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NUMERICAL STUDY OF BEAM BUNCH LENGTH MATCHING AT TRANSITION USING THE γ_t-JUMP METHOD

W. Lee and L. C. Teng

August 26, 1970

The bunch length equation (envelope equation) near transition for phase oscillation with space charge was derived by A. Sørenssen (CERN MPS/Int. MU/EP 67-2, 1967) and used by E. D. Courant (NAL Report FN-187, 1969) for a preliminary study of the γ_t-jump scheme for bunch length matching at transition. For completeness we shall give a brief outline of the derivation of the envelope equation here.

The equations for small phase oscillations of individual particles are

$$\begin{cases} \dfrac{d\psi}{dt} = aw & a = a(t) = \dfrac{h^2}{mR^2\gamma}\left(\dfrac{1}{\gamma_t^2} - \dfrac{1}{\gamma^2}\right) \\ \dfrac{dw}{dt} = -b\psi & b = b(t,\theta) \end{cases} \tag{1}$$

where ψ and w are small deviations of the phase and the conjugate "momentum" variable from their synchronous values. The first equation is a kinematic (geometric) relationship and does not depend on the space charge. The second is a dynamic equation where b is the sum of two terms:

RF term: This term is proportional to the time-slope of the cavity voltage at the synchronous phase and, therefore, changes sign at the RF phase jump.

Space charge term: This term is proportional to the second derivative of the linear charge density of the beam bunch. For the lowest moment of the linear charge density this term is proportional to $\frac{1}{\theta^3}$ where θ is the bunch half-length (envelope of ψ).

The envelope equation of Eq. (1) is

$$\frac{d}{dt}\left(\frac{1}{a}\frac{d\theta}{dt}\right) + b\theta - \left(\frac{S}{\pi}\right)^2 \frac{a}{\theta^3} = 0 \qquad (2)$$

where $S(>0)$ is the phase space area occupied by the beam bunch in $w\psi$ unit. At transition $a = 0$, $t = 0$. Near transition a is approximately proportional to t. Since the dependence of a on the transition γ_t is through the factor $\frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$, to incorporate γ_t-jump we shall write

$$a = \frac{2h^2\dot{\gamma}}{mR^2\gamma^4}\left(1 - \frac{\dot{\gamma}_t}{\dot{\gamma}}\right) t \equiv a_o A t \qquad a_o = \text{const} > 0.$$

Without γ_t-jump, $A = 1$ and a_o is just $\dot{a}$. With γ_t-jump $A = A(t) > 1$. To simplify the problem we shall assume that the γ_t-jump is turned on and off suddenly and that during the jump $A = \text{const} > 1$.

For b we shall write

$$b = \pm b_o + \frac{B}{\theta^3} = b_o\left(\delta + \frac{B}{b_o\theta^3}\right)$$

where

$$\begin{cases} b_o = \dfrac{ev}{2\pi h} \; |\cos \phi_s| = \text{const} > 0. \\ \delta = \begin{cases} +1 & \cos \phi_s < 0, \text{ after RF phase jump} \\ -1 & \cos \phi_s > 0, \text{ before RF phase jump} \end{cases} \\ B = \dfrac{3}{2} \; \dfrac{e^2 N g}{R\gamma^2} > 0 \; (g = \text{geometrical factor} \approx 4.5) \end{cases}$$

The envelope equation (2) becomes

$$\frac{d}{dt} \left(\frac{1}{t} \frac{d\theta}{dt}\right) + a_o b_o A \left(\delta + \frac{B}{b_o \theta^3}\right) \theta - \left(\frac{a_o S}{\pi}\right)^2 A^2 \frac{t}{\theta^3} = 0. \quad (3)$$

Now, we change the scale of t and θ to kill the complicated constants. Let

$$t = Tx \quad \text{and} \quad \theta = Ky$$

and rewrite Eq. (3) as

$$\frac{d}{dx} \left(\frac{1}{x} \frac{dy}{dx}\right) + a_o b_o T^3 A \left(\delta + \frac{B}{b_o K^3 y^3}\right) y - \left(\frac{a_o S}{\pi} \frac{T^2}{K^2}\right)^2 A^2 \frac{x}{y^3} = 0. \quad (4)$$

Setting $a_o b_o T^3 = 1$ and $\frac{a_o S}{\pi} \frac{T^2}{K^2} = 1$ we get

$$T^3 = \frac{1}{a_o b_o}, \quad K^3 = T^3 \left(\frac{a_o S}{\pi}\right)^{3/2} = \frac{a_o^{1/2}}{b_o} \left(\frac{S}{\pi}\right)^{3/2}$$

and the equation

$$\left(\frac{y'}{x}\right)' + A \left(\delta + \frac{C}{y^3}\right) y - A^2 \frac{x}{y^3} = 0 \quad (5)$$

where prime means $\frac{d}{dx}$, and

$$\begin{cases} \delta = \begin{cases} +1 & \cos\phi_s < 0, \text{ after RF phase jump} \\ -1 & \cos\phi_s > 0, \text{ before RF phase jump} \end{cases} \\ A = 1 - \dfrac{\dot{\gamma}_t}{\dot{\gamma}} \\ C = \dfrac{B}{b_o K^3} = \dfrac{B}{a_o^{1/2}} \left(\dfrac{\pi}{S}\right)^{3/2} \end{cases}$$

When all the parameters are substituted and reduced we get

$$\begin{cases} T^3 = \dfrac{1}{2} \dfrac{\beta^2\gamma^4}{\omega^3} \left(\dfrac{\omega}{\dot{\gamma}}\right)^2 \dfrac{1}{|\cot\phi_s|} \\ \quad (T = \text{same time scale factor used by Sørenssen}) \\ K = \dfrac{\theta_o}{k} \\ C = k^3 \eta_o \end{cases}$$

where

$$\begin{cases} k = \dfrac{3^{1/6}}{\pi^{1/2}} \Gamma(2/3) = 0.91749 \\ \theta_o^3 = \dfrac{k^3}{2} \left(\dfrac{2}{\pi}\right)^{3/2} \dfrac{S_p^{3/2}}{\gamma^2} \left(\dfrac{\omega}{\dot{\gamma}}\right)^{1/2} \dfrac{1}{|\cot\phi_s|} \\ \quad (\theta_o = \text{bunch half-length at transition without space charge}) \\ \eta_o = \dfrac{3}{k^3} \left(\dfrac{\pi}{2}\right)^{3/2} \dfrac{r_p}{R} \dfrac{Ng}{S_p^{3/2}} \left(\dfrac{\omega}{\dot{\gamma}}\right)^{1/2} = \text{Sørenssen parameter } \eta_o(o) \\ r_p = \text{classical proton radius} = 1.53 \times 10^{-18} \text{ m} \\ N = \text{total number of protons} \end{cases}$$

$$\begin{cases} S_p = \text{total phase space area in } \frac{p}{mc}\psi \text{ unit} = \frac{hS}{mcR} \\ \omega = \text{RF angular frequency} = \frac{hc\beta}{R} \end{cases}$$

Aside from the change in the sign-convention and the inclusion of $\dot{\gamma}_t$ (for treating γ_t-jump) Eq. (5) is identical to that given by A. Sørenssen. This equation is solved by numerical methods on a computer.

A. For $|x| \to \infty$ $\dot{\gamma}_t = 0$, $A = 1$ and we can neglect the derivative term $\left(\frac{y'}{x}\right)'$. This gives the algebraic equation

$$\delta y^4 + Cy - x = 0.$$

This equation gives the matched adiabatic solution of Eq. (5) at large $|x|$. Hence, starting with the solutions of this equation at some large values of $|x|$ as initial conditions and integrating Eq. (5) to $x = 0$ we can get the matched solutions for both below and above transition.

