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The Overshoot of Individual Bunch Longitudinal Instabilities as Overlapping of Several Resonating Modes

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Introduction

In this paper we want to show an example of a dynamical system with a behavior that can be described by the interaction of several resonances, and therefore it can be treated with the technique proposed by B. Chirikov[1] of many resonances overlapping. This procedure would then lead to a stochastic system.

At the same time we hope to suggest a solution to a problem which has been found typical in particle accelerators and storage rings. We refer here to the phenomena of lengthening and widening of bunches, in one word of bunch growth. In fact it has been found that high intense bunches could be unstable against coherent oscillations and grow in size until some critical spreads are reached[2]. Analytical theories have been provided that can predict the instabilities and can be used to estimate initial growth times and thresholds; but they are linear and are not capable of predicting when the bunch growth will eventually stop. Experiments and computer simulations have shown that indeed this occurs in a manner which is referred to as overshoot. The first overshoot formula was derived from Dory's computer simulation[3] and the main results are shown in Fig. 1. The formula which is commonly used is

$$\Delta_f \sim \Delta_i \sim \Delta_{th}^2 \tag{1}$$

which relates the initial spread Δ_i to the final spread Δ_f by means of a threshold value Δ_{th}. Yet so far not much of an explanation was given to (1). Only recently F. Sacherer[4] suggested that the observed bunch growth could be explained as the interaction of two modes of instability. We want to pursue this approach

