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EFFECTS OF FIELD ERRORS AND RIPPLES IN THE MAIN RING

A. G. Ruggiero

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SUMMARY

The purpose of this paper is to describe a possible beam instability caused by the coupling of the power supplies ripple and the remanent field errors.

Numerical application to the main ring of our analysis agrees more or less with the observed decay of 7 GeV coasting beam. There is the qualitative disagreement that no change in beam decay was observed when the quadrupole ripple was reduced by the addition of a filter. However, this test was performed rather hurriedly and did not take into account the sextupole components of the remanent field in the dipoles where the ripple was not reduced.

More careful experiments on the effect of reducing the ripple should be carried out.

The author wishes to thank S. Ohnuma and L. C. Teng for their valid and decisive contributions to this work.

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THE ANALYSIS OF MOTION

1. The equation of motion with magnetic field ripple and magnetic field random errors is

$$y'' + [K_o(s) + K_p(s)]y = F(s), \tag{1}$$

where F(s) is the random error function and $K_p(s)$ is the quadrupole error caused by the field ripple. Actually $K_p(s)$ is a function of s and of the time t; but for a particle with constant velocity v, the t-dependence of K_p can be converted in s-dependence by the relation: s = vt.

$K_o(s)$ is the main magnetic structure function.

We assume two independent solutions of the homogeneous equation associated to (1) can be written with the form

$$y(s) = w(s)e^{\pm i\psi(s)}, \quad \psi'(s) = \frac{1}{w^2}. \tag{2}$$

If $K_p(s) = 0$ and $K_o(s)$ being periodic, also the amplitude function w(s) is periodic. More generally, for $K_p(s) \neq 0$, w(s) has not the periodicity of $K_o(s)$ but that resulting by the combination of the ripple and revolution frequencies.

Let ν_o and β_o be respectively the betatron oscillations number per turn and the amplitude Courant-Snyder function for $K_p(s) = 0$.

Then, introduce the phase-advance ϕ and the total betatron oscillations number per turn ν by

$$d\phi = \frac{ds}{\nu_o\beta_o} = \frac{ds}{\nu w^2} . \tag{3}$$

Taking ϕ as running variable instead of s, the two independent solutions (2) can be written with the form

$$y(\phi) = w(\phi)e^{\pm i \int^{\phi} \nu(\phi)d\phi}, \qquad \nu = \frac{\nu_o \beta_o}{w^2} \tag{4}$$

According to (3), ϕ adds 2π each turn, and $\nu = \nu_o$ if $K_p = 0$, when $w^2 = \beta_o$.

With the Floquet transformations

$$\eta(\phi) = y/w \qquad \text{and} \qquad d\phi = \frac{ds}{\nu_o \beta_o},$$

Eq. (1) transforms to

$$\eta'' - \frac{\nu'}{\nu}\eta' + \nu^2\eta = w^3\nu^2 F(\phi), \tag{5}$$

where now prime denotes derivation to ϕ. It is easy to see the Eqs. (4) are two independent solutions of the homogeneous equation associated with (5).

A particular integral of the inhomogeneous Eq. (5) with initial conditions $\eta = 0$ and $\eta' = 0$ at $\phi = 0$ is

$$\eta(\phi) = \int_0^{\phi} \frac{(\nu_o \beta_o)^{\frac{3}{2}}}{\nu^{\frac{1}{2}}} F(\omega) \sin\left(\int_{\omega}^{\phi} \nu dx\right) d\omega.$$

In particular after m revolutions counted from $\phi = 0$ it is

$$\eta_m = \eta(2\pi m) = \int_0^{2\pi m} \frac{(\nu_o \beta_o)^{\frac{3}{2}}}{\nu^{\frac{1}{2}}} F(\omega) \sin\left(\int_{\omega}^{2\pi m} \nu dx\right) d\omega \tag{6}$$

2. Let us take for the random errors function

$$F(\omega) = \frac{L}{\rho\nu_o} \sum_{n=1}^{\infty} \frac{\varepsilon_n}{\beta_{o_n}} \delta\left(\omega - n\frac{2\pi}{M}\right) \tag{7}$$

L = bending magnet length,

ρ = bending magnets curvature radius,

ε_n = relative error of the magnetic field in the n-th magnet: $\varepsilon_n = \varepsilon_{n+M} = \varepsilon_{n+2M} = \ldots,$

β_{o_n} = β_o at the n-th magnet position,

M = number of magnets per turn.

The numbers $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_M$ have rms value $\langle\varepsilon\rangle$ and average value very close to zero. In the following we shall write also

$$\varepsilon_n = \alpha_n \langle\varepsilon\rangle. \tag{8}$$

Inserting (7) in (6) and antitransforming from η to y yield

$$y_m = \frac{L}{\rho} \frac{\nu_o \beta_{o_m}^{\frac{1}{2}}}{\nu_m^{\frac{1}{2}}} \sum_{n=1}^{mM} \beta_{o_n}^{\frac{1}{2}} \frac{\varepsilon_n}{\nu_n^{\frac{1}{2}}} \sin \int_{n\frac{2\pi}{M}}^{2\pi m} \nu d\phi.$$

If we take

$$\beta_{o_m} = \beta_{o_n} = \overline{\beta}_o \text{ and } \nu_m = \nu_n = \nu_o$$

we certainly do not make a large error and we have, with (8)

$$y_m = \frac{L}{\rho} \overline{\beta}_o \langle\varepsilon\rangle \sum_{n=1}^{mM} \alpha_n \sin \int_{n\frac{2\pi}{M}}^{2\pi m} \nu(\phi) d\phi.$$

Let us introduce complex notation with

$$P_m = \sum_{n=1}^{mM} \alpha_n \exp\left\{ i \int_{n\frac{2\pi}{M}}^{2\pi m} \nu(\phi)\,d\phi \right\}.$$

It is

$$y_m = \frac{L}{\rho}\,\overline{\beta}_o \langle\varepsilon\rangle \mathrm{Im}\left\{P_m\right\}, \qquad y_m' = \frac{L}{\rho}\,\overline{\beta}_o \langle\varepsilon\rangle \mathrm{Re}\left\{P_m\right\}.$$

3. Because the magnetic field ripple has the form of an oscillation, presumably we can specify ν also as an oscillation in ϕ with period N, which has the meaning of number of turns but is not necessarily an integer. We write

$$\nu = \nu_o + r \sin\frac{\phi}{N}, \qquad r \ll \nu_o,\ \nu_o \text{ not integer}$$

so that

$$P_m = e^{i\left(2\pi m\nu_o - rN\cos 2\pi\frac{m}{N}\right)} \times \sum_{n=1}^{mM} \alpha_n e^{-i\left(2\pi\frac{n}{M}\nu_o - rN\cos 2\pi\frac{n}{MN}\right)}. \tag{9}$$

Let us break the summation in two sums:

$$\sum_{n=1}^{mM} = \sum_{n=1}^{M} + \sum_{n=M+1}^{2M} + \ldots + \sum_{n=(k-1)M+1}^{kM} + \ldots + \sum_{n=(m-1)M+1}^{mM}$$

$$= \sum_{n=1}^{M} \sum_{k=1}^{m} \tag{10}$$

and make use of the following relation

$$e^{izcost} = \sum_{\ell=-\infty}^{+\infty} (i)^{\ell} J_{\ell}(z) e^{i\ell t} \tag{11}$$

where J_ℓ is the Bessel function of the ℓ-th order.