B. With $\eta_o = 0$ (no space charge), $A = 1$ (no γ_t-jump) and δ changing sign at $x = 0$ (RF phase jump at transition) the matched solution above transition is simply the reflection about the $x = 0$ axis of the matched solution below transition which is plotted in Figure 1 as Curve (0). Since $y'(x = 0) = 0$ the matched solutions above and below transition are automatically matched at transition.

C. With $\eta_o = 3.8$ (value for the NAL booster) and $A = 1$ (no γ_t-jump) the matched solutions below and above transition are shown as Curves (I) and (II) in Figure 1. The mismatch at

transition caused by space charge effect is indicated by the values

$$y_I(x = 0) = 1.502, \qquad y_{II}(x = 0) = 0.$$

D. Starting the γ_t-jump before transition does not change $y(x = 0)$ significantly. For example starting a γ_t-jump with $A = 1 - \frac{\dot{\gamma}_t}{\dot{\gamma}} = 5$ at $x = x_1$ we get for various values of x_1

x_1	$y(x = 0)$
0	1.50214
-0.1	1.50210
-0.2	1.50245
-0.3	1.50537
-0.4	1.51455
-0.5	1.53433
-1.0	1.80331

The conclusion is that starting the γ_t-jump before transition does not significantly change the status of mismatch at $x = 0$. We have, therefore, confined ourselves to cases where the γ_t-jump starts at transition. (In practice, since the start of γ_t-jump cannot be abrupt we will have to start the γ_t-jump just slightly before transition.)

E. Because $y'(x = 0) = 0$ to integrate across $x = 0$ we have to invoke the continuity of $\frac{y'}{x}(x = 0)$ or $y''(x = 0)$. With $A = 1$ (no γ_t-jump) and δ changing sign at $x = 0$ (RF phase jump at transition) the continuation of y_I above transition is plotted as Curve (III) in Figure 1 which shows the familiar bunch length oscillation due to space-charge mismatching.

With $A = 5$ and δ changing sign at $x = 0$ we get Curve (IV). Note that the first minimum of Curve (IV) is now above Curve (II). By adjusting A we obtain for $A = 2.446$ Curve (V) which is tangent to Curve (II) at $x = x_2 = 0.879$ near the first minimum. Thus, with $A = 2.446$ and turning off the γ_t-jump at $x_2 = 0.879$ the bunch length will be perfectly matched to Curve (II). For the NAL booster $\dot{\gamma} = 0.407 \times 10^3$ sec^{-1} and $T = 0.281 \times 10^{-3}$ sec, this value of A gives

$$\dot{\gamma}_t = -1.446\dot{\gamma} = -0.589 \times 10^3 \text{ sec}^{-1}$$

and this value of x_2 gives the corresponding real time

$$\Delta t = Tx_2 = 0.247 \times 10^{-3} \text{ sec.}$$

Together they give a total γ_t-jump of

$$\Delta\gamma_t = \dot{\gamma}_t \Delta t = -0.145$$

which is rather modest.

F. The strings of dots leading away from the first minima of Curves (III), (IV), and (V) in Figure 1 show the effects of delaying the RF phase jump. Delaying the RF phase jump to $x = x_3$ for Curve IV, for example, means integrating Eq. (5) with $A = 5$, $\delta = -1$ from $x = 0$ to $x = x_3$ and, then, with $A = 5$, $\delta = +1$ from $x = x_3$ on. The dots give the positions of the first minima of the respective curves for $x_3 = 0.1, 0.2, 0.3, \ldots$. The following conclusions can be drawn:

For $A = 5$ matching can be obtained with $x_3 \sim 0.96$ at $x_2 \sim 1.8$.

For A = 2.446 another matching exists with $x_3 \sim 0.77$ at $x_2 \sim 1.67$.

For A = 1 (no γ_t-jump) delaying RF phase jump cannot produce matching; however, it can reduce the amplitude of the bunch length oscillation. Detailed computation shows that at the optimum $x_3 = 0.92$ the maximum bunch length is reduced by about a factor of 1.4 from the case without RF phase jump delay ($x_3 = 0$).

In any case it is evident that delaying RF phase jump does not significantly relax the requirement on the γ_t-jump; in particular, it does not reduce the total jump $\Delta\gamma_t$.

To summarize for the NAL booster ($\eta_o = 3.8$) we should

1. Start the γ_t-jump at or near transition.
2. Jump the RF phase at transition.
3. The total γ_t-jump is $\Delta\gamma_t = -0.145$
4. The total time for the γ_t-jump is $\Delta t = 0.247$ msec.
5. After the γ_t-jump the jump quadrupoles can be turned off (γ_t returns to its value before the jump) slowly (adiabatically) some time (say, 5 msec) after transition.
6. With the small $\Delta\gamma_t$ the simple ν_x-jump (from $\nu_x = 6.7$ to $\nu_x = 6.555$) scheme is possible. However, the scheme of quadrupoles proposed in FN-207 giving a γ_t-jump without affecting ν_x is still desirable. Of course, the required gradient B' for the quadrupoles is much smaller than indicated in that report.

For the NAL Main Ring $\eta_o = 7.2$ the corresponding solutions of Eq. (5) are plotted in Figure 2. The parameters producing matching are

$$A = 2.180 \qquad x_2 = 0.864$$

The interesting point to note is that for $\eta_o \gtrsim 2.5$ the values of A and x_2 for matching do not change significantly with η_o. To understand this we observe that the effect of the γ_t-jump is to produce matching after the phase "ellipse" has rotated approximately 90°. In a crude manner of speaking the ψ-envelope before the γ_t-jump is matched onto the w-envelope after the γ_t-jump. One can, therefore, appreciate that although the mismatch in bunch length (ψ envelope) increases sharply with η_o, this particular mode of matching would not necessarily impose increasingly more stringent requirements on A and x_2 as η_o increases. However, the increased sharpness of the minima of Curves (III), (IV) and (V) in Figure 2 indicates that the precision of the γ_t-jump turn-off time x_2 required for matching becomes more critical with increasing η_o.