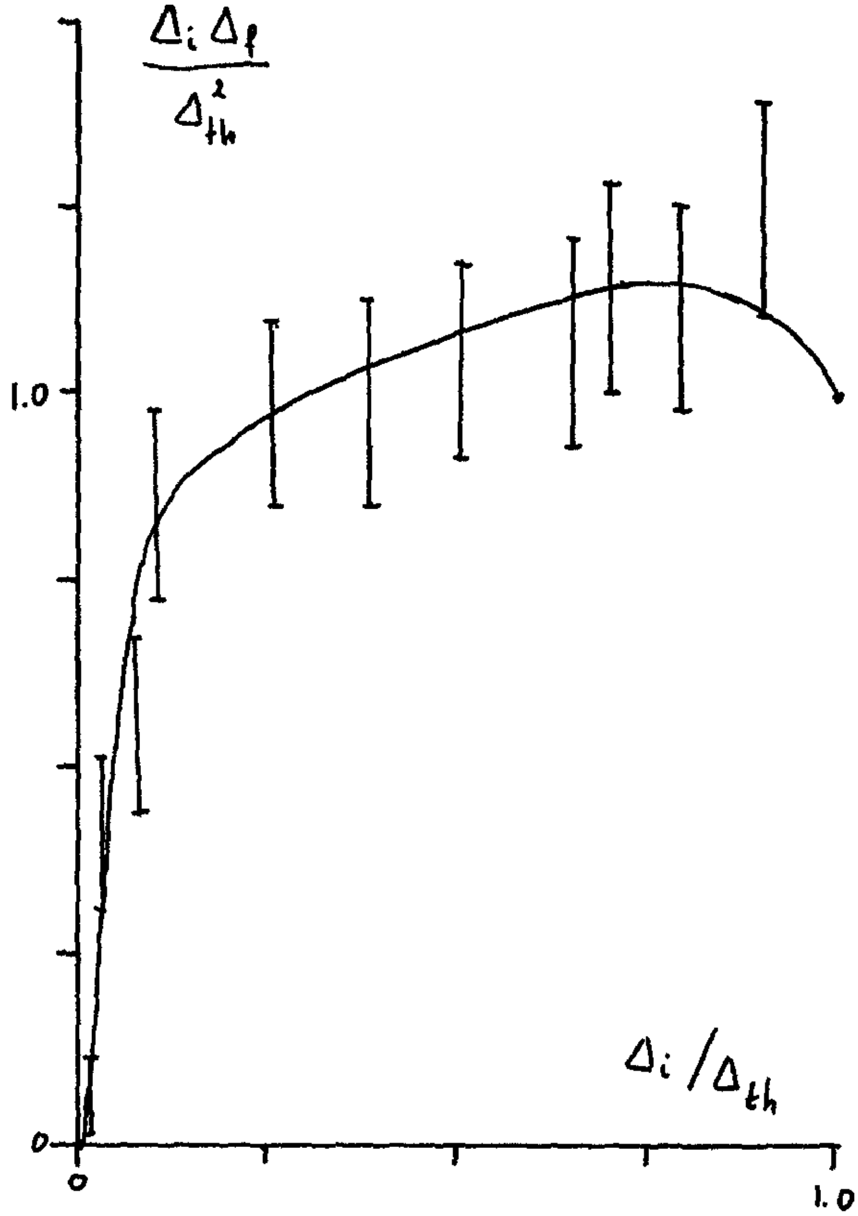

Figure 1. Dory's overshoot effect

a little closer here. Our outline is the following. A bunch can be unstable against coherent longitudinal oscillations, according to its original spread. Several modes of instabilities could be enhanced at the same time. Each of them of course can be treated independently only at the onset of the instability. But as long as the bunch grows these modes become larger and larger and at a certain moment they begin to interact with each other. This is the breaking point of the conventional theory. But if each mode can be described as a nonlinear resonance in the longitudinal phase-plane the interaction of the unstable modes would lead to the overlapping of the corresponding resonances. When this happens the motion becomes stochastic, random, and the coherence should disappear and the bunch would become stable. Clearly this approach would lead to a relation between the initial and final bunch spreads.

The Analysis

1. Consider a bunched beam of charged particles circulating in a conventional storage ring. Let $I(\theta,t)$ be the local current in the beam at the time t and at the azimuth θ around the ring. Because of the periodicity we can decompose the current in Fourier harmonics and write

$$I(\theta,t) = \sum_n I_n \cos n(\theta-\omega_o t + \phi_n) \qquad (2)$$

where ω_o is the angular revolution frequency.

In the case all current-dependent effects are neglected, the motion of the particles which make up the beam is determined only by the external forces. For instance the beam is bunched because of the presence of accelerating cavities. The degree of bunching and the shape and size of the bunches depend on the amount of RF forces and on the initial conditions of the particles in the longitudinal phase space. We can reasonably assume that the RF forces and the initial conditions are matched to each other so that the beam has a stationary, equilibrium configurations, from which one can derive a current distribution as given by (2) where the parameters involved I_n, ω_o and ϕ_n are constant. In the following we can consider only a "symmetric" bunch distribution and, therefore, take $\phi_n = 0$. We shall assume that there is only one bunch without this lacking of generality. By definition

$$I_n = \frac{1}{2\pi} \int_{-\pi}^{+\pi} I(\theta) \cos n\theta \, d\theta. \tag{3}$$

Denoting with I_p the peak value of the current and with θ_p the rms bunch spread, we can take a gaussian distribution so that

$$I(\theta) = I_p e^{\frac{\theta^2}{2\theta_p^2}} \tag{4}$$

which inserted in (3) gives, in the approximation $\theta_p \ll \pi$,

$$\begin{aligned} I_n &= \frac{I_p \theta_p}{\sqrt{2\pi}} e^{-\frac{1}{2} n^2 \theta_p^2} \\ &= I_o \, e^{-\frac{1}{2} n^2 \theta_p^2} \end{aligned} \tag{5}$$

One requires also a relation between the angular spread θ_p and the relative energy spread Δ. It is not really clear to us which relation to use, because this would depend on the properties of the system under consideration, namely whether it is conservative (protons) or not (electrons). Also it depends on the assumption we make for the phenomena we want to investigate: is the instability fast enough compared to one phase (synchrotron)oscillation period?

Would filamentation occur with consequent bunch area dilution? Here for sake of simplicity we shall assume the process is Liouvillian, so that

$$\theta_p \Delta = S\omega_o/E \tag{6}$$

where S is the invariant bunch area and E the beam energy. Therefore (5) and (6) combined give also a relation between the harmonic current and the bunch height.

2. Let us now introduce a complex impedance distributed around the ring. This will create a new longitudinal force which is dependent on the beam current. The initial current distribution (2) may not be matched to this extra force and the bunch can undergo changes of shape and size. In this situation, the three major quantities appearing in (2) for each harmonic, I_n, ω_o and ϕ_n, could be changing with time. The variation with time can be different for different modes (n). Some modes will be damped, which corresponds to decreasing I_n, whereas others will be excited. Also the angular frequency can be affected in different ways, with different shifts for different modes

$$\omega_n = \omega_o + \Delta\omega_n. \tag{7}$$

At the same time also the longitudinal electric field, induced by the beam, will change accordingly

$$E_z = - \sum_n \frac{|Z_n| I_n(t)}{2\pi R} \cos(n\theta - n\omega_n t + \alpha_n) \tag{8}$$

where $2\pi R$ is the machine circumference, and

$$Z_n = |Z_n| e^{i\alpha_n}$$

is the complex impedance at the angular frequency $n\omega_o$. For simplicity we have neglected the phase factors ϕ_n, which eventually can be included with α_n.