With (10) and (11) we have for P_m

$$P_m = e^{i\left(2\pi m\nu_o - rN\cos 2\pi\frac{m}{N}\right)} \sum_{\ell=-\infty}^{+\infty} \sum_{n=1}^{M} \left\{(i)^{\ell} \alpha_n J_{\ell}(rN) \times \right.$$

$$\left. \times\, e^{i2\pi\left(\frac{\ell}{N}-\nu_o\right)\frac{n}{M}} \; e^{i\pi(m-1)\left(\frac{\ell}{N}-\nu_o\right)} \frac{\sin\pi m\left(\frac{\ell}{N}-\nu_o\right)}{\sin\pi\left(\frac{\ell}{N}-\nu_o\right)} \right\}. \tag{12}$$

4. We distinguish two possible cases:

(a) N and ν_o are such that $\frac{\ell}{N}-\nu_o$ is not an integer for whatever value of ℓ. Then the quantity

$$\sin\pi\left(\frac{\ell}{N}-\nu_o\right)$$

can never be zero. The ℓ-th term in the summation expressing P_m is thus oscillating with m with periodicity of $N/(\ell-N\nu_o)$ turns and amplitude

$$\frac{J_{\ell}(rN)}{2\sin\pi\left(\frac{\ell}{N}-\nu_o\right)} \left| \sum_{n=1}^{M} \alpha_n e^{i2\pi\left(\frac{\ell}{N}-\nu_o\right)\frac{n}{M}} \right|.$$

P_m is, then, a limited quantity for whatever value of m, and, hence, the motion is stable, although P_m can have also a very large magnitude depending on how much $\frac{\ell}{N}-\nu_o$ is close to an integer.

(b) N and ν_o are such that $\frac{\ell}{N}-\nu_o$ is an integer at least for one value of ℓ. The resonance condition is

$$\frac{\ell}{N}-\nu_o = g \tag{13}$$

where ℓ and g are integer numbers.

Assume that (13) is satisfied by the couple of values $\ell = \ell_o$ and $g = g_o$, namely

$$\frac{\ell_o}{N}-\nu_o = g_o. \tag{14}$$

Subtracting (14) from (13) side by side gives

$$\frac{\ell-\ell_o}{g-g_o} = N. \tag{15}$$

Thus any other couple of values ℓ,g must satisfy (15). Because $\ell-\ell_o$ and $g-g_o$ are integer numbers, the condition (13) can be satisfied by no more than one couple of values ℓ,g if N is an <u>irrational</u> number.

If N is a rational number we can write

$$N = \frac{S}{I}$$

where S and I are integer numbers, relatively prime. In this case (15) can be replaced by

$$\ell = \ell_o+hS \qquad\qquad g = g_o+hI \tag{16}$$

where h is any integer number.

Special values of ℓ_o, g_o are obtained, splitting ν_o in its integer p and fractional δ part ($\nu_o = p+\delta$), so that

$$g_o = -p \qquad\qquad \ell_o = \delta N. \tag{17}$$

The second of these relations is the resonance condition. We can say that there is a resonance when the quantity δN is an integer number.

Let us neglect the steady oscillatory terms in the summation (12). This is a good approximation for P_m, of course, after a large number of turns m. We have with the help of (16) and (17)

$$P_m \simeq e^{i\left(2\pi m\nu_o - rN\cos\frac{2\pi m}{N}\right)} \times$$

$$\times\, m \sum_{h=-\infty}^{+\infty} \sum_{n=1}^{M} \alpha_n e^{i2\pi g\frac{n}{M}} J_{\delta N+hS}(rN)\, e^{i\left[\frac{\pi}{2}(\nu_o+g)N+\pi(m-1)g\right]}. \tag{18}$$

5. For g<<M we have

$$\sum_{n=1}^{M} \alpha_n e^{i2\pi g\frac{n}{M}} \approx \left(\sum_{n=1}^{M} \alpha_n{}^2\right)^{\frac{1}{2}} = \sqrt{M}\ . \tag{19}$$

On the other hand, the Bessel function with constant argument depends rather strongly on the order, with maximum usually for low orders. Thus we can retain only the term h = 0 in the summation (18) and neglect all the others.

It is, neglecting also the ripple modulation and any phase angle,

$$P_m \simeq m\sqrt{M}\, J_{\delta N}(rN)\, e^{i\pi m(p+2\delta)}. \tag{20}$$

From (20) we infer that the particle executes an oscillation with amplitude increasing linearly with the number of turns performed m,

$$y_m = a_m \sin[\pi m(p+2\delta)],$$

$$a_m = \frac{L}{\rho}\, \bar{\beta}\langle\varepsilon\rangle m\sqrt{M}\, J_{\delta N}(rN). \tag{21}$$

APPLICATIONS

6. General. To check our analytical results we performed also a simulation of the process on the computer. This takes randomly M numbers with normal distribution of unit variance. These numbers are used in the place of the α_n's to calculate P_m, according to Eq. (9) but with the summation split as in Eq. (10). The only difference is that now ν is modulated as follows

$$\nu = \nu_o - r\sin(\phi/N).$$

Except one case we shall mention later, we always took N as a rational number and required the computer to output P_m every S=NI turns. Indeed, as proved in the Appendix, P_m adds a constant quantity each S turns if we are on a resonance.

Vice versa this is a very simple way to check on the computer whether a special case is a resonance.

(a) The results shown in Fig. 1 are to check that δN = integer is the resonance condition. We ran 6 cases with several N and δN, and all the other quantities constant. In particular, we used the same series of M = 10 random numbers α_n's.

We found confirmation that the slope of the resonance is essentially given by the Bessel function $J_{\delta N}$. All the cases with the same δN have the same slope and this decreases as δN increases.

(b) All the cases in Fig. 1 satisfy also the condition $\nu_o N$ = integer. To prove that this is not an essential condition for resonance, we ran other 6 cases. The results are shown in Fig. 2.