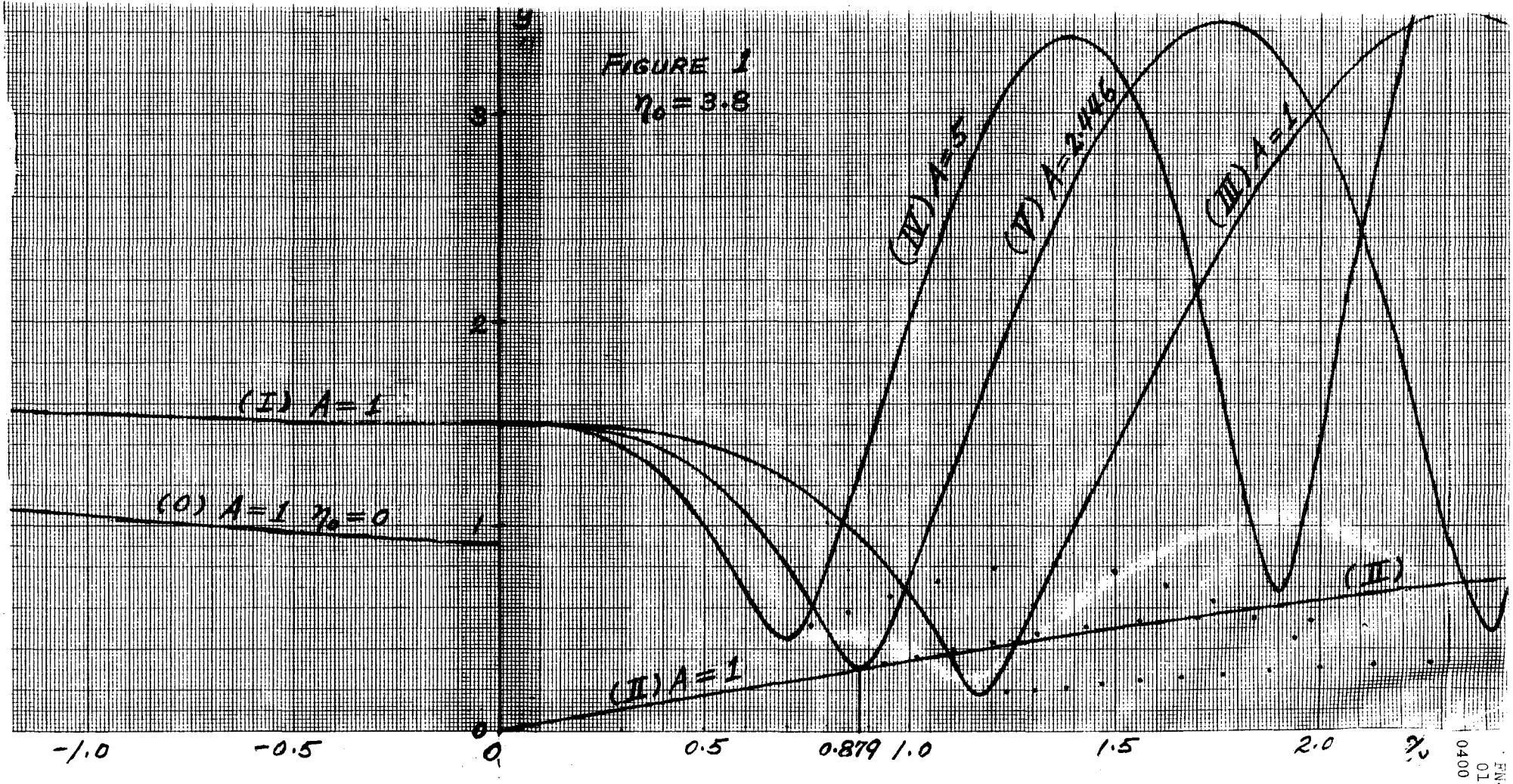
For the Main Ring $\dot{\gamma} = 0.135 \times 10^3$ sec^{-1} and $T = 2.44 \times 10^{-3}$ sec. The above values of A and x_2 give

$$\dot{\gamma}_t = -1.180\,\dot{\gamma} = -0.159 \times 10^3 \text{ sec}^{-1}$$

$$\Delta t = Tx_2 = 2.11 \times 10^{-3} \text{ sec}$$

$$\Delta\gamma_t = \dot{\gamma}_t \Delta t = -0.336$$

which is, again, not excessive.



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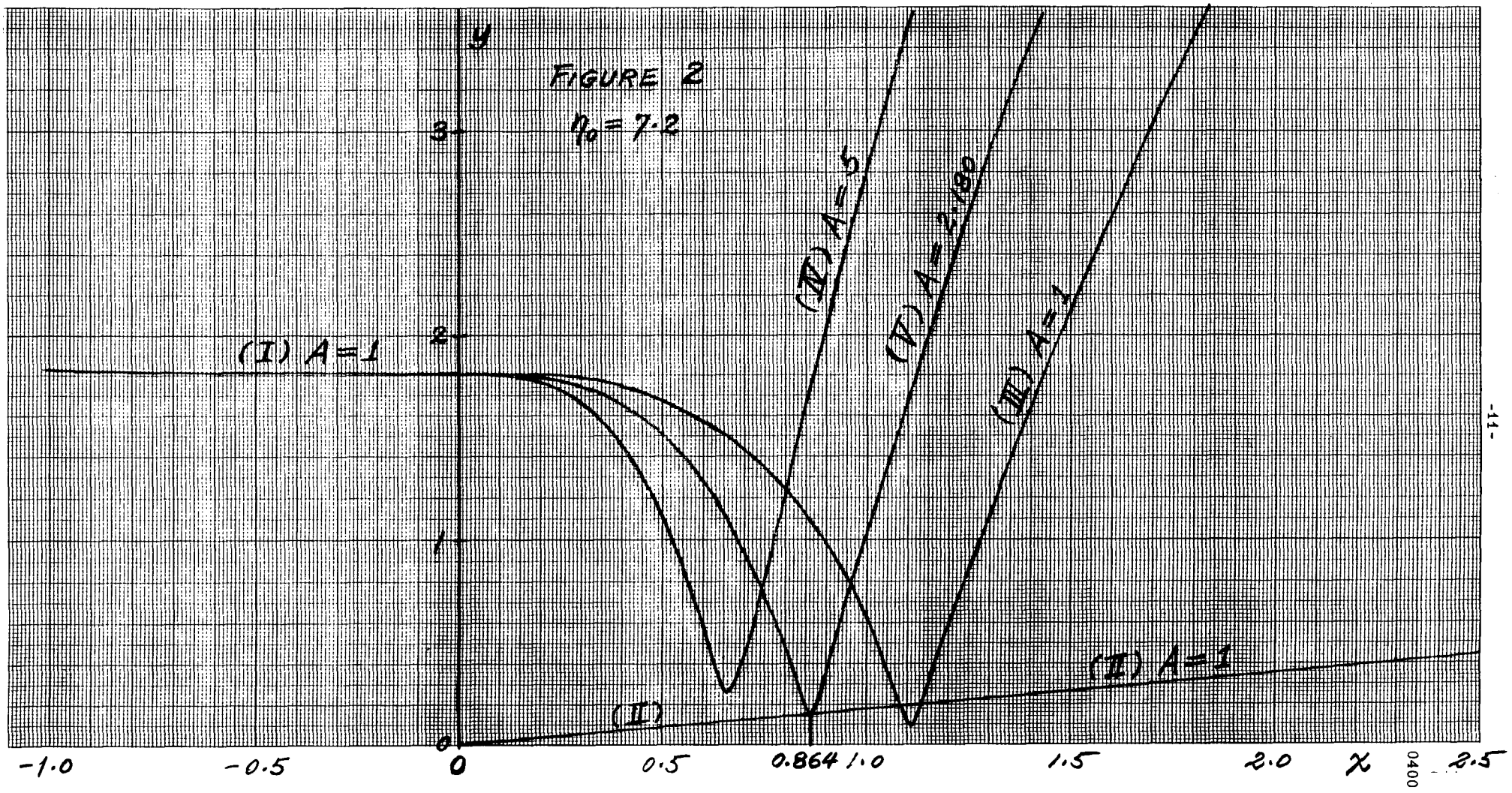

FIGURE 2

$\eta_0 = 7.2$

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ADDENDUM

W. Lee and L. C. Teng

October 16, 1970

W. Hardt pointed out an error in our report FN-215. The after-transition Curves (II) given in Figures 1 and 2 of that report are matched solutions corresponding to γ_t values before the jump (γ_{to}). In Equ. (5) of FN-215 the origin x = 0 is defined as the "time" when a = 0; namely, when γ_t intersects γ. The matched curve for the after-jump value of γ_t (γ_{tf} or