Because of the impedance a loop is established between the motion of the beam and the beam induced field. Because of the latter, the current distribution within the bunching changes; from (5) and (6), this in turn means that the bunch also continuously changes size. On the other hand, as a consequence of this also the electric field, as can be seen from (8), continuously changes.

The amplitude time variation and the frequency shift $\Delta\omega_n$ can be calculated at the onset with the "coasting beam"[5] techniques, where one mode at a time is taken and assumed to be decoupled from the others. If one knows the complex impedance Z_n for this mode it is then possible to calculate the complex shift, of which the real part determines $I_n(t)$ and the imaginary part gives $\Delta\omega_n$ in Eq. (7). For instance, from the coasting beam theory, it is well known that in the case of uniform energy distribution, the complex shift is

$$\Delta\omega_n(\text{complex}) = \left\{ k_o^2 \frac{\Delta_i^2}{\omega_o^2} E^2 - ie \frac{k_o I_o Z_n/n}{2\pi} \right\}^{1/2} \tag{9}$$

where

$$k_o = \omega_o \frac{d\omega_o}{dE} \tag{10}$$

is a machine parameter related to the momentum compaction factor. I_o is the average beam current (n=0) and Δ_i is the initial (rms) energy spread.

A beam with uniform energy distribution is always unstable no matter what its initial size. But for more realistic distribution with tails, the expression for the complex frequency shift is more complicated than the one given by (9). In this case, though, the beam can be entirely stable against coherent oscillations if its initial spread Δ_i is large enough. There will be some modification, of course, of its initial shape toward one which matches to the external as well as beam induced fields; but the bunch area should be preserved during the change. Here we are interested in the case where the initial spread Δ_i is small enough so that the beam is unstable, at least several modes are. Then we can disregard the details of the initial bunch shape and make use of (9) for the frequency shift. Again the real part of (9) gives the growth rate of the harmonic amplitude I_n, and therefore of the beam width according to (5) and (6), the imaginary part of (9) only enters our definition (7).

3. Consider now a test particle and let us write down its equations of motion

$$\dot{\theta} = \omega(w) = \omega_o + k_o W \tag{11a}$$

$$w = \frac{e}{2\pi} \sum_n |Z_n| I_n(t) \cos(n\theta - n\omega_n t + \alpha_n) + \text{ external fields} \tag{11b}$$

where w is the angular momentum canonically conjugated to θ.

In absence of any force the particle would simply drift at the speed given by (11a) where w is a constant. The drift speed is the revolution frequency which is a function of the particle energy, or w.

There are two kinds of forces. The external forces which keep the motion of all the particles, including our test particle, tight up together in one bunch, and a current dependent force which is a consequence of the bunching. In the second of the equations of motion, (11b), we have shown explicitely the second kind of forces as one could have derived it from the discussion of the previous section and in particular from (7).

By inspecting Eqs. (11a and b) we quickly see that, as long as the various modes have different frequency shifts $\Delta\omega_n$, the test particle is experiencing a field that can be decomposed in many resonances. Each resonance would correspond to one collective mode n and is caused by the beating between the revolution frequency of the test particle $\omega(w)$ and the collective frequency ω_n. Since there are different frequencies ω_n for different modes n, there are several resonances located at different places in the longitudinal phase plane (θ,w). The situation is sketched in Fig. 2. There is a large island which corresponds to the external field "bucket" which itself is a resonance. The beam populates the inside of this bucket. But inside this bucket and within the bunch, there are other islands each of them corresponding to one collective mode n of the expansion at the r.h. side of (11b). Thus this is a typical example of interaction of several resonances in the same phase space. We can solve our problem, therefore, by making use of the techniques devised for this field. The approach is made of two steps:

i. Each mode or resonance n is treated individually and independently of all others. This is correct at the onset of the growth or instability, when the islands in Fig. 2 are reasonably small and separated from each other. In this case we can investigate the motion around a single resonance, and calculate the corresponding island size and location. The resonance width is the island width.

