\; \mathrm{Max}\,\gamma(t)\,]^{1/2} . \tag{2-7}$$

The soution of (1-3) with the constant $\lambda = \lambda_2$ is know:

$$\beta(t) = \sqrt{A^2+B^2+1/\lambda_2} + A\cos[2\sqrt{\lambda_2}(t-t_2)] + B\sin[2\sqrt{\lambda_2}(t-t_2)] \tag{2-8}$$

$$\dot{\beta}(t) = 2\sqrt{\lambda_2}\{-A\sin[2\sqrt{\lambda_2}(t-t_2)] + B\cos[2\sqrt{\lambda_2}(t-t_2)]\}, \tag{2-9}$$

where A and B satisfy the boundary condition

$$\beta(t_2) = \sqrt{A^2+B^2+1/\lambda_2} + A , \tag{2-10}$$

$$\dot{\beta}(t_2) = 2\sqrt{\lambda_2}\, B . \tag{2-11}$$

From (2-10) and (2-11), A and B are written in the terms

$$A = -\frac{1}{2\beta(t_2)}\left[\dot{\beta}(t_2)^2/4\lambda_2 + 1/\lambda_2 - \dot{\beta}(t_2)\right], \qquad (2\text{-}12)$$

$$B = \dot{\beta}(t_2)/2\sqrt{\lambda_2}\,. \qquad (2\text{-}13)$$

Further we know the point (β(t), $\dot{\beta}$(t)) traverses an ellipse in the auxiliary phase space (β, $\dot{\beta}$) described in the form

$$\frac{\left(\beta - \sqrt{A^2+B^2+1/\lambda_2}\right)^2}{A^2+B^2} + \frac{\dot{\beta}^2}{4\lambda_2(A^2+B^2)} = 1, \qquad (2\text{-}14)$$

from (2-8) and (2-9). Therefore it is trivial to obtain the maximum values of β(t) and γ(t). We write these values in the terms

$$\mathrm{Max}\ \beta(t) = \sqrt{A^2+B^2+1/\lambda_2} + \sqrt{A^2+B^2}, \qquad (2\text{-}15)$$

$$\mathrm{Max}\ \gamma(t) = 1 \Big/ \left(\sqrt{A^2+B^2+1/\lambda_2} - \sqrt{A^2+B^2}\right). \qquad (2\text{-}16)$$

So that the adiabatic ratio becomes

$$r = \left\{\left(\sqrt{A^2+B^2+1/\lambda_2} + \sqrt{A^2+B^2}\right)\Big/\left(\sqrt{A^2+B^2+1/\lambda_2} - \sqrt{A^2+B^2}\right)\right\}^{1/2}. \qquad (2\text{-}17)$$

We note that Eq.(2-17) represents geometrically the relative extent on the β axis of the auxiliary ellipse (2-14). If we know the values of β(t), $\dot{\beta}$(t) at t=t_2 when the variation of the Hamiltonian is completed, we can evaluate exactly the adiabatic ratio r by Eq.(2-17). Therefore, it is necessary only to solve the auxiliary equation (1-3) with the boundary condition ($1/\sqrt{\lambda_1}$,0).

3. Example and Discussion

As a example, we shall consider the cosine-like change of the coefficient $\lambda(t)$

$$\lambda(t) = \frac{1}{2}\left[\lambda_1 + \lambda_2 - (\lambda_2 - \lambda_1)\cos\frac{\pi}{T}(t-t_1)\right] \quad t_1 \leqq t \leqq t_2, \tag{3-1}$$

where $T(=t_2-t_1)$ is the variation time.
Introducing a parameter ϵ

$$\lambda_2 = \lambda_1(1+\epsilon), \tag{3-2}$$

and substituting (3-2) into (3-1), we have

$$\lambda(t) = \lambda_1\left[1 + \frac{\epsilon}{2} - \frac{\epsilon}{2}\cos\frac{\pi}{T}(t-t_1)\right]. \tag{3-3}$$