All these cases, as well as all the other cases shown in Figs. 3, 4 and 5, have been processed with the same identical series of M = 20 random numbers α_n's. We can see in Fig. 2 that also the two cases with p = 3 and pN $\neq$ integer are resonances because also for them δN = integer. Nevertheless, it must be noted that the quantity $\nu_o N$ (integer or not) seems to have an effect on the slope of the resonance. This can be seen also in Fig. 3 where the special case of

$N = 10+1/3$ has been run for $p = 3$ and $p = 4$. This could be explained by the fact that $M = 20$ is not enough larger than $p = 4$ or $p = 3$, so that (19) results in a too poor approximation.

(c) Fig. 3 shows results of the cases we ran with all the parameters constant, with the exception of N. The resonance is for $N = 10+1/3$, and we wanted to show that as N moves away from this value we do not have a resonance any more, but the slope bends and even oscillates. Nevertheless, values of N very close to the resonance value lead to oscillations of very large amplitude.

(d) A group of cases with N constant and ν_o changing are shown in Fig. 4. The result is the same effect observed in (c).

(e) Finally, we processed a case with N irrational. Because now $S \to \infty$, we had to require the computer to output P_m every turn. The results are shown in Fig. 5. As all the other cases, also this has been run with the Double Precision feature available with the computer. Thus the first 17 significant digits of N and δ have been surely assimilated by the computer. Only in this way and because we carried on the calculation up to $m = 100$, the definition of "irrational number" makes sense.

The slope observed in Fig. 5 is very close to the slope of the cases in Fig. 2 with $p = 3$ and $N\delta = 2$.

7. <u>NAL Main Ring</u>. We have

$\frac{L}{\rho}\overline{\beta}_o \sim 40$ cm $\qquad N = 70$

$M = 774 \qquad r = 0.1$

$\langle\varepsilon\rangle = 5\mathrm{x}10^{-4} \qquad \delta = 0.2$

which give

$$a_m = m\ 0.114 \times 10^{-3}\ \mathrm{cm}. \tag{22}$$

If the main ring semiaperture is about 5 cm the beam will hit the wall, according to (22), after $\sim 4 \cdot 10^4$ turns corresponding to ~0.8 seconds.

These numbers agree fairly well with the experimental beam observations.

The numerical results of the process simulated in the computer are shown in Fig. 6 only for m up to 200. For such low values of m, of course, as expected, the steady oscillatory terms in (12) are predominant and the rising terms in (18) are negligible to all.

The trajectory is mainly an oscillation at the frequency $\sim 1/\delta$ turns. This frequency is modulated by the field ripple at the frequency of N turns. Besides, we can observe that the oscillation level is modulated too at the same ripple frequency of N turns.

When m increases the oscillation level also increases at constant speed. For instance, from our numerical computation we got

m	P_m
70	0.0262
140	0.0523
210	0.0786
.	.
.	.
.	.

which averaging gives

$$a_m = m\ 0.0075 \cdot 10^{-3}\ \text{cm}.$$

This quantity is a factor 15 smaller than that in (22), but it is just one case. We ran the computer 10 times with 10 different series of random numbers, α_n, and we carried out P_m at m = 70. These are the results

	P_{70}	a_m/m (in cm)
1	0.0262	0.0075×10^{-3}
2	-0.0603	-0.0172
3	0.160	0.0457
4	0.310	0.0886
5	-0.0256	-0.0073
6	-0.164	-0.0469
7	-0.424	-0.1211
8	-0.0967	-0.0276
9	-0.157	-0.0449
10	0.236	0.0674
	rms value	0.0588×10^{-3}

The rms value is in good agreement with the analytical result (Eq. 21). Then, we infer that in order to have an order of magnitude for P_m, Eq. (21) does very well.

8. The most important parameters r, δ and N enter all together the expression

$$J_{\delta N}(rN).$$

Values of this quantity are reported in Table 1 for some numbers r, δ, N.

TABLE 1

N	r	δ	$J_{\delta N}(rN)$ =	....
60	0.1	0.2	$J_{12}(6)$	$0.5452\ 10^{-3}$
70	0.1	0.2	$J_{14}(7)$	$0.2052\ 10^{-3}$
80	0.1	0.2	$J_{16}(8)$	$0.7801\ 10^{-4}$
70	0.05	0.2	$J_{14}(3.5)$	$\sim 0.2\ 10^{-7}$
70	0.10	0.2	$J_{14}(7.0)$	$0.2052\ 10^{-3}$
70	0.15	0.2	$J_{14}(10.5)$	$\sim 0.2\ 10^{-1}$
70	0.1	0.1	$J_{7}(7)$	0.2336
70	0.1	0.2	$J_{14}(7)$	$0.2025\ 10^{-3}$
70	0.1	0.3	$J_{21}(7)$	$0.2966\ 10^{-8}$

Irrelevant is the dependence on the periodicity N. There is only a factor 2 or 3 when N changes from 70 down to 60 or from 70 up to 80.

Significant, instead, is the dependence on the amplitude of the ripple r and the tuning of the machine δ. Reducing the ripple only by a factor 2 has the effect of also reducing a_m, but by a factor 10^4, which means the beam (if it can survive) hits the wall after ~$4 \cdot 10^8$ turns or ~10^4 seconds.

Also, the dependence on the tuning δ is very critical. Assuming that the unperturbed ν_o of the machine is 20.2, detuning the machine down to 20.1 has the effect of having a more unstable beam, which should hit the wall after only ~40 turns or ~1 ms. But tuning the machine up to 20.3 keeps the beam much more stable. In this case a_m decreases by a factor 10^5, and the beam would hit the wall after 10^5 seconds. That because the stability of the beam depends on how much ν_o is close to an integer number.

9. The above numbers refer to the case all the particles in the beam have the same ν_o. Let us consider the graph in Fig. 7 for a beam with a ν_o-spread. To continue to refer to the NAL main ring we assume that ν_o = 20.20 is the average value across the beam and to have a ν_o-spread of about ±0.05. If N = 70 (but the situation does not change very much if N has any other value around 70), we see from Fig. 7 that the beam contains 7 resonances all corresponding to δN between 11 and 17 included. This means, as also one can see from Fig. 8, that the beam is scattered by the resonances in many streams with different slopes. The particles with low ν_o will hit the wall first, and the particles with higher ν_o will hit the wall at a later time.