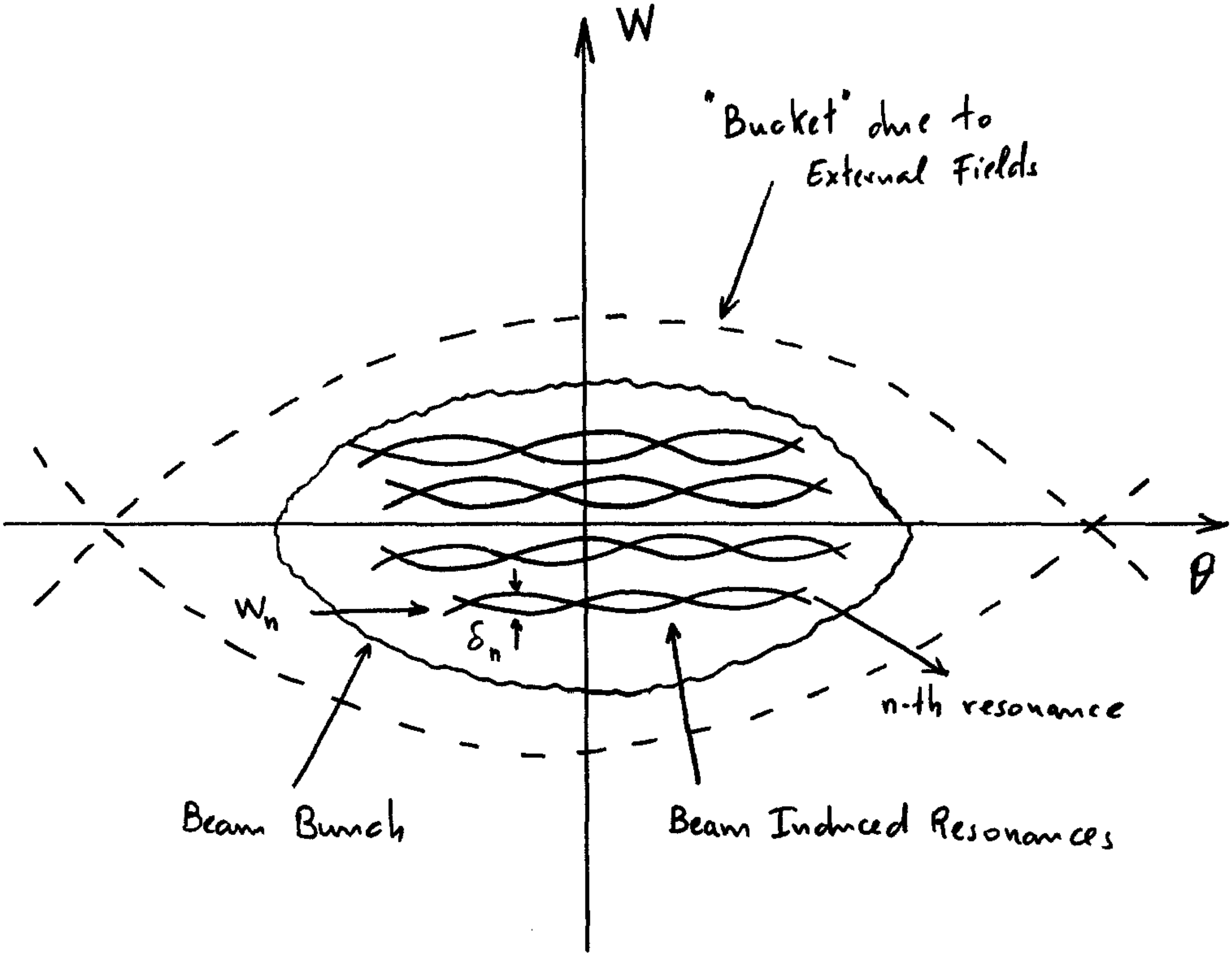

Figure 2. Distribution of Resonances in the (θ,w)-Phase Plane

Introduce the rotating phase angle

$$\psi_n = n\theta - n\omega_n t + \alpha_n - \frac{\pi}{2} \tag{12}$$

then the equation of motion around the n-th resonance is obtained by retaining only the n-th Fourier mode at the r.h. side of (11b).