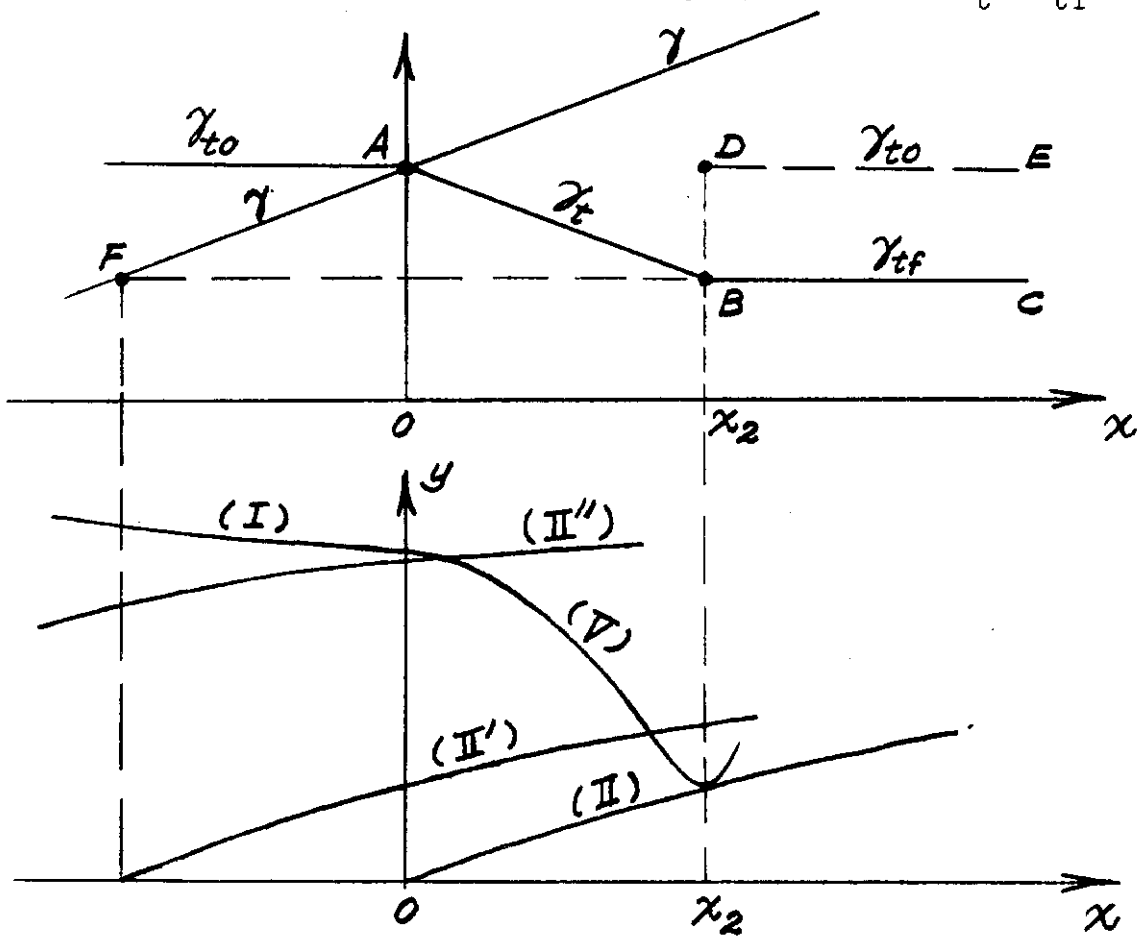

Figure 1

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line BC in Figure 1) should be Curve (II') with origin shifted to point F. Curve (V) is clearly not matched to Curve (II').

A. First we want to show that if the γ_t-jump quadrupoles can indeed be turned off abruptly at x_2 so that γ_t jumps from γ_{tf} back to γ_{to} (B to D) infinitely fast, Curve (V) is a fairly good match to Curve (II).

For infinitely fast γ_t-jumps $A = \infty$. Therefore we rewrite Equ. (5) of FN-215 as

$$\left(\frac{y'}{f}\right)' + \left(\delta + \frac{C}{y^3}\right) y - \frac{f}{y^3} = 0 \tag{1}$$

or

$$\begin{cases} y' = fp_y \\ p_y' = \dfrac{f}{y^3} - \left(\delta + \dfrac{C}{y^3}\right) y \end{cases} \tag{2}$$

where

$$f = f(x) \equiv \frac{\gamma - \gamma_t}{\gamma'} .$$

Equation (1) or (2) avoids the troubles of infinite A and shifting origins. Furthermore, it can be shown that

$$\begin{cases} \theta = \text{envelope of } \psi = Ky \\ W = \text{envelope of } w = \dfrac{K}{a_o T^2} \left(p_y^2 + \dfrac{1}{y^2}\right)^{1/2} \end{cases} \tag{3}$$

It is interesting to point out that matching only θ and W leaves an ambiguity in the sign of p_y. This corresponds to the two possible orientations of an ellipse with given area inscribed in a given rectangle. Therefore we must match y and

p_y or y and y' when there is no discontinuity in f.

For a step-function discontinuity in f Equ. (1) or (2) shows that $p_y = \frac{y'}{f}$ is constant. The effect of the jump from B to D is thus to produce a kink in Curve (V) at x_2. The slope after the kink is reduced to

$$\frac{\gamma - \gamma_{to}}{\gamma - \gamma_{tf}} y'_V = \frac{1}{A} y'_V$$

which is no longer equal to y'_{II} at x_2. However, since the slope of Curve (V) is increasing very rapidly at x_2 it is easy to find a point a little beyond x_2 where $y'_V = Ay'_{II}$ and where y_V is still $\cong y_{II}$. As a matter of fact we can always reduce A slightly and increase x_2 slightly to re-establish perfect matching after the kink. The matched values for the NAL booster ($\eta_o = 3.8$, $\gamma' = 0.11445$, $T = 0.2812 \times 10^{-3}$ sec) are given in the second boxed row in Table 1.

If one allows γ_{tf} to be different from γ_{to} perfect matching can be obtained with any prescribed A value. These values for the NAL booster are also given in Table 1 where $\Delta\gamma_t$ and

TABLE 1

A	x_2	$\Delta\gamma_t$	$\delta\gamma_t$	
1	1.1780	0	0.0753	Case 1
1.5	1.0448	-0.0598	0.0424	
2.0	0.9606	-0.1099	0.0175	
2.416	0.9096	-0.1474	0	Case 2
2.5	0.9004	-0.1546	-0.0033	
3.0	0.8540	-0.1955	-0.0213	

$\delta\gamma_t$ are defined in Figure 2. Aside from the case of A = 2.416 where $\delta\gamma_t = 0$ the other interesting case is that of A = 1.