APPENDIX

Let us write $m = gS = gIN$ and break the summation in Eq. (9) into two sums

$$\sum_{n=1}^{mM} = \sum_{n=1}^{SM} + \sum_{n=SM+1}^{2SM} + \sum_{n=(k-1)SM+1}^{kSM} + \ldots + \sum_{n=(g-1)SM+1}^{gSM}$$

$$= \sum_{k=1}^{g} \sum_{n=(k-1)SM+1}^{kSM}.$$

Then we have

$$P_{gS} = e^{i(2\pi gS\nu_o - rN)} \times$$

$$\times \sum_{k=1}^{g} \sum_{n=1}^{SM} \alpha_n e^{-i\left[2\pi\frac{(k-1)SM+n}{M}\nu_o - rN\cos 2\pi\frac{n}{MN}\right]}.$$

If the fractional part of ν_o, δ, satisfies the resonance condition

$$N\delta = \text{integer} \qquad \text{(A)}$$

the above equation writes also

$$P_{gs} = e^{-irN} g \sum_{n=1}^{SM} \alpha_n e^{-i\left[2\pi\frac{n}{M}\nu_o - rN\cos 2\pi\frac{n}{MN}\right]}$$

$$= g\, A_s.$$

This means that, if the resonance condition (A) is satisfied, the displacement P_m adds the quantity

$$A_s = e^{-irN} \sum_{n=1}^{SM} \alpha_n e^{-i\left[2\pi\frac{n}{M}\nu_o - rN\cos 2\pi\frac{n}{MN}\right]}$$

each $S = NI$ turns.

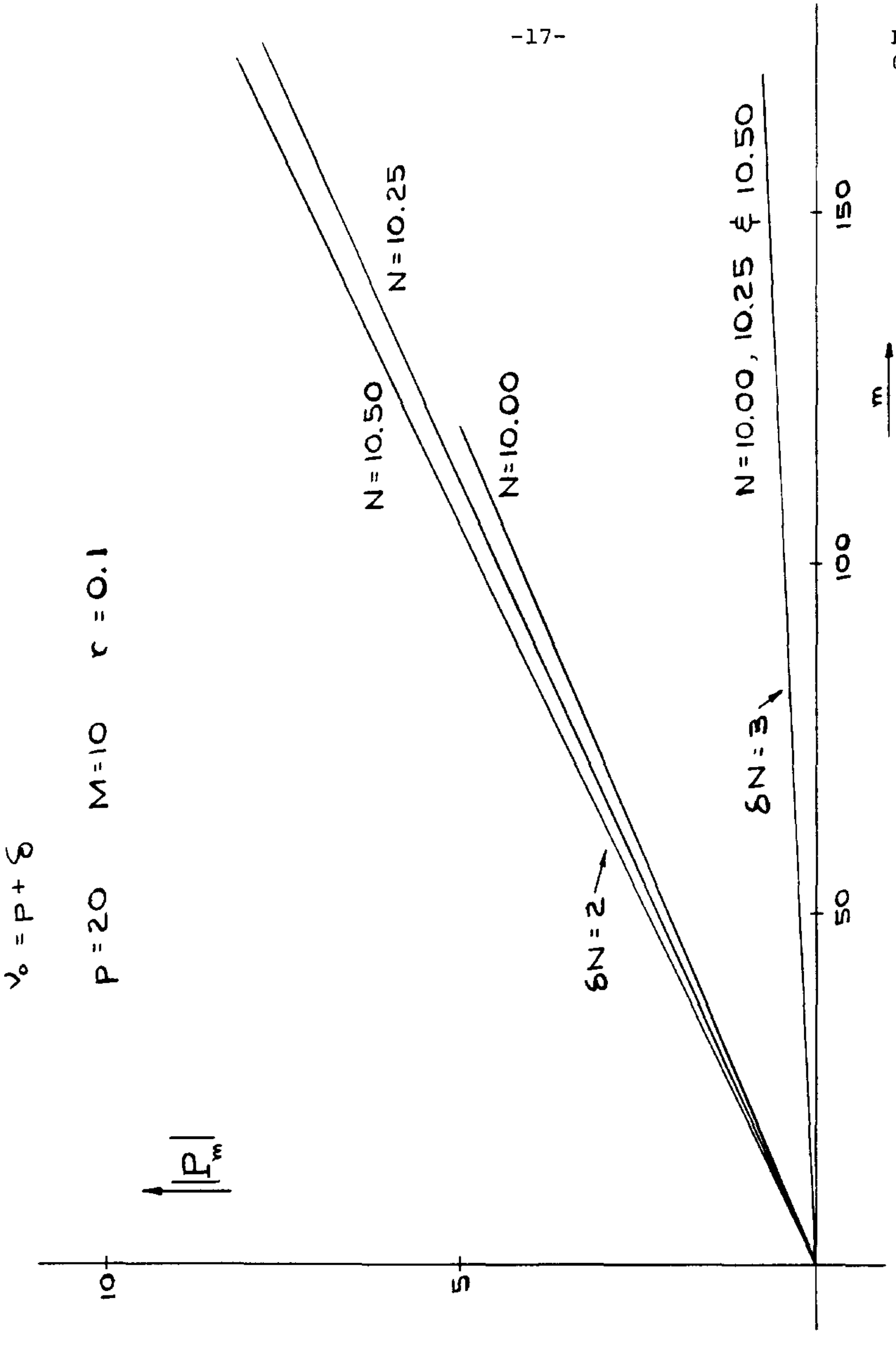
$\nu_0 = P + \delta$
P = 20
M = 10
r = 0.1
$|P_m|$
N = 10.25
N = 10.50
N = 10.00
$\delta N = 2$
$\delta N = 3$
N = 10.00, 10.25 & 10.50
m
50
100
150
5
10