$$\dot{\psi}_n = n(\omega_o - \omega_u) + nk_o w \tag{13a}$$

$$w = \frac{e}{2\pi}|Z_n| I_n(t) \sin \psi_n . \tag{13b}$$

It is well known that these equations lead to an island in the (θ-w), phase-plane located at

$$w_n = \frac{\Delta\omega_n}{k_o} \tag{14}$$

and having full width

$$\delta_n = \sqrt{8 \frac{e|Z_n| I_n(t)}{n|k_o|}} \quad . \tag{15}$$

The location of the resonance is then given by the shift $\Delta\omega_n$ and does not change, whereas the width δ_n depends on the harmonic current, and therefore an increase of I_n leads also to an increase of δ_n.

ii. Resonance Overlapping. At certain moments of the growth two neighboring islands will overlap. This could be a local phenomena involving only some region of the phase-plane. When this occurs resonances cannot be treated isolated any longer; they are now interacting with each other. From the instability point of view this means that now several collective modes are coupled to each other and because their amplitude is too large, they are no longer normal to each other, but the development of one depends on (and influences the development of) the others.

According to Boris Chirikov[1] the condition of resonance overlapping is essential for the motion of a test particle to become "stochastic". The randomness of the motion, we could infer then, could be also regarded as a condition to reduce by mixing the amount of coherence in the motion within the beam bunch. Simply put, we draw a plausible assumption that the bunch growth will stop when the motion of a test particle becomes to some degree stochastic, namely when neighboring resonances associated to neighboring modes overlap. From (15), (5) and (6) it is then possible to estimate the final stability in the sense so specified.

4. The separation between two adjacent resonances in proximity of the mode is obviously given by

$$s_n = \frac{\Delta\omega_{n+1} - \Delta\omega_n}{k_o} \sim \frac{\Delta\omega_n - \Delta\omega_{n-1}}{k_o} .$$

In case the first term is predominant compared to the second one at the r.h. side of (9), we have

$$s_n \sim \frac{eI_o\omega_o}{4\pi|k_o|\Delta_i E}\left(\frac{X_{n+1}}{n+1} - \frac{X_n}{n}\right) \tag{16}$$

where X_n is the imaginary part of Z_n, namely the reactance.

The Chirikov condition for resonance overlapping is

$$\delta_n \sim s_n . \tag{17}$$

Combining (15) and (16) in (17) gives for the final harmonic amplitude when overlapping occurs

$$I_{nf} = \frac{eI_o^{\,2}\ \omega_o^{\,2}}{128\pi^2|k_o|\Delta_i^{\,2}E^2}\ G_n \tag{18}$$

where

$$G_n = \frac{\left(\frac{X_{n+1}}{n+1} - \frac{X_n}{n}\right)^2}{\frac{|Z_n|}{n}} \ . \tag{19}$$

Let us introduce the final relative energy spread Δ_f corresponding to I_{nf}, then from (4) and (5)

$$I_{nf} = I_o\ e^{-n^2 \frac{S^2\omega_o^{\,2}}{2E^2\Delta_f^{\,2}}} \tag{20}$$

then from (18) and (20) we finally derive

$$\Delta_i^{\,2}\ e^{-n^2 \frac{S^2\omega_o^{\,2}}{2E^2\Delta_f^{\,2}}} = \frac{eI_o\omega_o^{\,2}\ G_n}{128\pi^2|k_o|E^2} \ . \tag{21}$$

This is an overshoot formula in the sense it relates the initial bunch spread to the final one. It still depends on n, the harmonic number, because it was derived from the assumption of a local overlapping. It can be applied, of course, only when the initial spread Δ_i is less than some critical value (the threshold).

Let us investigate (21) in more detail. Define

$$x = \frac{2E^2\Delta^2}{n^2S^2\omega_o^{\,2}} \quad , \qquad A = \frac{eI_o\ G_n}{64\pi^2|k_o|n^2S^2} \ .$$

Then we can plot x_f versus x_i as shown in Figure 3.