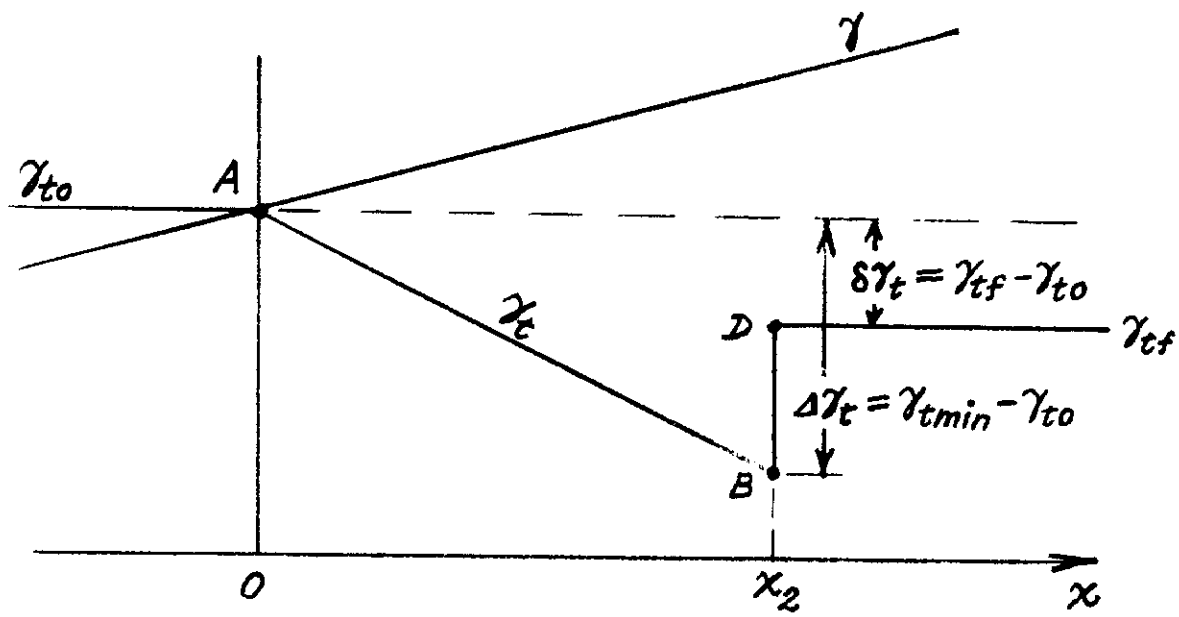


Figure 2

In this case γ_t simply continues the value $\gamma_{to} = 5.4458$ to $x_2 = 1.178$ where it is <u>jumped up</u> (infinitely rapidly) by $\delta\gamma_t = 0.0753$ and, then, stays at the value $\gamma_{tf} = \gamma_{to} + \delta\gamma_t = 5.5211$.

Although the jump from B to D (Figure 2) is treated as infinitely fast, in practice, a jump which completes in, say, $\Delta x = 10^{-2}$ ($\Delta t \sim 10^{-2}$ T = 2.8 μsec for NAL booster) is quite adequate. Nevertheless, this rapid jump together with the high precision required on the "timing" x_2 of the jump make these matched solutions impractical.

B. It is clear that by adjusting the parameters it is

possible to obtain perfect matching even with finite-speed return jumps. To limit the number of possibilities we concentrate on only two simple cases corresponding to the two boxed cases in Table 1.

Case 1 - Here we continue γ_{to} to a given x_2, jump γ_t upward at x_2 with slope γ_t' to x_4. The value γ_{tf} at x_4 is, then, held constant.

For the NAL booster we obtained a matched solution with

$$\begin{cases} x_2 = 1.100 & x_4 = 1.240 \\ \gamma_t' = 5.518\ \gamma' & \delta\gamma_t = 0.0884 \qquad \Delta\gamma_t = 0 \end{cases}$$

Since there are 3 adjustable parameters x_2, x_4, and γ_t', other matched solutions exist for other given values of x_2. One can choose a convenient γ_t' or a convenient $\delta\gamma_t$. This case is plotted in Figure 3.

Case 2 - Here we jump γ_t downward from $x = 0$ to $x = x_2$ with slope $-\gamma_t'$, then upward from $x = x_2$ to $x = 2x_2 \equiv x_4$ with slope γ_t' (We could use different slopes for jumping down and up, but for simplicity we give here only the case of equal slopes.) and end up with $\gamma_{tf} = \gamma_{to}$. For the NAL booster the matched solution has

$$\begin{cases} x_2 = 0.4884 & x_4 = 2x_2 = 0.9768 \\ \gamma_t' = 4.1651\ \gamma' & \Delta\gamma_t = -0.2328 \qquad \delta\gamma_t = 0 \end{cases}$$

Other matched solutions exist with unequal slopes for jumping

down and up. But the case of equal slopes is, perhaps, the simplest in terms of hardware requirements. This case is plotted in Figure 4.

For both cases the Curve (V) dips down below the matched Curves (II) or (II'), turns upward, and matches Curves (II) or (II') from below. This behavior is easily understood by comparison with the cases 1 and 2 given in Table 1 for infinitely fast jumps. In those cases the corresponding Curves (V) have to come up from below and kink downward (reducing y') to match Curves (II) or (II').

One observes also from Figures 3 and 4 that near the matching point (x_4) Curve (V) runs closely along Curves (II) or (II') for a fair stretch. This implies that the "timing" for x_4 is not very critical. This makes the practical operation of the system easy.

For the NAL main ring ($\eta_o = 7.2$, $\gamma' = 0.3290$, $T = 2.444 \times 10^{-3}$ sec) the corresponding matched cases are

<u>Case 1</u>

$$\begin{cases} x_2 = 1.0800 & x_4 = 1.1346 \\ \gamma_t' = 13.715\ \gamma' & \delta\gamma_t = 0.2464 \qquad \Delta\gamma_t = 0 \end{cases}$$

<u>Case 2</u>

$$\begin{cases} x_2 = 0.3576 & x_4 = 2x_2 = 0.7152 \\ \gamma_t' = 9.5434\ \gamma' & \Delta\gamma_t = -1.1228 \qquad \delta\gamma_t = 0 \end{cases}$$

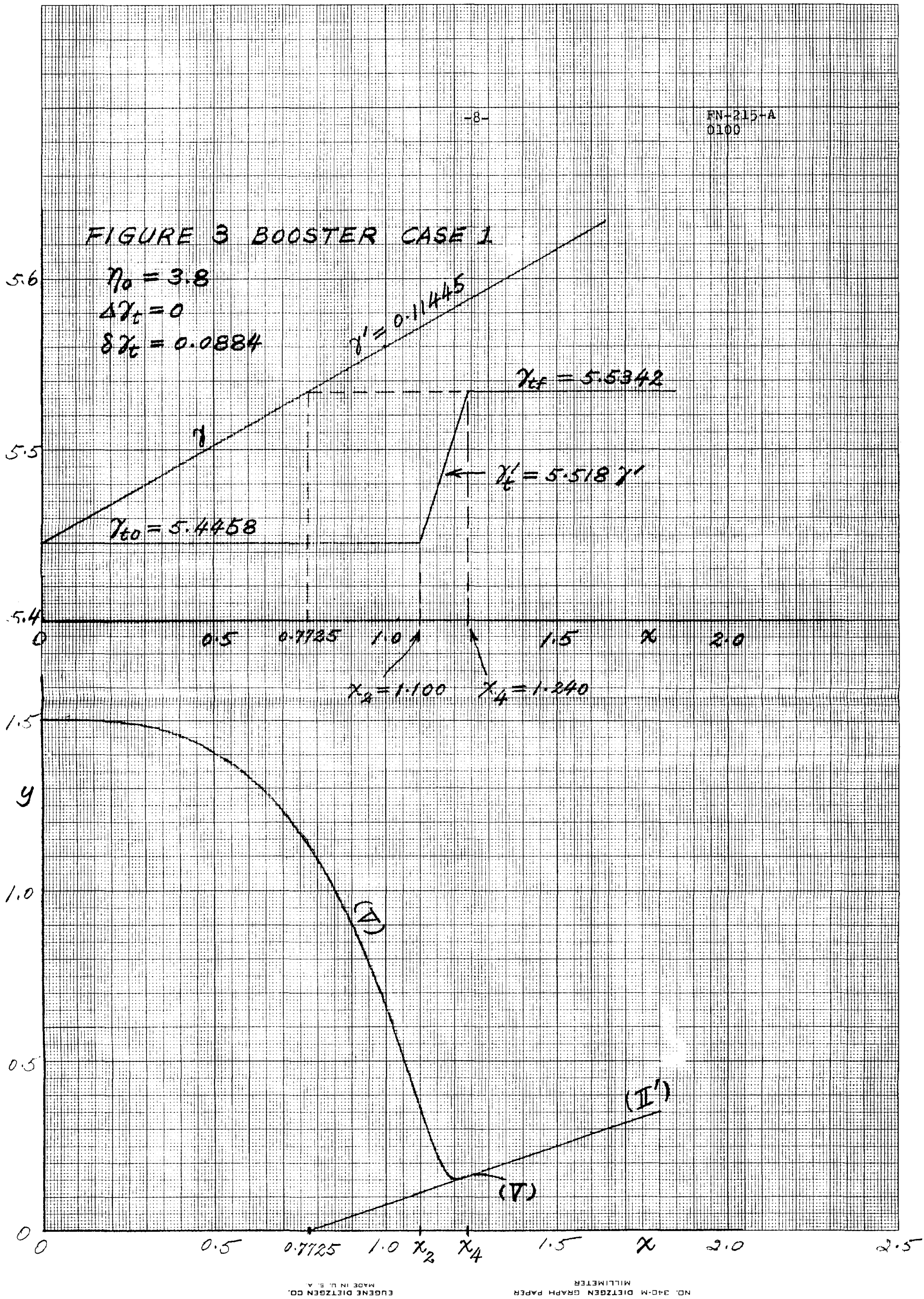
These cases are plotted in expanded scale in Figures 5 and 6.