Furthermore, if time-scale change

$$t' = \frac{\pi}{T}(t-t_1), \tag{3-4}$$

is made, we have the equation of motion

$$\frac{d^2x}{dt'^2} + \frac{\lambda_1 T^2}{\pi^2}\left[1 + \frac{\epsilon}{2} - \frac{\epsilon}{2}\cos t'\right]x = 0. \tag{3-5}$$

Then, introducing a characteristic time T_0

$$T_0 = 2\pi/\sqrt{\lambda_1}, \tag{3-6}$$

which corresponds to the oscillation period at $t \leqq t_1$, and a parameter μ

$$T = \mu T_0, \tag{3-7}$$

we obtain the normalized equation of motion for the system (1-1)

$$\frac{d^2x}{dt'^2} + 4\mu^2\left[1 + \frac{\epsilon}{2} - \frac{\epsilon}{2}\cos t'\right]x = 0. \tag{3-8}$$

For simplicity, we shall consider the case of $\epsilon=1$. The auxiliary differential equation for β can be described in the form

$$\frac{1}{2}\beta\ddot{\beta} - \frac{1}{4}\dot{\beta}^2 + 2\mu^2(3 - \cos t')\beta^2 = 1 \quad \left(\dot{} \equiv \frac{d}{dt'}\right), \tag{3-9}$$

which has the boundary condition

$$\beta(0) = 1/\sqrt{\lambda_1}, \qquad \dot{\beta}(0) = 0. \tag{3-10}$$

Unfortunately, we do not know the explicit analytical solution of (3-9). However, it is easy to obtain its solution with a help of a computer. Now we particularly need the solution at $t'=t'_2=\pi$

$$\beta(\pi), \qquad \dot{\beta}(\pi). \tag{3-11}$$

It is trivial to calculate the adiabatic ratio, after $\beta(\pi)$ and $\dot{\beta}(\pi)$ are obtained. Thus, the adiabatic ratio r is shown as a function of μ (See Fig.4).

Using the present method, the adiabatic ratio for an arbitrary time-varying function $\lambda(t)$ can be obtained easily. Further we note that it gives an answer even in the region where traditional approximate methods are not applicable either in principle or because of intricacies of calculation.

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Appendix

Approximate Calculation of the Adiabatic Ratio

With the introduction of a function

$$\rho(t) = \sqrt{\beta(t)}\,, \tag{A-1}$$

we have a nonlinear differential equation for $\rho(t)$ equivalent to (1-3) /8/

$$\ddot{\rho} + \lambda(t)\rho = \frac{1}{\rho^3}. \tag{A-2}$$

Instead of solving Eq.(1-3), we shall attempt to solve approximately Eq.(A-2) in the region $t_1 \leqq t \leqq t_2$, which has the initial condition

$$\rho(t_1) = (\lambda_1)^{-1/4}\,, \qquad \dot{\rho}(t_1) = 0. \tag{A-3}$$

Assumption of a nearly adiabatic system permits us to linearize (A-2) about the guiding center

$$\rho_0(t) = [\lambda(t)]^{-1/4}. \tag{A-4}$$

We write the linearized equation as follows

$$\ddot{\epsilon} \quad + \quad 4\,\lambda(t)\,\epsilon \;=\; -\,\ddot{\rho}_0\,, \tag{A-5}$$

with the initial condition

$$\epsilon(t_1) = 0 \quad , \qquad \dot{\epsilon}(t_1) = 0\,, \tag{A-6}$$

where $\epsilon(t)$ is $\rho(t)$ - $\rho_0(t)$.