FIG 1

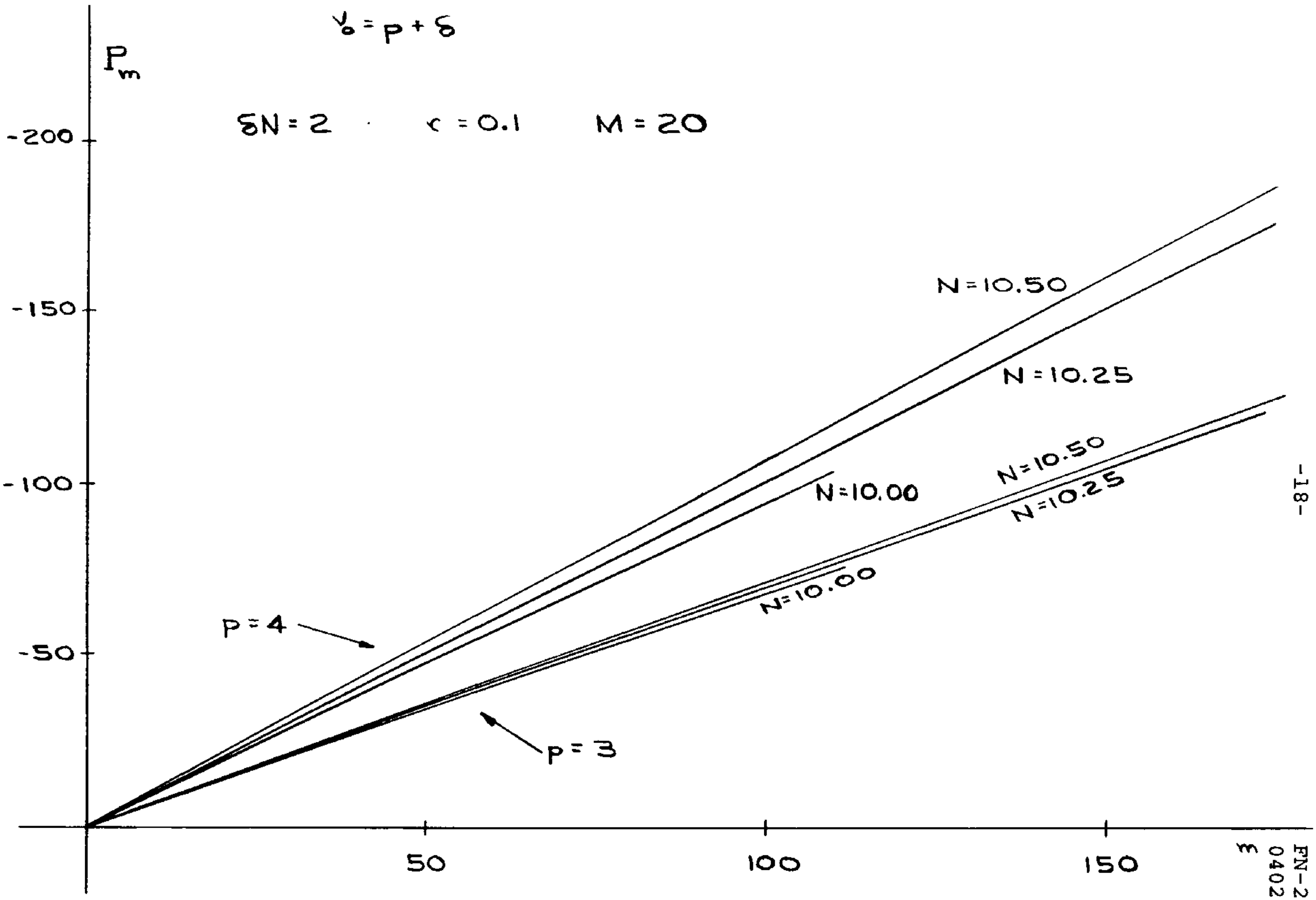
$\nu_0 = p + \delta$
P_m
$\delta N = 2$
$\epsilon = 0.1$
$M = 20$
-200
-150
-100
-50
50
100
150
m
N=10.50
N=10.25
N=10.00
N=10.50
N=10.25
N=10.00
p=4
p=3

FIG 2

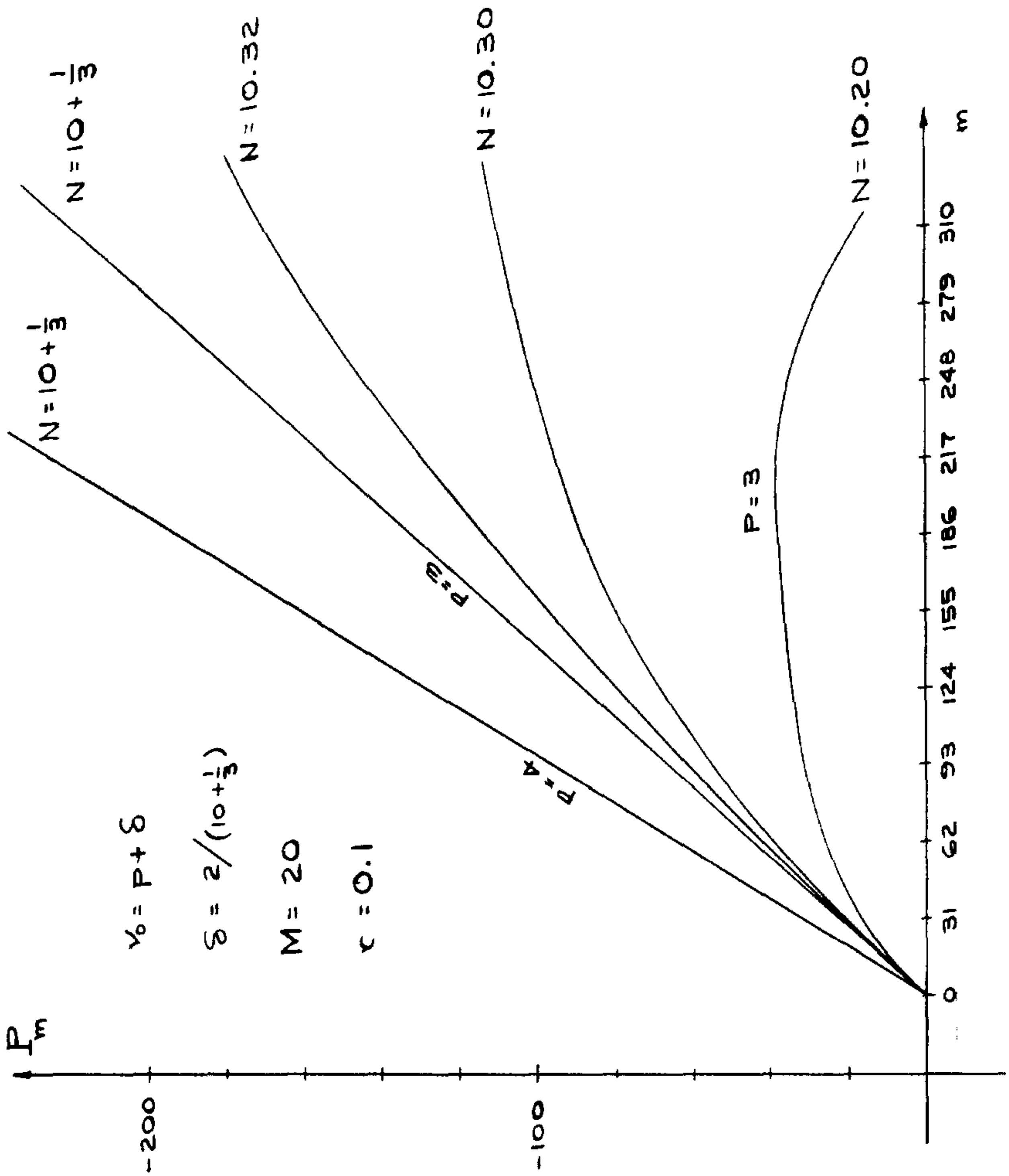
N = 10 + 1/3
N = 10.32
N = 10.30
N = 10.20
N = 10 + 1/3
P = 3
P = 3
P = 4
m
P_m
0
31
62
93
124
155
186
217
248
279
310
-100
-200
ν₀ = P + δ
δ = 2/(10 + 1/3)
M = 20
r = 0.1