With increasing η_o the precision required in x_2, x_4, and γ_t' for good matching becomes more critical. Also, Case 1 involves matching at much smaller y value ("bunch length") and is, therefore, more critical to operate than Case 2.

It is interesting to note that in the straightforward Q-jump scheme studied, for example, by D. Möhl (CERN-ISR/300/GS/69-62) one makes one big fast jump downward similar to that shown in Fig. 1. The magnitude of the jump is roughly equivalent to shifting the origin far enough to the left so that the matched curve for the after-jump value of γ_t is Curve (II") which has approximately the same y value as Curve (V) immediately after the fast jump.

Although the impractical infinitely fast γ_t-jumps are avoided in the cases studied here, they still involve abrupt turn-ons and turn-offs. The realistic cases of smooth turn-on and turn-off will be investigated. But we do not expect to find any qualitative difference. In addition, cases of starting the γ_t-jump before transition and of delayed or advanced RF-phase jump will also be studied further, and the results will be given in another report.

We are grateful to Dr. Werner Hardt of CERN for pointing out the error in our report FN-215.



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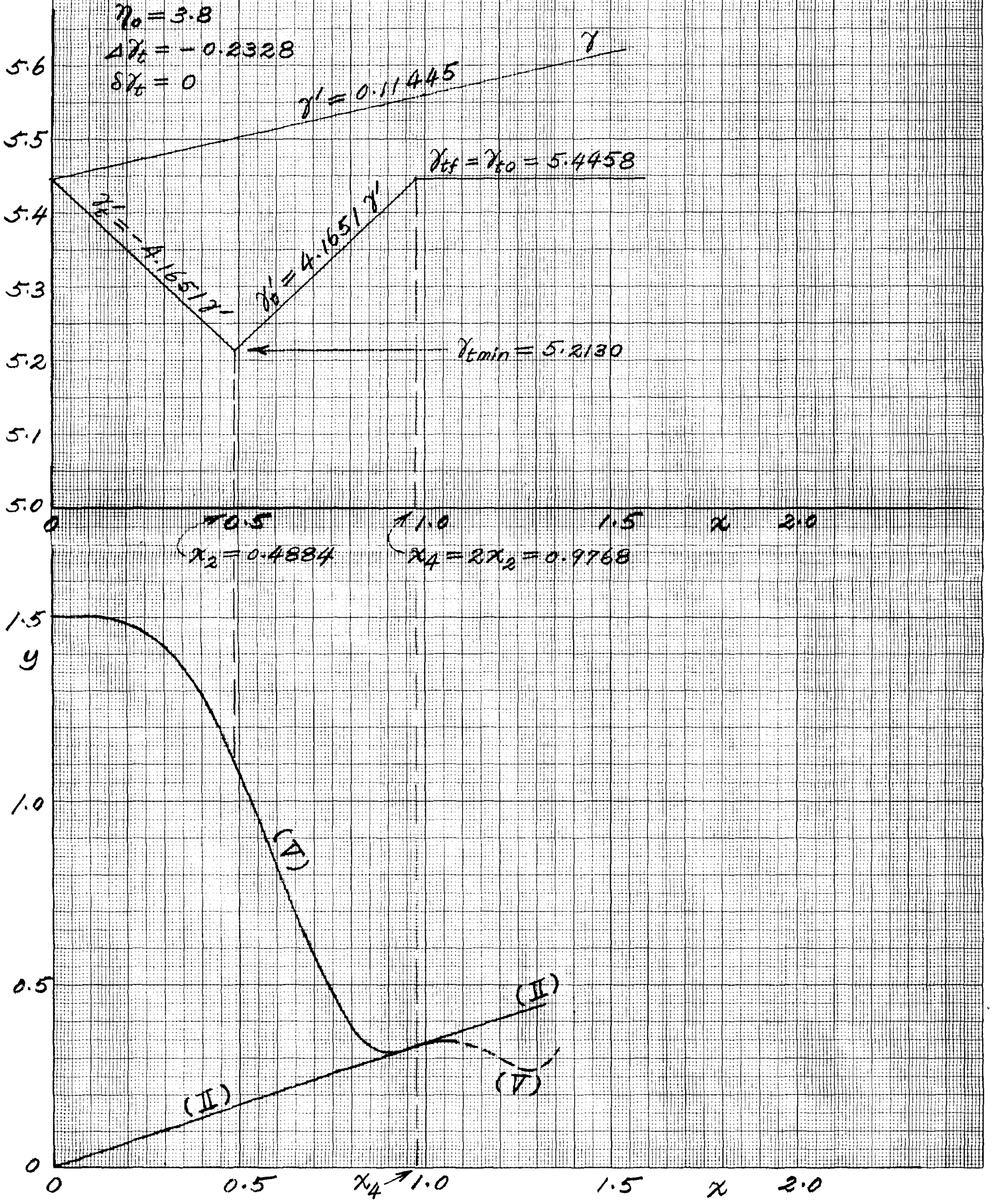