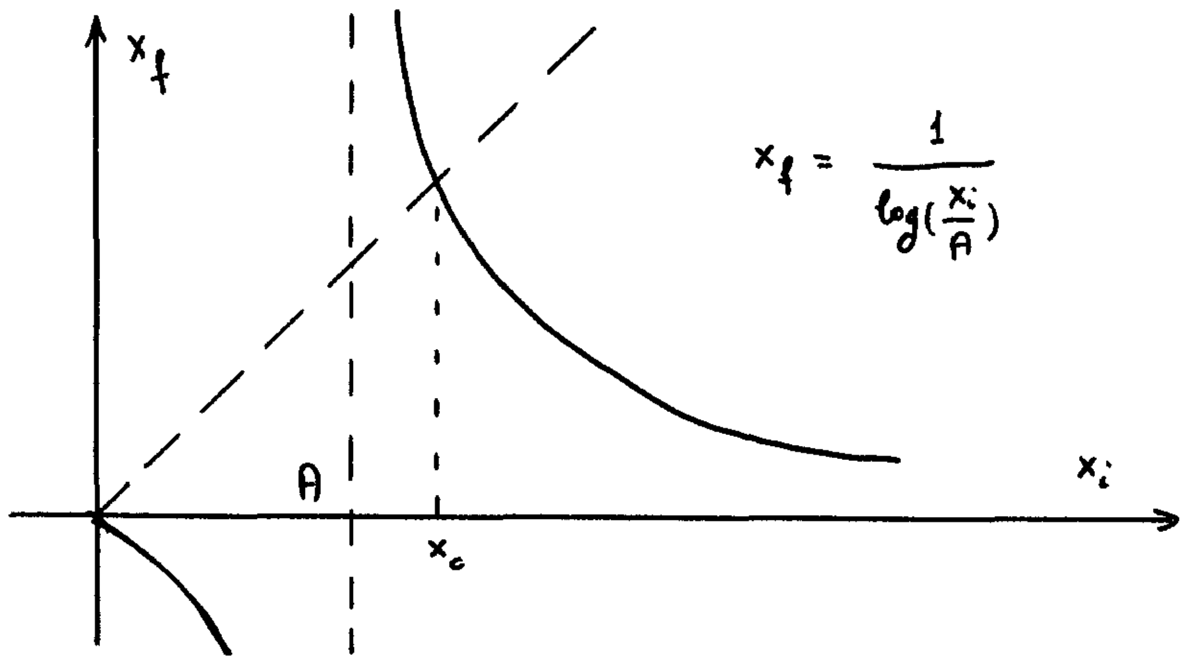

Figure 3. Graphic view of the overshoot

Both x_i and x_f, of course are positive quantities. Therefore for $x_i<A$ no overshoot is possible. This corresponds to the fact that the width of a resonance is limited by the maximum value the Fourier harmonic I_n can reach. This is I_o as one can see from (20). For $x_i>x_c$, where x_c is the solution of

$$x_c = 1/\log(\frac{x_c}{A}),$$

the final spread would be smaller than the initial one which corresponds to the stability of the bunch. In conclusion the overshoot applies only in the range

$$A<x_i<x_c$$

where, if x_i is not too large compared to A,

$$x_f(x_i - A) \approx A$$

which is similar to the one commonly used.

5. In the case the impedance is made of a pure inductance (plus a resistance) and, therefore, increases linearly with the frequency, the G-factor, Eq. (19), is identically zero and overlapping does not occur. In fact, in this case all the modes suffer the same amount of frequency shift, and the corresponding islands will be sitting on top of each other. It is well known that this case corresponds then to one single resonance with highly nonlinear driving term.

As an alternative we want, though, to show another case where islands are produced sitting around different frequencies.

This time we shall assume that the impedance is lumped in one spot of the ring, say at $\theta = 0$, then the longitudinal field induced by the beam harmonic (8) is now replaced by

$$E_z = -\delta_p(\theta) \sum_n \frac{|Z_n| I_n(t)}{R} \cos(n\theta - n\omega_n t + \alpha_n) \tag{22}$$

where $\delta_p(\theta)$ is a periodic delta function with period 2π, which can be decomposed in the following way

$$\delta_p(\theta) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_m \cos m\theta$$

which included in (22) gives

$$E_z = -\sum_{mn}\sum \frac{|Z_n| I_n(t)}{2\pi R} \cos(m\theta - n\omega_n t + \alpha_n).$$

The case we have analyzed before, eventually, would correspond to having singled out the term m = n from all the others. Here we could do the opposite and consider only one harmonic n, so that

$$E_z = -\frac{|Z_n| I_n(t)}{2\pi R} \sum_m \cos(m\theta - n\omega_n t + \alpha_n). \tag{24}$$

which can be inserted at the r.h. side of Eq. (11b). And again we would have an example of a dynamical system composed of several interacting resonances. The approach of course remains the same.