FIG. 3

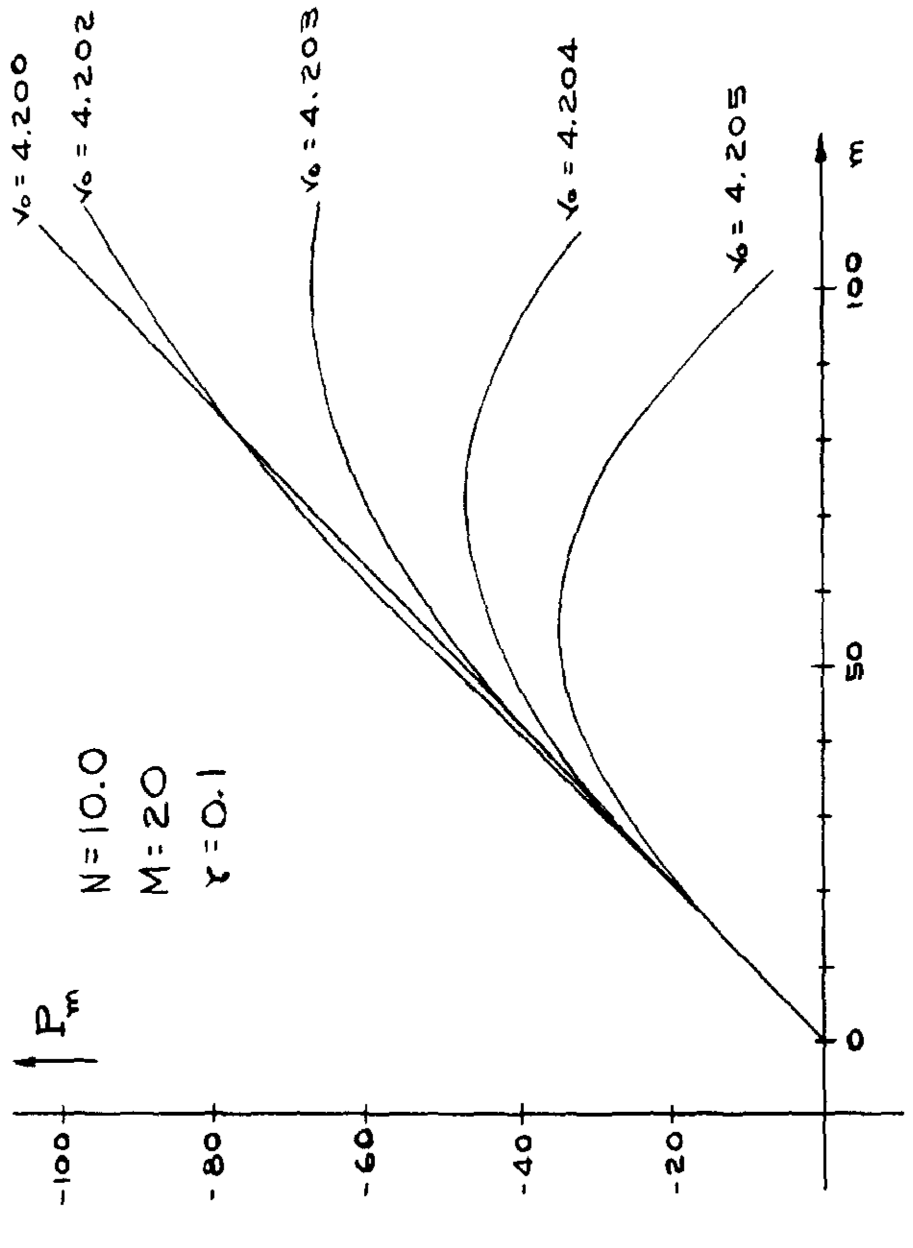
$y_o = 4.200$
$y_o = 4.202$
$y_o = 4.203$
$y_o = 4.204$
$y_o = 4.205$
P_m
m
N=10.0
M=20
τ=0.1
0
50
100
-20
-40
-60
-80
-100

FIG. 4

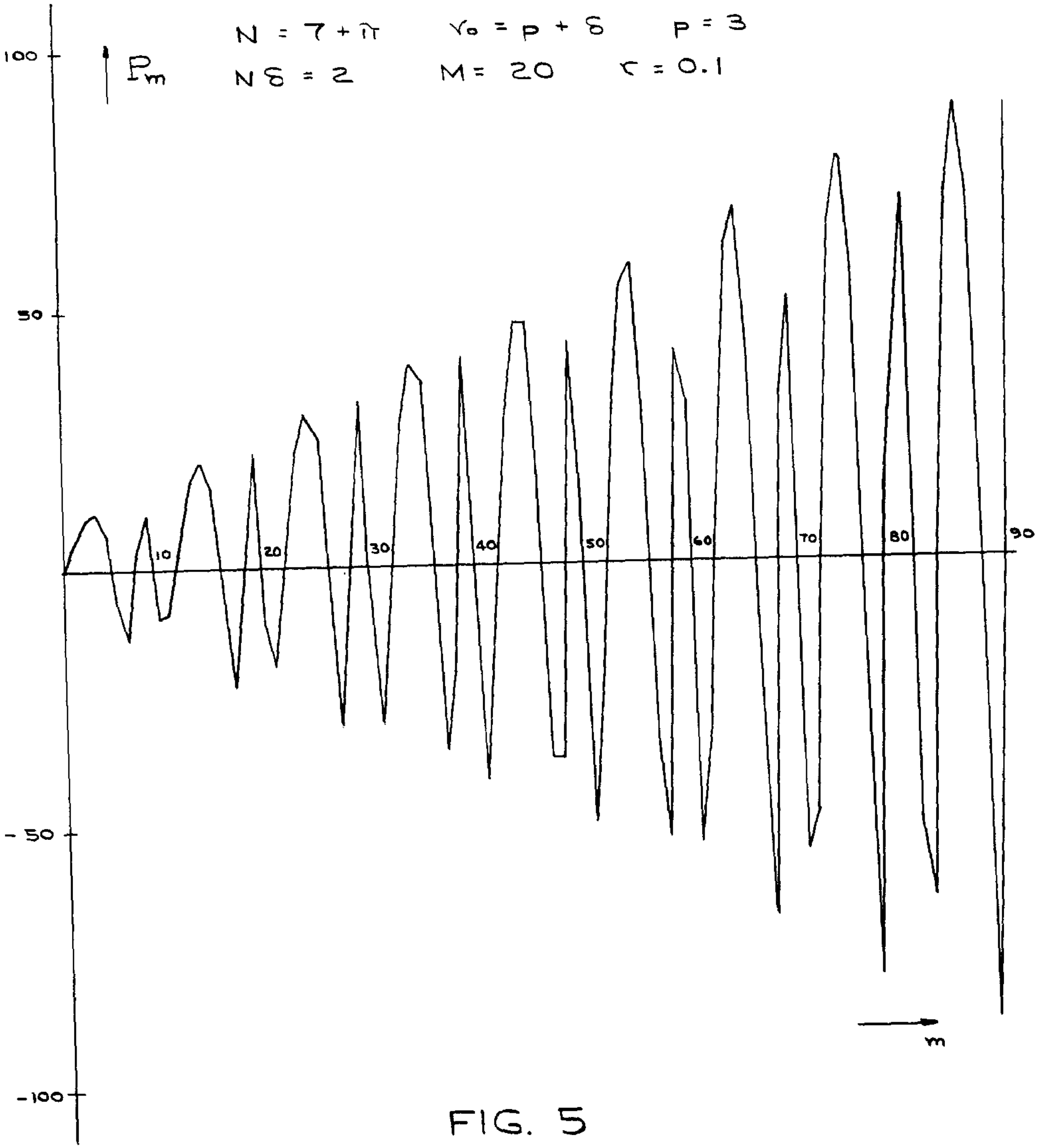
N = 7 + π
ν_0 = p + δ
p = 3
P_m
Nδ = 2
M = 20
τ = 0.1
100
50
-50
-100
10
20
30
40
50
60
70
80
90
m