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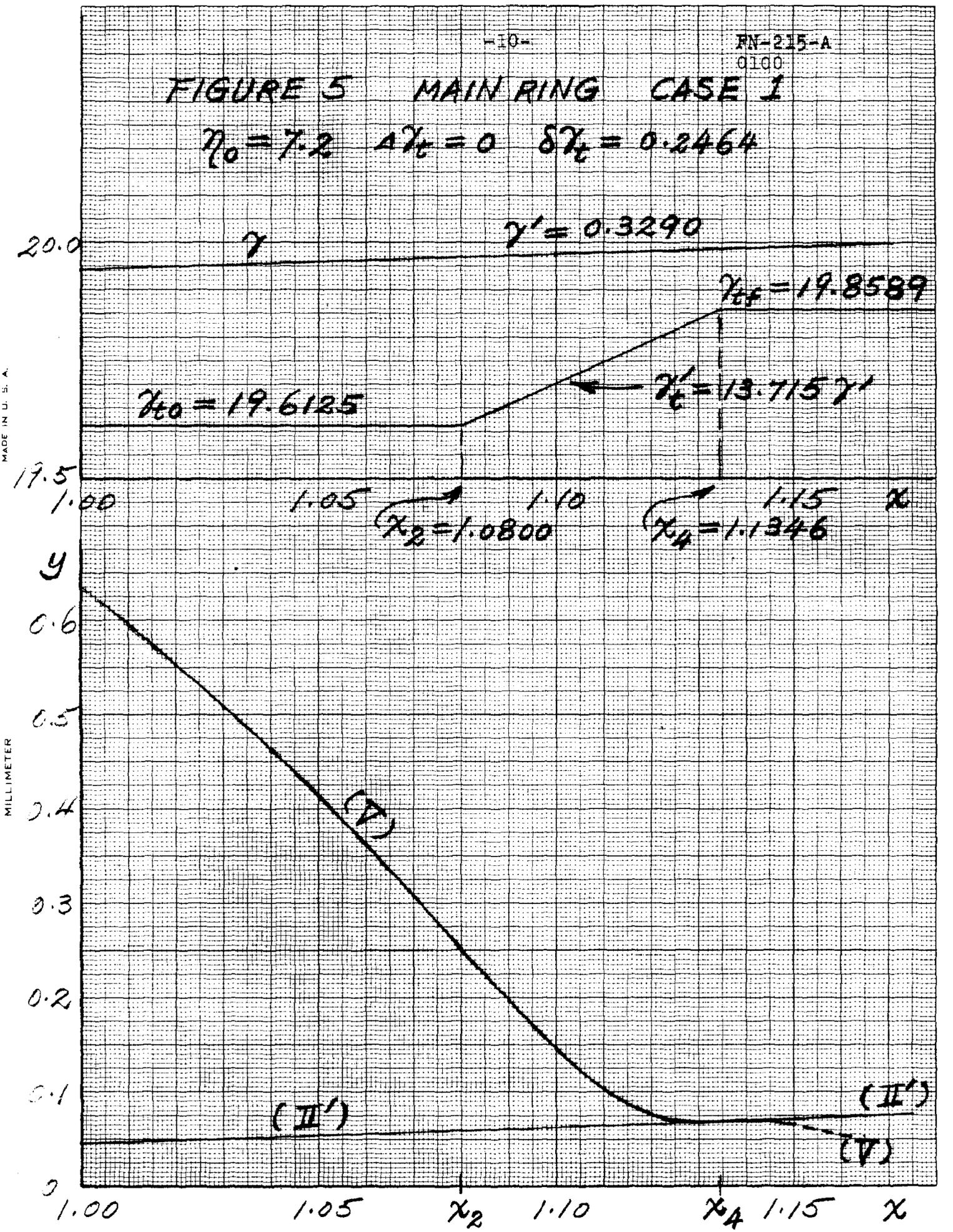
FIGURE 5 MAIN RING CASE 1

$\eta_0 = 7.2 \quad \Delta\gamma_t = 0 \quad \delta\gamma_t = 0.2464$



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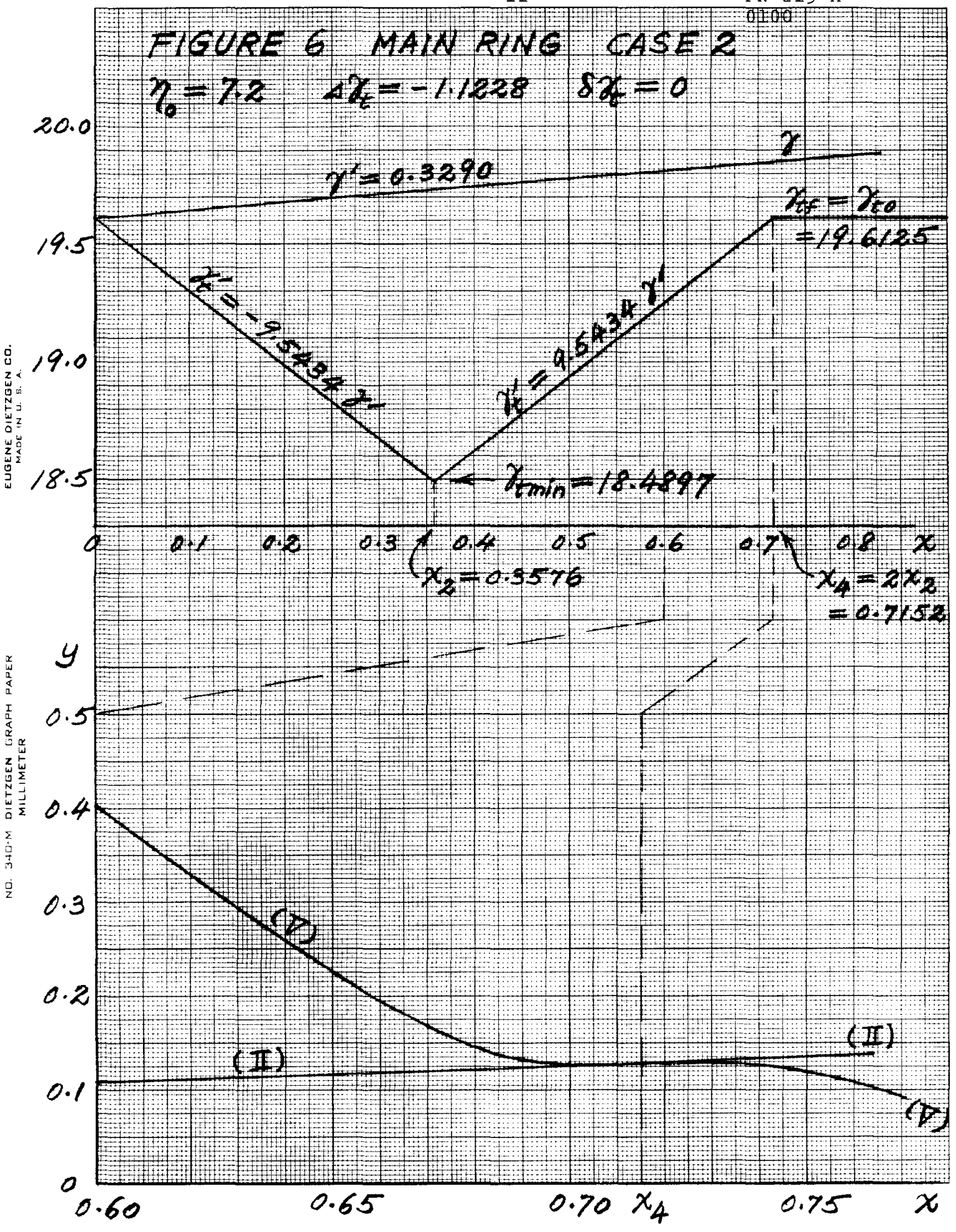