When two linearly independent solutions of the homogenious equation

$$\ddot{\epsilon} \quad + \quad 4\,\lambda(t)\,\epsilon \;=\; 0, \tag{A-7}$$

are known, we can easily obtain a general solution of (A-5) by the method of variation of parameters. Unfortunately, we in general don't know such linearly independent solutions. Nevertheless, it may be reasonable to take an approximate solution obtained by the WKBJ method as particular solutions of (A-7), to the present approximation. Setting

$$4\,\lambda(t) \;=\; k^2 f(t)\,, \tag{A-8}$$

where k is an asymptotic parameter, we have two approximate solutions to the order (1/k)

$$\epsilon_{\pm} \;=\; f^{-\frac{1}{4}} \exp(\pm\, i\, k\, \tau)\,, \tag{A-9}$$

with $$\tau \;=\; \int f^{\frac{1}{2}}\, dt\,. \tag{A-10}$$

Using (A-9), we write the approximate general solution in the form

$$\epsilon(t) = C_+\epsilon_+ + C_-\epsilon_- + \frac{\epsilon_-}{\Delta}\int \epsilon_+\,\ddot{\rho}_0\,dt - \frac{\epsilon_+}{\Delta}\int \epsilon_-\,\ddot{\rho}_0\,dt\,, \tag{A-11}$$

where C_+, C_- are arbitrary coefficients and Δ is -2ik. Thus, the solution satisfying the initial condition (A-6) can be described as follows

$$\epsilon(t) = \frac{1}{\Delta}\left\{ \epsilon_- \int_{t_1}^{t} \epsilon_+ \ddot{\rho}_0 \, dt' - \epsilon_+ \int_{t_1}^{t} \epsilon_- \ddot{\rho}_0 \, dt' \right\}. \tag{A-12}$$

Repeated integration by parts yields

$$\epsilon(t_2) = \frac{1}{\Delta}\left\{ -\epsilon_-(t_2)\int_{t_1}^{t_2} \dot{\epsilon}_+ \dot{\rho}_0 \, dt' + \epsilon_+(t_2)\int_{t_1}^{t_2} \dot{\epsilon}_- \dot{\rho}_0 \, dt' \right\}, \tag{A-13}$$

$$\dot{\epsilon}(t_2) = \frac{1}{\Delta}\left\{ -\dot{\epsilon}_-(t_2)\int_{t_1}^{t_2} \dot{\epsilon}_+ \dot{\rho}_0 \, dt' + \dot{\epsilon}_+(t_2)\int_{t_1}^{t_2} \dot{\epsilon}_- \dot{\rho}_0 \, dt' \right\}. \tag{A-14}$$

Finally, using (A-13),(A-14), we can obtain

$$\beta(t_2) = \left[\lambda(t_2)^{-1/4} - \epsilon_-(t_2) I_+ + \epsilon_+ I_- \right]^2, \tag{A-15}$$

$$\dot{\beta}(t_2) = 2\left[\quad '' \quad \right]\left[-\dot{\epsilon}_-(t_2) I_+ + \dot{\epsilon}_+ I_- \right], \tag{A-16}$$

where

$$I_\pm = \frac{1}{\Delta}\int_{t_1}^{t_2} \dot{\epsilon}_\pm \dot{\rho}_0 \, dt'. \tag{A-17}$$

Substituting these results into (2-12),(2-13), we have a formal expression for the maximum change in the adiabatic invariant for the system (1-1).

It may be difficult to evaluate asymptotically the integral $I_\pm$ by the so-called saddle point method/6/, since the range of integration is not over infinite but finite. Therefore we will not examine this problem extensively.