FIG. 5

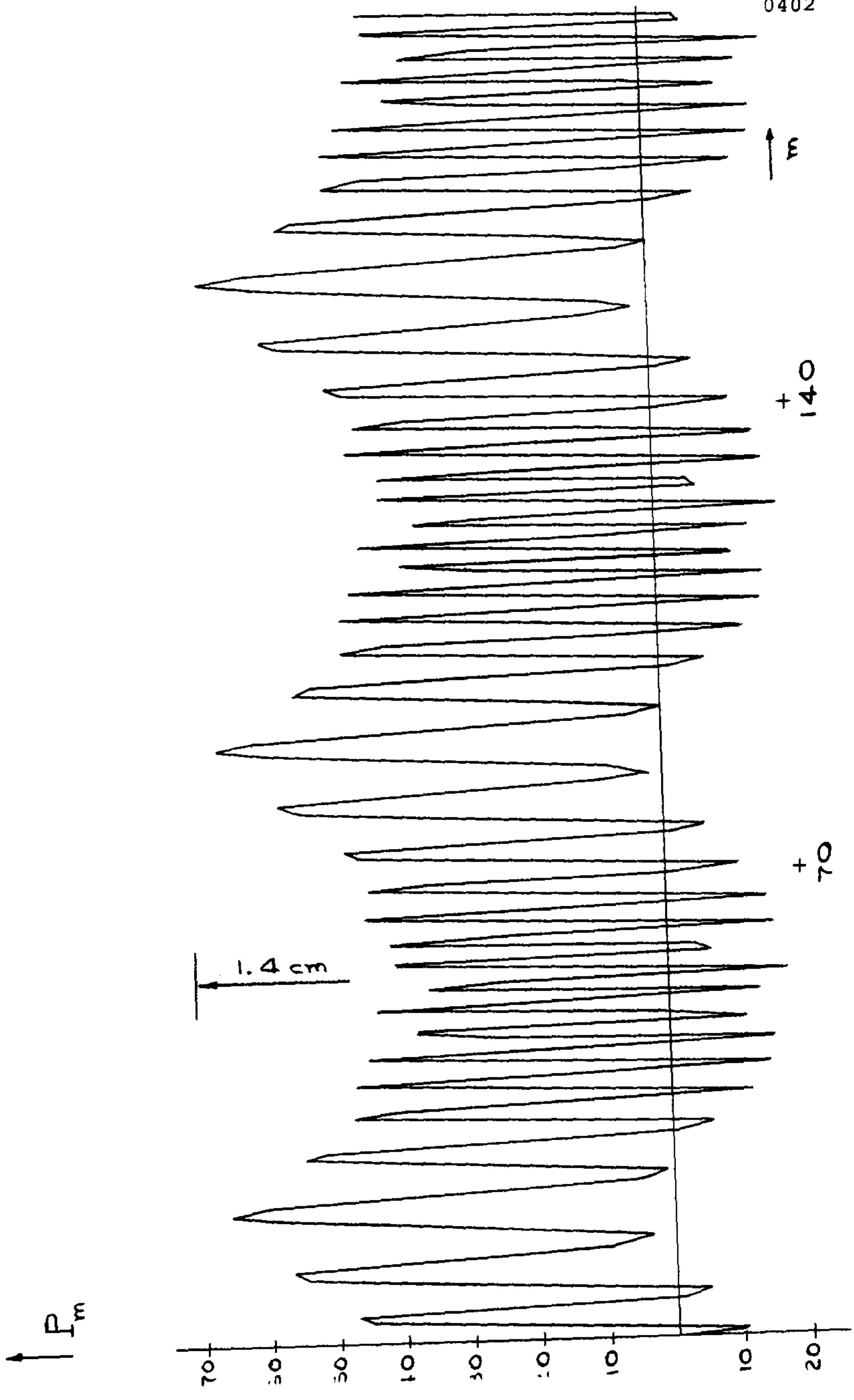

FIG. 6

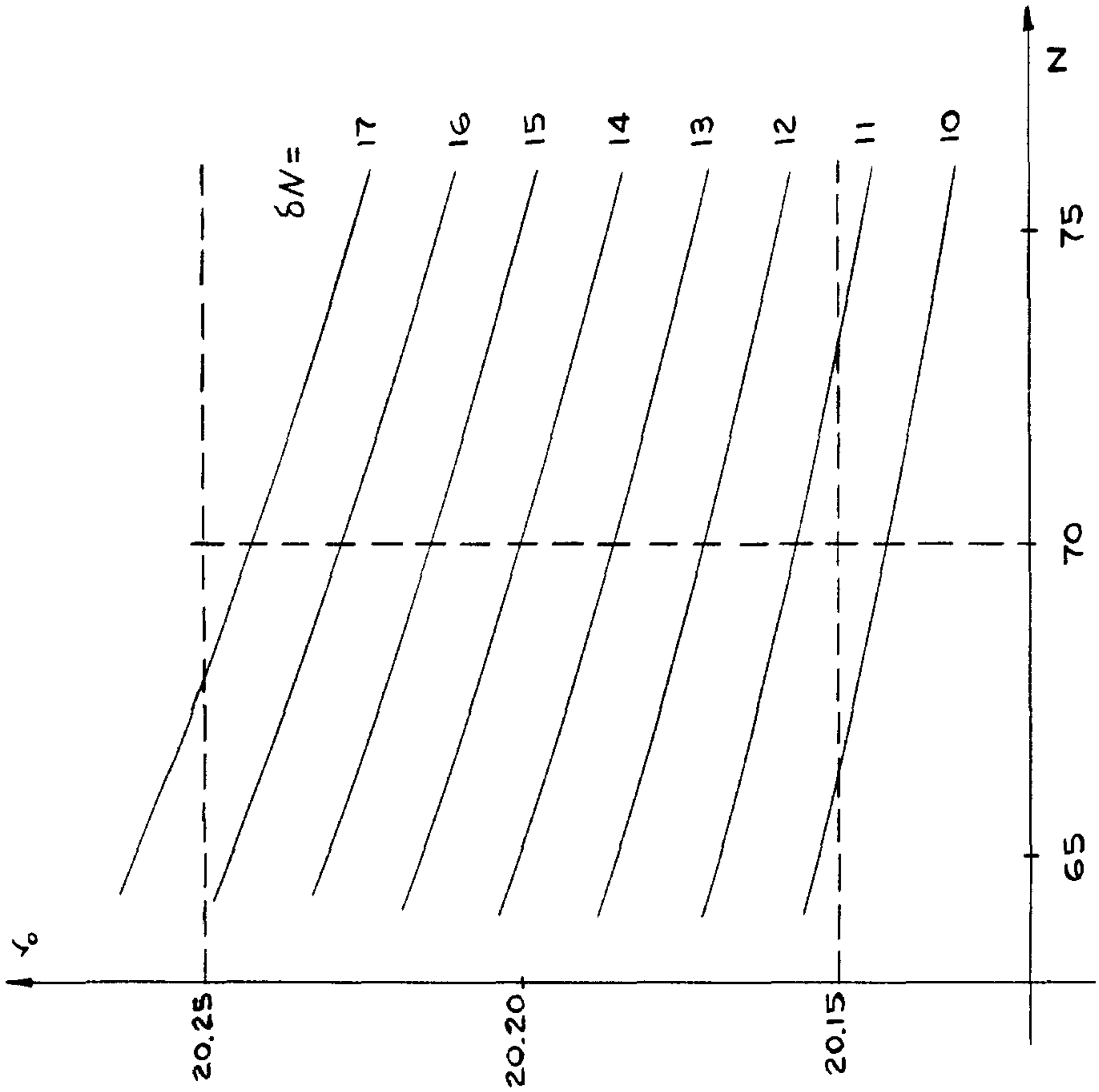

FIG. 7

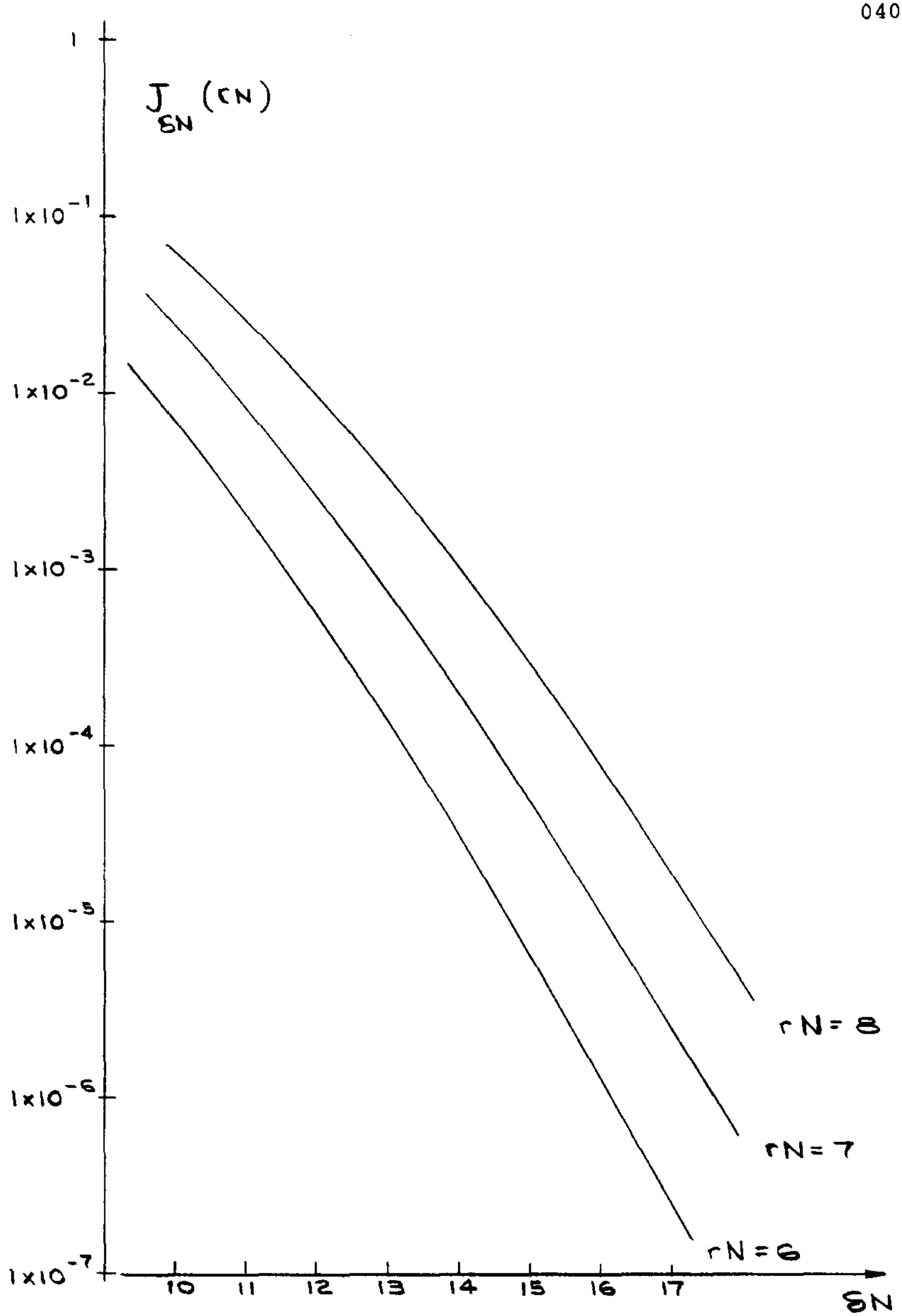
$J_{\delta N}(rN)$
1
1x10⁻¹
1x10⁻²
1x10⁻³
1x10⁻⁴
1x10⁻⁵
1x10⁻⁶
1x10⁻⁷
10
11
12
13
14
15
16
17
δN
rN = 8
rN = 7
rN = 6

FIG. 8