For the first step where a single resonance is treated, it is convenient to introduce the following rotating phase angle

$$\psi_{nm} = m\theta - n\omega_n t + \alpha_n - \frac{\pi}{2} . \tag{25}$$

Then the equations of motion for the single resonance are

$$\dot{\psi}_{nm} = (m-n)\omega_o - n\Delta_n + mk_o w \tag{26a}$$

$$\dot{w} = \frac{e}{2\pi} |Z_n| I_n(t) \sin\psi_{nm}. \tag{26b}$$

The island is located at

$$w_{nm} = \frac{n\Delta\omega_n + (n-m)\omega_o}{mk_o} \tag{27}$$

and has a full width

$$\delta_{nm} = \sqrt{8\frac{e|Z_n| I_n(t)}{m|k_o|}} \tag{28}$$

In the approximation that

$$\Delta\omega_n << \omega_o$$

and

$$m \sim n >> 1$$

the separation between two resonances next to each other and generated by the same mode of instability is

$$s_n \sim \frac{\omega_o}{n|k_o|} \tag{29}$$

independent of the beam parameters: current and spreads.

Let us consider the local overlapping of the resonances produced by the same mode n ($m \sim n$); then this is reached for

$$I_{nf} = \frac{\omega_o^2}{8en|Z_n||k_o|} \tag{30}$$

which is obtained by combining (17) with (28) and (29). Also from (20) and (30)

$$e^{-n^2 \frac{S^2 \omega_o^2}{2E^2 \Delta_f^2}} = \frac{\omega_o^2}{8enI_o|Z_n||k_o|} \quad .$$

Again define

$$x = \frac{2E^2\Delta^2}{n^2\delta^2\omega_o^2}$$

and

$$B = \frac{8enI_o|Z_n||k_o|}{\omega_o^2}$$

then

$$x_f = \frac{1}{\log B} \tag{31}$$

which we plotted in Fig. 4

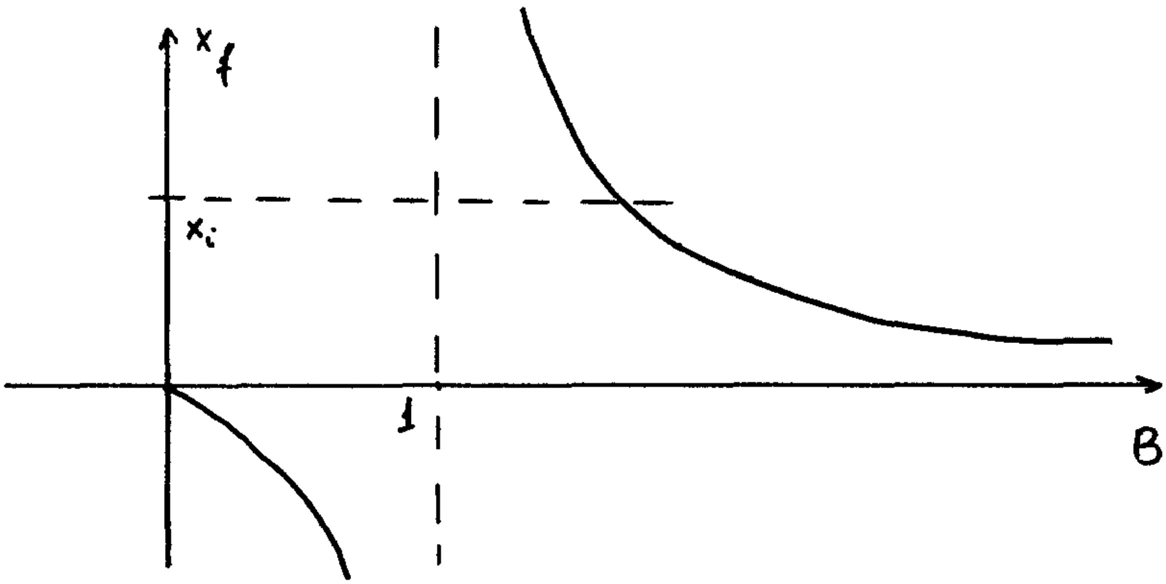

Figure 4. Graphic view of the overshoot

When B<1 the overshoot is not possible, at least locally. Eq. (31) gives the final beam size only in the case $x_f > x_i$ as shown in Fig. 4.

We realize that our presentation has been sketchy and sometimes inconsistent. But our main interest was to propose a new approach to explain bunch growth in particle accelerators and storage rings. We are confident that a closer look at the problem, as we have outlined here, but with more detail, can provide a reasonable explanation of the overshoot phenomena.

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