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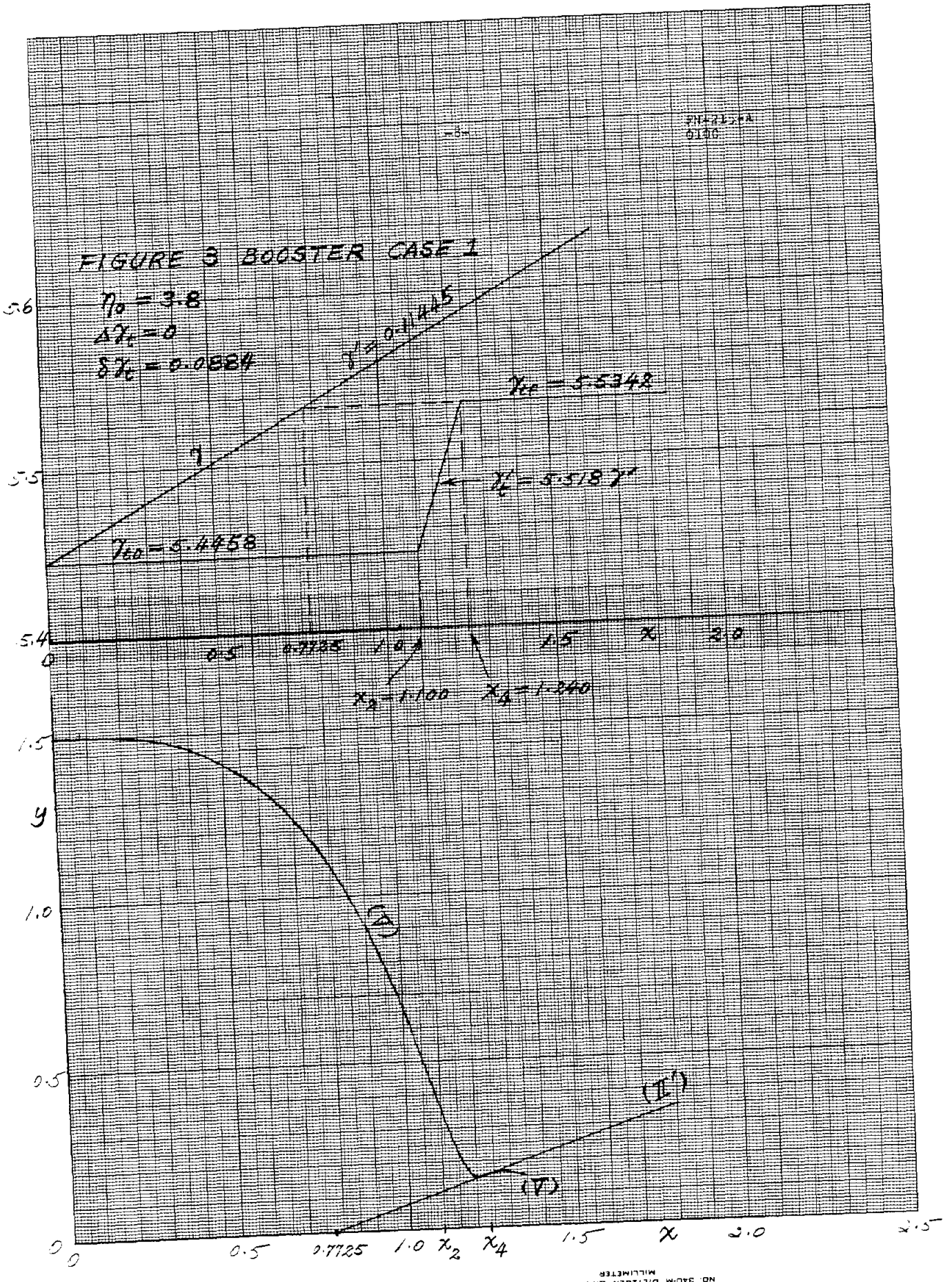
FIGURE 3 BOOSTER CASE 1



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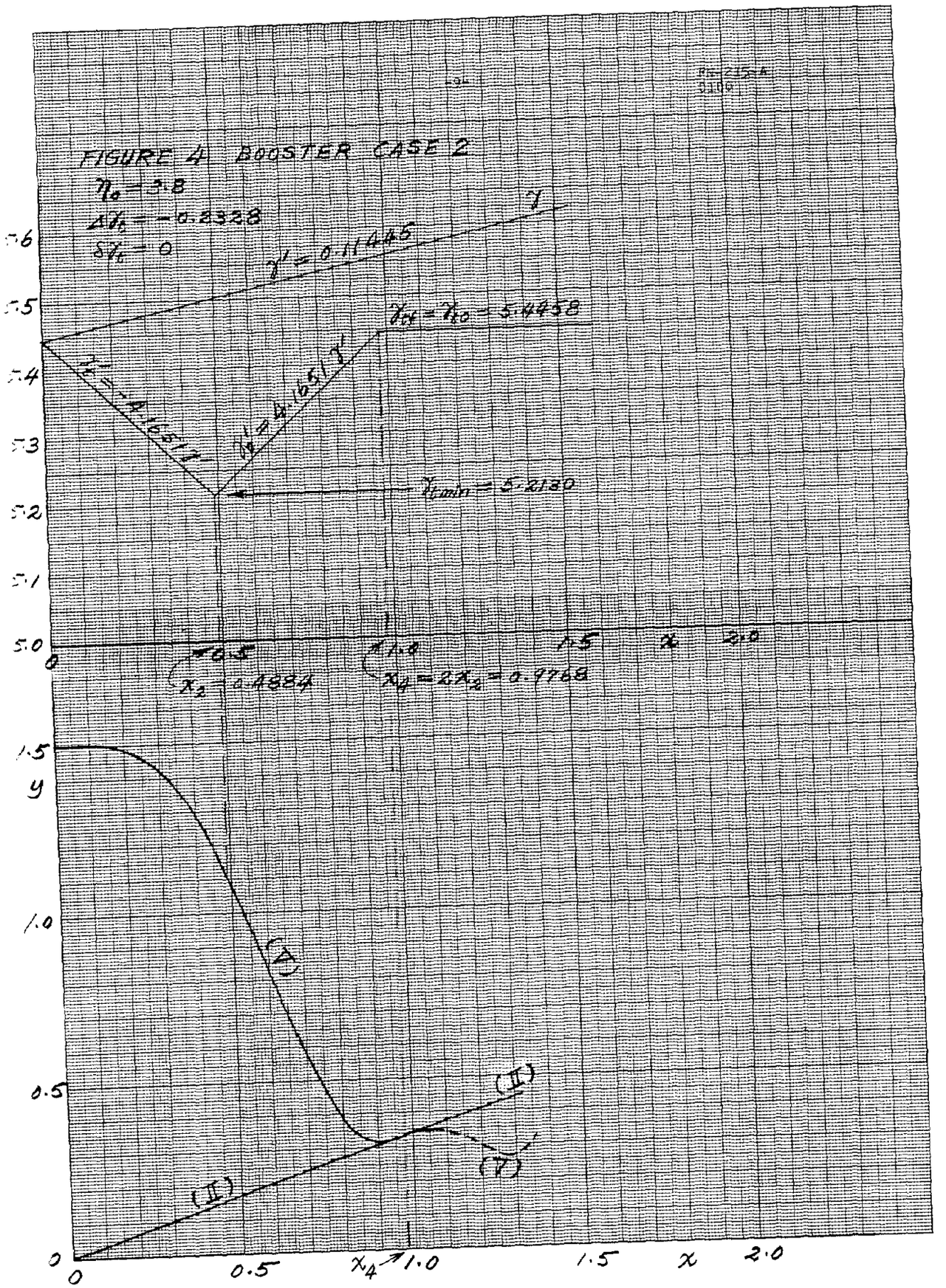
FIGURE 4 BOOSTER CASE 2
$\eta_0 = 3.8$
$\Delta\gamma_t = -0.2328$
$\delta\gamma_t = 0$
$\gamma' = 0.11445$
γ
$\gamma_{tf} = \gamma_{t0} = 5.4458$
$\gamma_t' = -4.1651\gamma'$
$\gamma_0' = 4.1651\gamma'$
$\gamma_{t\,min} = 5.2130$
5.6
5.5
5.4
5.3
5.2
5.1
5.0
0
0.5
1.0
1.5
x
2.0
$x_2 = 0.4884$
$x_4 = 2x_2 = 0.9768$
1.5
y
1.0
0.5
0
(I)
(II)
(V)
x_4