References and Footnotes

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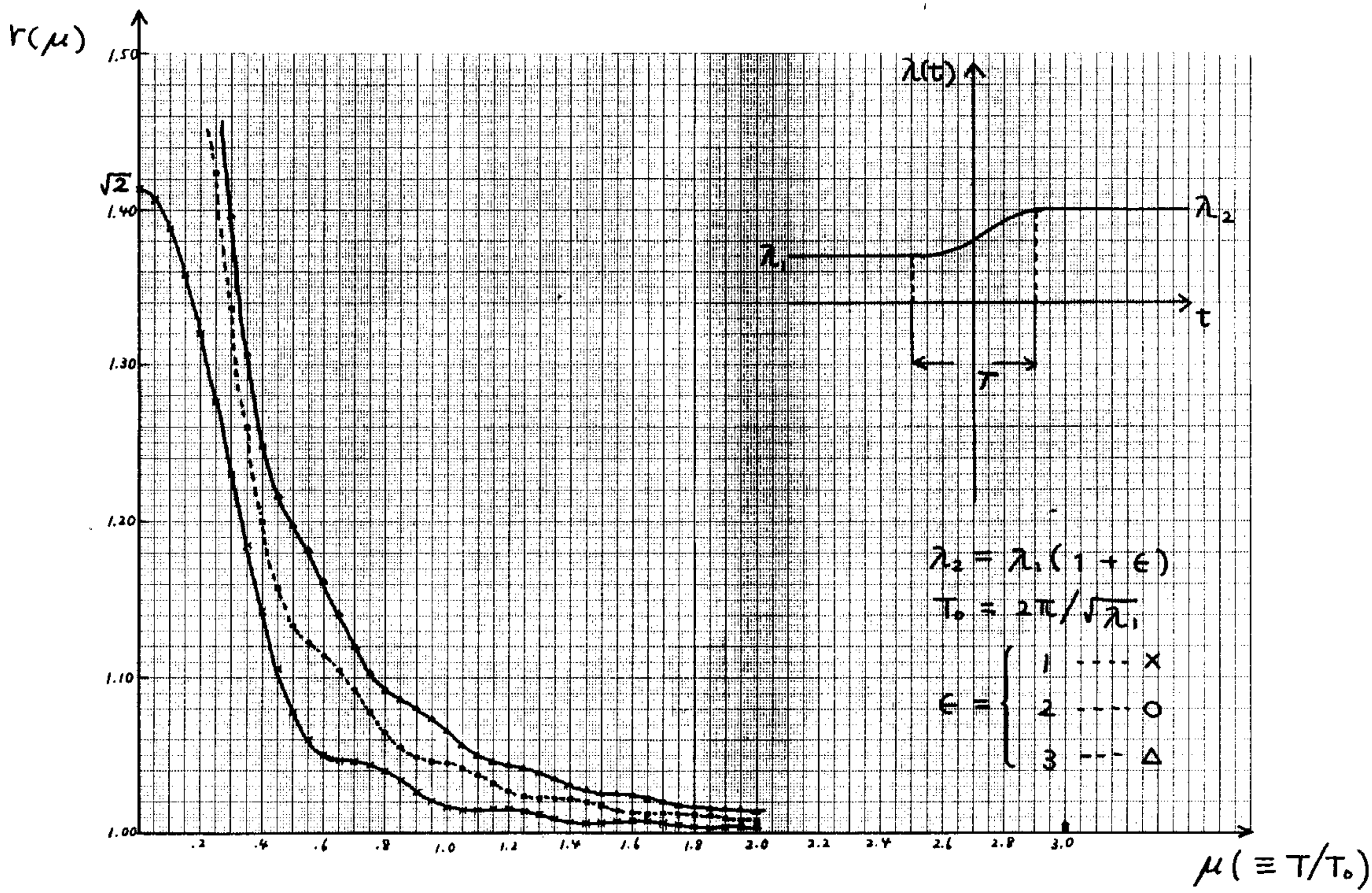
Examples of eq.(3-2) which can be strictly solved are given in this paper.



Adiabatic Ratio
$r(\mu)$
1.50
$\sqrt{2}$
1.40
1.30
1.20
1.10
1.00
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0
$\lambda(t)$
λ_1
λ_2
t
τ
$\lambda_2 = \lambda_1(1+\epsilon)$
$T_0 = 2\pi/\sqrt{\lambda_1}$
$\epsilon = \begin{cases} 1 & \times \\ 2 & \circ \\ 3 & \triangle \end{cases}$
$\mu\,(\equiv T/T_0)$
Normalized Time

Fig.4